CONGRUENCES FOR k DOTS BRACELET PARTITION FUNCTIONS

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Abstract. Andrews and Paule introduced broken k-diamond partitions by using MacMahon's partition analysis. Recently, Fu found a generalization which he called k dots bracelet partitions and investigated some congruences for this kind of partitions. In this paper, by finding congruence relations between the generating function for 5 dots bracelet partitions and that for 5-regular partitions, we get some new congruences modulo 2 for the 5 dots bracelet partition function. Moreover, for a given prime p, we study arithmetic properties modulo p of k dots bracelet partitions.

1. Introduction

Andrews and Paule [1] studied broken k-diamond partitions by using MacMahon's partition analysis, and gave the generating function for $\Delta_k(n)$ which denotes the number of broken k-diamond partitions of n:

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty^2 (-q^{2k+1};q^{2k+1})_\infty}.$$

They [1] proved the following arithmetic theorem for $\Delta_1(n)$.

Theorem 1.1. *[1, Theorem 5] For $n \geq 0$,*

$$\Delta_1(2n+1) \equiv 0 \pmod 3.$$

Meanwhile, they posed some conjectures related to $\Delta_2(n)$. Arithmetic properties of broken k-diamond partitions have been the subject of many studies, see, for example [3, 7, 9–12, 14, 17]. Recently, Fu [4] found a combinatorial proof of Theorem 1.1 and introduced a generalization of broken k-diamond partitions which he called k dots bracelet partitions. The generating function for the number of this kind of partitions of n, denoted by $\mathfrak{B}_k(n)$, is given by

$$\sum_{n=0}^{\infty} \mathfrak{B}_k(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty^{k-1} (-q^k;q^k)_\infty}, \ k \geq 3.$$

Fu [4] proved the following congruences for $\mathfrak{B}_k(n)$.

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Theorem 1.2. *[4, Theorem 3.3] For $n > 0$, $k \geq 3$, if $k = p^r$ is a prime power, we have*

$$\mathfrak{B}_k(2n+1) \equiv 0 \pmod{p}.$$

Theorem 1.3. *[4, Theorem 3.5] For any $k \geq 3$, s an integer between 1 and $p-1$ such that $12s+1$ is a quadratic nonresidue modulo p, and any $n \geq 0$, if $p \mid k$ for some prime $p \geq 5$ say $k = pm$, then we have*

$$\mathfrak{B}_k(pn+s) \equiv 0 \pmod{p}.$$

Theorem 1.4. *[4, Theorem 3.6] For $n \geq 0$, $k \geq 3$ even, say $k = 2^m l$, where l is odd, we have*

$$\mathfrak{B}_k(2n+1) \equiv 0 \pmod{2^m}.$$

Later, Radu and Sellers [13] found some new congruences for $\mathfrak{B}_k(n)$.

Theorem 1.5. *[13, Theorem 1.4] For all $n \geq 0$,*

$$\begin{aligned} \mathfrak{B}_5(10n+7) &\equiv 0 \pmod{5^2}, \\ \mathfrak{B}_7(14n+11) &\equiv 0 \pmod{7^2}, \ \textit{and} \\ \mathfrak{B}_{11}(22n+21) &\equiv 0 \pmod{11^2}. \end{aligned}$$

In this paper, we continue to study arithmetic properties of k dots bracelet partitions. First, we recall two kinds of partitions which are used in this paper.

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. We have

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

If ℓ is a positive integer, then a partition is called an ℓ-regular partition if there is no part divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. The generating function for $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}.$$

In section 2, in view of an identity given by Ramanujan [16] and a congruence given by Hirschhorn and Sellers [8], we obtain two congruences modulo 2 for $\mathfrak{B}_5(n)$. Meanwhile, by finding a congruence relation between $\mathfrak{B}_5(n)$ and $b_5(n)$, we derive many infinite families of congruences modulo 2 for $\mathfrak{B}_5(n)$. In section 3, for a given prime p, by means of the p-dissection identity for $f(-q)$ given by the authors [6] and three classical congruences for $p(n)$ given by Ramanujan [15,16], we deduce more congruences modulo p for $\mathfrak{B}_k(n)$.

As usual, we follow the standard q-series notation [5]

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \quad \text{and} \quad (a_1,a_2,\ldots,a_m;q)_\infty = \prod_{j=1}^{m}(a_j;q)_\infty, \quad |q|<1.$$

The Legendre symbol is defined as

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p}, \\ -1, & \text{if } a \text{ is a quadratic non-residue modulo } p, \\ 0, & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

Let $f(a,b)$ be Ramanujan's general theta function given by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \qquad |ab| < 1.$$

In light of Jacobi's triple product identity [2, Theorem 1.3.3]

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq, -q/z, q^2; q^2)_\infty, \quad |q| < 1,$$

a special case of $f(a,b)$ is stated as follows:

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

2. Congruences Modulo 2 for $\mathfrak{B}_5(n)$

We recall Ramanujan's identity [16, p. 212]

$$(q;q)_\infty = \frac{(q^{10}, q^{15}, q^{25}; q^{25})_\infty}{(q^5, q^{20}; q^{25})_\infty} - q(q^{25}; q^{25})_\infty - q^2 \frac{(q^5, q^{20}, q^{25}; q^{25})_\infty}{(q^{10}, q^{15}; q^{25})_\infty}. \tag{2.1}$$

For convenience, set

$$a(q) = \frac{(q^{10}, q^{15}; q^{25})_\infty}{(q^5, q^{20}; q^{25})_\infty} \qquad \text{and} \qquad b(q) = \frac{(q^5, q^{20}; q^{25})_\infty}{(q^{10}, q^{15}; q^{25})_\infty} = \frac{1}{a(q)}.$$

Then we rewrite (2.1) as

$$(q;q)_\infty = (q^{25}; q^{25})_\infty \left(a(q) - q - q^2 b(q) \right). \tag{2.2}$$

In addition, Hirschhorn and Sellers [8] showed that

$$\sum_{n=0}^{\infty} b_5(2n) q^n \equiv (q^2; q^2)_\infty \pmod{2}. \tag{2.3}$$

By means of (2.2) and (2.3), we derive the following results.

Theorem 2.1. *For $n \geq 0$, we have*

$$\mathfrak{B}_5(10n+6) \equiv 0 \pmod{2},$$
$$\mathfrak{B}_5(10n+8) \equiv 0 \pmod{2}.$$

Proof. We show that

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty^4 (-q^5; q^5)_\infty}$$

$$
\begin{aligned}
&= \frac{(q^2;q^2)_\infty (q^5;q^5)_\infty}{(q;q)_\infty^5 (q^{10};q^{10})_\infty} \\
&\equiv \frac{(q^2;q^2)_\infty (q^5;q^5)_\infty}{(q^4;q^4)_\infty (q^{10};q^{10})_\infty (q;q)_\infty} \pmod 2 \\
&\equiv \frac{1}{(q^2;q^2)_\infty (q^{10};q^{10})_\infty} \frac{(q^5;q^5)_\infty}{(q;q)_\infty} \pmod 2 \\
&= \frac{1}{(q^2;q^2)_\infty (q^{10};q^{10})_\infty} \cdot \sum_{n=0}^{\infty} b_5(n) q^n.
\end{aligned}
$$

Then

$$
\begin{aligned}
\sum_{n=0}^{\infty} \mathfrak{B}_5(2n) q^n &\equiv \frac{1}{(q;q)_\infty (q^5;q^5)_\infty} \cdot \sum_{n=0}^{\infty} b_5(2n) q^n \pmod 2 \\
&\equiv \frac{(q^2;q^2)_\infty}{(q;q)_\infty (q^5;q^5)_\infty} \pmod 2 \qquad \text{by (2.3)} \\
&\equiv \frac{(q;q)_\infty}{(q^5;q^5)_\infty} \pmod 2.
\end{aligned}
$$

According to (2.2), it follows that

$$
\sum_{n=0}^{\infty} \mathfrak{B}_5(2n) q^n \equiv \frac{(q^{25};q^{25})_\infty}{(q^5;q^5)_\infty} \left(a(q) - q - q^2 b(q) \right) \pmod 2. \tag{2.4}
$$

Therefore, we get

$$
\begin{aligned}
\mathfrak{B}_5(2(5n+3)) = \mathfrak{B}_5(10n+6) \equiv 0 \pmod 2, \\
\mathfrak{B}_5(2(5n+4)) = \mathfrak{B}_5(10n+8) \equiv 0 \pmod 2.
\end{aligned}
$$

□

Lemma 2.2. *For $n \geq 0$, we have*

$$
\sum_{n=0}^{\infty} \mathfrak{B}_5(10n+2) q^n \equiv \sum_{n=0}^{\infty} b_5(n) q^n \pmod 2.
$$

Proof. Applying (2.4) yields that

$$
\sum_{n=0}^{\infty} \mathfrak{B}_5(2(5n+1)) q^n = \sum_{n=0}^{\infty} \mathfrak{B}_5(10n+2) q^n \equiv \frac{(q^5;q^5)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty} b_5(n) q^n \pmod 2.
$$

□

The authors [6] found that for any prime $p \geq 5$, $\left(\frac{-10}{p}\right) = -1$, $\alpha \geq 1$, and $n \geq 0$,

$$
b_5\left(4 \cdot p^{2\alpha} n + \frac{(24i+7p)p^{2\alpha-1}-1}{6}\right) \equiv 0 \pmod 2, \quad i = 1, 2, \ldots, p-1. \tag{2.5}
$$

Meanwhile, for $\alpha \geq 0$ and $n \geq 0$, there exist

$$
b_5\left(4 \cdot 5^{2\alpha+1} n + \frac{31 \cdot 5^{2\alpha} - 1}{6}\right) \equiv 0 \pmod 2, \tag{2.6}
$$

$$b_5\left(4\cdot 5^{2\alpha+1}n+\frac{79\cdot 5^{2\alpha}-1}{6}\right)\equiv 0 \pmod 2, \tag{2.7}$$

$$b_5\left(4\cdot 5^{2\alpha+2}n+\frac{83\cdot 5^{2\alpha+1}-1}{6}\right)\equiv 0 \pmod 2, \tag{2.8}$$

$$b_5\left(4\cdot 5^{2\alpha+2}n+\frac{107\cdot 5^{2\alpha+1}-1}{6}\right)\equiv 0 \pmod 2. \tag{2.9}$$

Therefore, The combination of Lemma 2.2 and (2.5)-(2.9) gives more congruences for $\mathfrak{B}_5(n)$.

Theorem 2.3. *For any prime $p\geq 5$, $\left(\frac{-10}{p}\right)=-1$, $\alpha\geq 1$, and $n\geq 0$, we have*

$$\mathfrak{B}_5\left(40\cdot p^{2\alpha}n+\frac{5\cdot(24i+7p)p^{2\alpha-1}+1}{3}\right)\equiv 0 \pmod 2,$$

where $i=1,2,\ldots,p-1$.

For example, setting $p=17$, $i=6$, and $\alpha=1$ in Theorem 2.3, we deduce that

$$\mathfrak{B}_5(11560n+7452)\equiv 0 \pmod 2.$$

Theorem 2.4. *For $\alpha\geq 1$ and $n\geq 0$, we have*

$$\mathfrak{B}_5\left(8\cdot 5^{2\alpha}n+\frac{31\cdot 5^{2\alpha-1}+1}{3}\right)\equiv 0 \pmod 2,$$

$$\mathfrak{B}_5\left(8\cdot 5^{2\alpha}n+\frac{79\cdot 5^{2\alpha-1}+1}{3}\right)\equiv 0 \pmod 2,$$

$$\mathfrak{B}_5\left(8\cdot 5^{2\alpha+1}n+\frac{83\cdot 5^{2\alpha}+1}{3}\right)\equiv 0 \pmod 2,$$

$$\mathfrak{B}_5\left(8\cdot 5^{2\alpha+1}n+\frac{107\cdot 5^{2\alpha}+1}{3}\right)\equiv 0 \pmod 2.$$

3. Congruences Modulo p for $\mathfrak{B}_k(n)$

The authors [6] derived that for a given prime $p\geq 5$,

$$f(-q)=\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}}(-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}},-q^{\frac{3p^2-(6k+1)p}{2}}\right)+(-1)^{\frac{\pm p-1}{6}}q^{\frac{p^2-1}{24}}f(-q^{p^2}), \tag{3.1}$$

where

$$\frac{\pm p-1}{6}:=\begin{cases}\frac{p-1}{6}, & p\equiv 1 \pmod 6,\\ \frac{-p-1}{6}, & p\equiv -1 \pmod 6.\end{cases}$$

Furthermore, for $-(p-1)/2\leq k\leq (p-1)/2$ and $k\neq(\pm p-1)/6$,

$$\frac{3k^2+k}{2}\not\equiv\frac{p^2-1}{24} \pmod p.$$

With the aid of the above result, we obtain the following lemma.

Lemma 3.1. *For any prime $p\geq 5, n\geq 0$, and $r\geq 1$, if $k=p^r$ is a prime power, then we have for $1\leq\alpha\leq (r+1)/2$,*

$$\sum_{n=0}^{\infty}\mathfrak{B}_k\left(p^{2\alpha-1}n+\frac{p^{2\alpha}-1}{12}\right)q^n\equiv\left((-1)^{\frac{\pm p-1}{6}}\right)^{\alpha}\frac{(q^{2p};q^{2p})_\infty}{\left(q^{2p^{r-(2\alpha-1)}};q^{2p^{r-(2\alpha-1)}}\right)_\infty}\pmod p.$$

Proof. We prove the lemma by induction on α. For $k=p^r$, Fu [4] showed that

$$\sum_{n=0}^{\infty}\mathfrak{B}_k(n)q^n\equiv\frac{(q^2;q^2)_\infty}{(q^{2k};q^{2k})_\infty}\pmod p.$$

In light of (3.1), it can be seen that

$$\sum_{n=0}^{\infty}\mathfrak{B}_k\left(pn+\frac{p^2-1}{12}\right)q^n\equiv(-1)^{\frac{\pm p-1}{6}}\frac{(q^{2p};q^{2p})_\infty}{\left(q^{2p^{r-1}};q^{2p^{r-1}}\right)_\infty}\pmod p,$$

which is the case when $\alpha=1$. Suppose that the lemma holds for α. We prove the case for $\alpha+1$. Since

$$\sum_{n=0}^{\infty}\mathfrak{B}_k\left(p^{2\alpha-1}n+\frac{p^{2\alpha}-1}{12}\right)q^n\equiv\left((-1)^{\frac{\pm p-1}{6}}\right)^{\alpha}\frac{(q^{2p};q^{2p})_\infty}{\left(q^{2p^{r-(2\alpha-1)}};q^{2p^{r-(2\alpha-1)}}\right)_\infty}\pmod p,$$

it can be shown that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_k\left(p^{2\alpha-1}(pn)+\frac{p^{2\alpha}-1}{12}\right)q^n&=\sum_{n=0}^{\infty}\mathfrak{B}_k\left(p^{2\alpha}n+\frac{p^{2\alpha}-1}{12}\right)q^n\\&\equiv\left((-1)^{\frac{\pm p-1}{6}}\right)^{\alpha}\frac{(q^2;q^2)_\infty}{\left(q^{2p^{r-2\alpha}};q^{2p^{r-2\alpha}}\right)_\infty}\pmod p.\qquad(3.2)\end{aligned}$$

Using (3.1) again, we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathfrak{B}_k\left(p^{2\alpha}\left(pn+\frac{p^2-1}{12}\right)+\frac{p^{2\alpha}-1}{12}\right)q^n\\=&\sum_{n=0}^{\infty}\mathfrak{B}_k\left(p^{2\alpha+1}n+\frac{p^{2\alpha+2}-1}{12}\right)q^n\\\equiv&\left((-1)^{\frac{\pm p-1}{6}}\right)^{\alpha+1}\frac{(q^{2p};q^{2p})_\infty}{\left(q^{2p^{r-(2\alpha+1)}};q^{2p^{r-(2\alpha+1)}}\right)_\infty}\pmod p.\end{aligned}$$

Therefore, the lemma holds for $\alpha+1$. □

Theorem 3.2. *For any prime $p\geq 5, n\geq 0$, and $r\geq 1$, if $k=p^r$ is a prime power, then we have*

$$\mathfrak{B}_k\left(p^{2\alpha}n+\frac{(12i+p)p^{2\alpha-1}-1}{12}\right)\equiv 0\pmod p,\qquad(3.3)$$

where $i=1,2,\cdots,p-1$ and $1\leq\alpha\leq r/2$. We also have

$$\mathfrak{B}_k\left(p^{2\alpha+1}n+\frac{(12j+1)p^{2\alpha}-1}{12}\right)\equiv 0\pmod p,$$

where $1\leq j\leq p-1$ is an integer such that $\left(\frac{12j+1}{p}\right)=-1$ and $1\leq\alpha\leq(r-1)/2$.

Proof. When $1 \leq \alpha \leq r/2$, from Lemma 3.1, it follows that for $i = 1, 2, \cdots, p-1$,

$$\mathfrak{B}_k\left(p^{2\alpha-1}(pn+i) + \frac{p^{2\alpha}-1}{12}\right) \equiv 0 \pmod{p},$$

which implies (3.3). Applying (3.1) to (3.2), we consider

$$j \equiv 2 \cdot \frac{3k^2+k}{2} \pmod{p},$$

namely,

$$12j + 1 \equiv (6k+1)^2 \pmod{p}.$$

Since $12j+1$ is a quadratic nonresidue modulo p, we conclude that

$$\mathfrak{B}_k\left(p^{2\alpha}(pn+j) + \frac{p^{2\alpha}-1}{12}\right) \equiv 0 \pmod{p}.$$

□

Based on Lemma 3.1 and generating functions for $p(n)$ and $b_\ell(n)$, we get the following congruence relations.

Theorem 3.3. *For any prime $p \geq 5$, $\alpha \geq 1$, and $n \geq 0$, if $k = p^{2\alpha-1}$ is a prime power, then we have*

$$\sum_{n=0}^{\infty} \mathfrak{B}_k\left(2p^{2\alpha-1}n + \frac{p^{2\alpha}-1}{12}\right) q^n \equiv \left((-1)^{\frac{\pm p-1}{6}}\right)^{\alpha} \sum_{n=0}^{\infty} b_p(n) q^n \pmod{p},$$

$$\sum_{n=0}^{\infty} \mathfrak{B}_k\left(2p^{2\alpha-1}n + \frac{p^{2\alpha}-1}{12}\right) q^n \equiv \left((-1)^{\frac{\pm p-1}{6}}\right)^{\alpha} (q^p;q^p)_\infty \sum_{n=0}^{\infty} p(n) q^n \pmod{p}. \quad (3.4)$$

Proof. Set $r = 2\alpha - 1$ in Lemma 3.1. Then $k = p^{2\alpha-1}$. So we derive that

$$\sum_{n=0}^{\infty} \mathfrak{B}_k\left(p^{2\alpha-1}n + \frac{p^{2\alpha}-1}{12}\right) q^n \equiv \left((-1)^{\frac{\pm p-1}{6}}\right)^{\alpha} \frac{(q^{2p};q^{2p})_\infty}{(q^2;q^2)_\infty} \pmod{p}.$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{B}_k\left(p^{2\alpha-1}(2n) + \frac{p^{2\alpha}-1}{12}\right) q^n &= \sum_{n=0}^{\infty} \mathfrak{B}_k\left(2p^{2\alpha-1}n + \frac{p^{2\alpha}-1}{12}\right) q^n \\ &\equiv \left((-1)^{\frac{\pm p-1}{6}}\right)^{\alpha} \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \pmod{p}.\end{aligned}$$

□

Combining (3.4) with three famous congruences for $p(n)$ given by Ramanujan [15,16]

$$p(5n+4) \equiv 0 \pmod{5}, \quad (3.5)$$

$$p(7n+5) \equiv 0 \pmod{7}, \quad (3.6)$$

$$p(11n+6) \equiv 0 \pmod{11}, \quad (3.7)$$

we obtain the following results.

Corollary 3.4. *For $\alpha \geq 1$ and $n \geq 0$, we have*

$$\mathfrak{B}_{5^{2\alpha-1}}\left(2\cdot 5^{2\alpha}n+\frac{101\cdot 5^{2\alpha-1}-1}{12}\right)\equiv 0 \pmod 5,$$
$$\mathfrak{B}_{7^{2\alpha-1}}\left(2\cdot 7^{2\alpha}n+\frac{127\cdot 7^{2\alpha-1}-1}{12}\right)\equiv 0 \pmod 7,$$
$$\mathfrak{B}_{11^{2\alpha-1}}\left(2\cdot 11^{2\alpha}n+\frac{155\cdot 11^{2\alpha-1}-1}{12}\right)\equiv 0 \pmod{11}.$$

Proof. With the aid of (3.4), we arrive at

$$\sum_{n=0}^{\infty}\mathfrak{B}_{5^{2\alpha-1}}\left(2\cdot 5^{2\alpha-1}n+\frac{5^{2\alpha}-1}{12}\right)q^n\equiv(-1)^{\alpha}(q^5;q^5)_\infty\sum_{n=0}^{\infty}p(n)q^n \pmod 5,$$
$$\sum_{n=0}^{\infty}\mathfrak{B}_{7^{2\alpha-1}}\left(2\cdot 7^{2\alpha-1}n+\frac{7^{2\alpha}-1}{12}\right)q^n\equiv(-1)^{\alpha}(q^7;q^7)_\infty\sum_{n=0}^{\infty}p(n)q^n \pmod 7,$$
$$\sum_{n=0}^{\infty}\mathfrak{B}_{11^{2\alpha-1}}\left(2\cdot 11^{2\alpha-1}n+\frac{11^{2\alpha}-1}{12}\right)q^n\equiv(q^{11};q^{11})_\infty\sum_{n=0}^{\infty}p(n)q^n \pmod{11}.$$

Applying (3.5), (3.6), and (3.7) yields

$$\mathfrak{B}_{5^{2\alpha-1}}\left(2\cdot 5^{2\alpha-1}(5n+4)+\frac{5^{2\alpha}-1}{12}\right)\equiv 0 \pmod 5,$$
$$\mathfrak{B}_{7^{2\alpha-1}}\left(2\cdot 7^{2\alpha-1}(7n+5)+\frac{7^{2\alpha}-1}{12}\right)\equiv 0 \pmod 7,$$
$$\mathfrak{B}_{11^{2\alpha-1}}\left(2\cdot 11^{2\alpha-1}(11n+6)+\frac{11^{2\alpha}-1}{12}\right)\equiv 0 \pmod{11}.$$

□

Another congruence modulo p for $\mathfrak{B}_k(n)$ can be directly obtained from Lemma 3.1.

Theorem 3.5. *For any prime $p \geq 5$, $\alpha \geq 1$, and $n \geq 1$, if $k = p^{2\alpha}$ is a prime power, then we have*

$$\mathfrak{B}_k\left(p^{2\alpha-1}n+\frac{p^{2\alpha}-1}{12}\right)\equiv 0 \pmod p.$$

Proof. Set $r = 2\alpha$ in Lemma 3.1. Then we have $k = p^{2\alpha}$. Therefore,

$$\sum_{n=0}^{\infty}\mathfrak{B}_k\left(p^{2\alpha-1}n+\frac{p^{2\alpha}-1}{12}\right)q^n\equiv\left((-1)^{\frac{\pm p-1}{6}}\right)^{\alpha} \pmod p.$$

□

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