Skew spectra of oriented bipartite graphs

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July 1, 2013

Abstract

A graph G is said to have a parity-linked orientation ϕ if every even cycle C_{2k} in G^{ϕ} is evenly (resp. oddly) oriented whenever k is even (resp. odd). In this paper, this concept is used to provide an affirmative answer to the following conjecture of D. Cui and Y. Hou [D. Cui, Y. Hou, On the skew spectra of Cartesian products of graphs, *The Electronic J. Combin.* 20(2) (2013), #P19]: Let $G = G(X, Y)$ be a bipartite graph. Call the $X \to Y$ orientation of G, the canonical orientation. Let ϕ be any orientation of G and let $Sp_S(G^{\phi})$ and $Sp(G)$ denote respectively the skew spectrum of G^{ϕ} and the spectrum of G. Then $Sp_S(G^{\phi}) = \mathbf{i}Sp(G)$ if and only if ϕ is switching-equivalent to the canonical orientation of G. As an illustration of this result, we determine the switch for a special family of oriented hypercubes Q_d^{ϕ}, $d \geq 1$. Moreover, we give an orientation of the Cartesian product of a bipartite graph and a graph, and then determine the skew spectrum of the resulting oriented product graph, which generalizes a result of Cui and Hou. Further this is used to construct new families of oriented graphs with maximum skew energy.

Keywords: Oriented bipartite graphs; Skew energy and skew spectrum of an oriented graph; Canonical orientation of bipartite graphs;

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Parity-linked orientation of a graph; Switching-equivalence between orientations of a graph; Cartesian product; Oriented hypercube.
AMS Subject Classification: 05C20, 05C50, 05C75.

1 Introduction

Let $G = (V, E)$ be a finite simple undirected graph of order n with $V = \{v_1, v_2, \ldots, v_n\}$ as its vertex set and E as its edge set. An orientation ϕ of E results in the oriented graph $G^\phi = (V, \Gamma)$, where Γ is the arc set of G^ϕ. The adjacency matrix of G is the $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ if $(v_i, v_j) \in E$ and $a_{ij} = 0$ otherwise. As the matrix A is real and symmetric, all its eigenvalues are real. The spectrum of G, denoted by $Sp(G)$, is the spectrum of A. The energy $\mathcal{E}(G)$ of a graph G of order n, introduced by Ivan Gutman [8] in 1978, is defined as the sum of the absolute values of its eigenvalues. The skew adjacency matrix of an oriented graph G^ϕ is the $n \times n$ matrix $S(G^\phi) = (s_{ij})$, where $s_{ij} = 1 = -s_{ji}$ whenever $(v_i, v_j) \in \Gamma(G^\phi)$ and $s_{ij} = 0$ otherwise. As the matrix $S(G^\phi)$ is real and skew symmetric, its eigenvalues are all pure imaginary. The skew spectrum of G^ϕ is the spectrum of $S(G^\phi)$. The concept of graph energy was recently generalized to oriented graphs as skew energy by Adiga, Balakrishnan and Wasin So in [1]. The skew energy $\mathcal{E}_S(G^\phi)$ of an oriented graph G^ϕ is defined as the sum of the absolute values of all the eigenvalues of $S(G^\phi)$. For the properties of the energy and spectrum of a graph, the reader may refer to [3, 9, 12], and for skew energy and skew spectrum of an oriented graph, to [1, 4, 10, 11, 13]. We follow [3] for standard graph theoretic notation.

By a cycle in G^ϕ, we refer to not necessarily a directed cycle. An oriented even cycle is classified into two types based on its structure. An even cycle C of G^ϕ is said to be evenly or oddly oriented according as the number of arcs of C in each direction is even or odd [10].

Let $\mathscr{G}$ denote the family of graphs without even cycles. In [4], Cavers et al. have proved the following result.

Theorem 1.1 (Cavers et al. [4])**.** *The skew spectrum of G^ϕ remains invariant under any orientation ϕ of G if and only if G contains no even cycles, that is, $G \in \mathscr{G}$.*

2 Oriented bipartite graphs G^{ϕ} with $Sp_S(G^{\phi}) = \mathbf{i}Sp(G)$

Let $G = G(X,Y)$ be a bipartite graph with bipartition (X,Y). The *canonical orientation* of G is that orientation which orients all the edges from one partite set to the other. It is immaterial if it is from X to Y or from Y to X. Shader and So [13] have shown that for the canonical orientation σ of $G(X,Y)$,

$$Sp_S(G^{\sigma}) = \mathbf{i}Sp(G). \tag{1}$$

From this point onward, σ stands for the canonical orientation of a bipartite graph G with a fixed bipartition (X,Y).

Let G^{ϕ} be an oriented graph of order n. An even cycle C_{2k} of length $2k$ in G^{ϕ} is said have a *parity-linked orientation* if it is evenly oriented whenever k is even and oddly oriented whenever k is odd. If every even cycle in G^{ϕ} has a parity-linked orientation, then the orientation ϕ is defined to be a parity-linked orientation of G. (The parity-linked orientation is termed as uniform orientation in [6].)

In [6], Cui and Hou have given a characterization of oriented bipartite graphs G^{ϕ} that satisfy Equation (1) by using the parity-linked orientation of graphs.

Theorem 2.1 ([6]). *Suppose G^{ϕ} is an oriented bipartite graph with G as its underlying graph. Then $Sp_S(G^{\phi}) = \mathbf{i}Sp(G)$ if and only if the orientation ϕ of G is parity-linked.*

Let U be any proper subset of $V(G)$ of an oriented graph G^{ϕ_1} and let $\overline{U} = V(G) \setminus U$ be its complement. Reversing the orientations of all the arcs between U and $\overline{U}$ results in another oriented graph G^{ϕ_2}. This process is called the *switch* of G^{ϕ_1} with respect to U. The oriented graph got by two successive switches with respect to U_1 and U_2 is just the oriented graph obtained from G by the switch with respect to the set $U_1 \Delta U_2$, the symmetric difference of U_1 and U_2.

Suppose ϕ_1 and ϕ_2 are two orientations of a graph G. Then G^{ϕ_1} and G^{ϕ_2} are said to be *switching-equivalent* if G^{ϕ_2} can be obtained from G^{ϕ_1} by a switch. It is clear that switching-equivalence among the set $\mathscr{O}$ of all orientations of a graph G is indeed an equivalence relation on $\mathscr{O}$. The following result is proved in [1].

Theorem 2.2 ([1]). *Let ϕ_1 and ϕ_2 be two orientations of a graph G. If G^{ϕ_1} and G^{ϕ_2} are switching-equivalent, then $Sp_S(G^{\phi_1}) = Sp_S(G^{\phi_2})$.*

We mention that the converse of Theorem 2.2 is not true for non-bipartite graphs.

Example 2.3. Consider the two orientations ϕ_1 and ϕ_2 of the cycle graph C_5 as given in Figure 1.

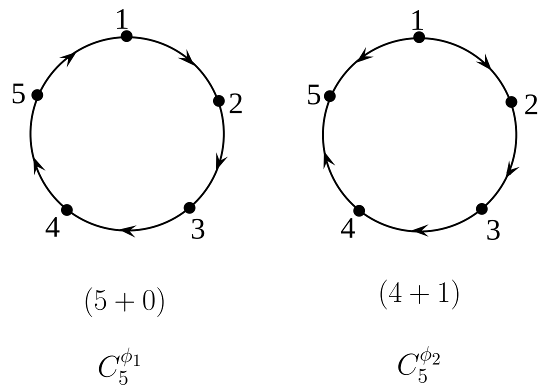


Figure 1: Two orientations of the cycle graph C_5

Since $C_5 \in \mathscr{G}$, the family of graphs without even cycles, by Theorem 1.1,

$$Sp_S(C_5^{\phi_1}) = Sp_S(C_5^{\phi_2}).$$

The oriented cycle $C_5^{\phi_1}$ has 5 arcs in one direction (clockwise) while $C_5^{\phi_2}$ has 4 arcs in the same direction for the given labeling. Any switch in $C_5^{\phi_1}$ will cause an even number of changes in the number of arcs in both the directions. Hence the 5 arcs in the clockwise direction can only become either 3 arcs or 1 arc in the clockwise direction after any switch but never 4 arcs in the clockwise direction. Therefore ϕ_1 and ϕ_2 are not switching-equivalent in C_5. (Clearly, 5 can be replaced by any odd number $2p+1$, $p \geq 1$.)

In [6], Cui and Hou conjectured that for an oriented *bipartite* graph G^ϕ, $Sp_S(G^\phi) = \mathbf{i}Sp(G)$ if and only if G^ϕ is switching-equivalent to G^σ, where σ is the canonical orientation of G. In this paper, we settle the above conjecture in the affirmative and present it as the following theorem.

Theorem 2.4 (Conjectured in [6])**.** *Suppose ϕ is an orientation of a bipartite graph $G = G(X, Y)$. Then $Sp_S(G^\phi) = \mathbf{i}Sp(G)$ if and only if G^ϕ is switching-equivalent to G^σ, where σ is the canonical orientation of G.*

Proof. Without loss of generality, we may assume that G is a connected graph.
Sufficiency. If G^ϕ and G^σ are switching-equivalent, then by Theorem 2.2, $Sp_S(G^\phi) = Sp_S(G^\sigma) = \mathbf{i}Sp(G)$ (where the second equality follows from (1)).

Necessity. We prove by induction on the number of edges m of the bipartite graph G. The result is trivial for $m = 1$.

Assume that the result is true for all bipartite graphs with at most $m - 1 (m \geq 2)$ arcs. Let G be a bipartite graph with m edges and (X, Y) be the bipartition of the vertex set of G. Suppose that ϕ is an orientation of G such that $Sp_S(G^\phi) = \mathbf{i}Sp(G)$. We have to prove that ϕ is switching-equivalent to σ. Let e be any edge of G. By Theorem 2.1, ϕ is a parity-linked orientation of G^ϕ and hence of $(G-e)^\phi$. Consequently, $(G-e)^{\phi_e}$ has a parity-linked orientation, where ϕ_e is the restriction of ϕ to the graph $G - e$. So again by Theorem 2.1,

$$Sp_S((G-e)^{\phi_e}) = \mathbf{i}Sp(G-e).$$

Consequently, by induction hypothesis, $(G-e)^{\phi_e}$ is switching-equivalent to $(G-e)^{\sigma_e}$, where σ_e is the restriction of σ to the graph $G - e$.

Let α be the switch that takes $(G-e)^{\phi_e}$ to $(G-e)^{\sigma_e}$ effected by the subset U of $V(G-e) = V(G)$. We claim that α takes ϕ to σ in G. If not, then the resulting oriented graph $G^{\phi'}$ will be of the following type: All the arcs of $G - e$ will be oriented from one partite set (say, X) to the other (namely, Y) while the arc e will be oriented in the reverse direction, that is, from Y to X (See Figure 2).

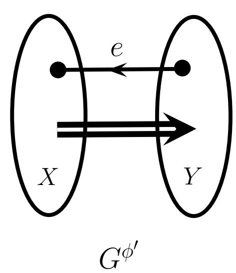
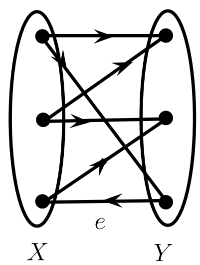


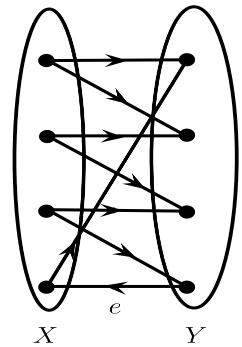
Figure 2: The oriented bipartite graph $G^{\phi'}$ in Theorem 2.4

Consider first the case when e is a cut edge of G. The subgraph $G - e$ will then consist of two components with vertex sets, say, S_1 and S_2. Now switch with respect to S_1. This will change the orientation of the only arc e and the resulting orientation is σ. Consequently, ϕ is switching-equivalent to σ.

Note that the above argument also takes care of the case when G is a tree since each edge of G will then be a cut edge. Hence we now assume that G contains an even cycle C_{2k} containing the arc e and complete the proof. But then any such C_{2k} has $k - 1$ arcs in one direction and $k + 1$ arcs in the opposite direction (see Figure 3) thereby not admitting a parity-linked orientation. Hence this case can not arise. Consequently, ϕ is switching-equivalent to σ in G. $\square$



(a) An evenly oriented C_6 in $G^{\phi'}$



(b) An oddly oriented C_8 in $G^{\phi'}$

Figure 3: Cycle C_{2k} for $k = 3, 4$ in $G^{\phi'}$

Theorem 2.1 provides a nice characterization for an oriented bipartite graph G^ϕ to have the property that $Sp_S(G^\phi) = \mathbf{i}Sp(G)$. But it requires to check if every cycle in G^ϕ possesses a parity-linked orientation. A natural question is the following: Is it possible to reduce the number of checks to determine whether an oriented graph G^ϕ has a parity-linked orientation? Our next result provides an answer in this direction.

Theorem 2.5. *Let G be a bipartite graph and ϕ be an orientation of G. If ϕ induces a parity-linked orientation on every chordless (even) cycle of G, then $Sp_S(G^\phi) = \mathbf{i}Sp(G)$.*

Proof. By virtue of Theorem 2.1, it suffices to show that if ϕ induces a parity-linked orientation on every chordless (even) cycle of G, then ϕ induces a parity-linked orientation on every cycle of G. If the result were not true, then there exists a cycle $C_{2\ell}$ in G^ϕ of least length 2ℓ such that ϕ does not induce a parity-linked orientation on $C_{2\ell}$. This of course means that $C_{2\ell}$ is evenly (resp. oddly) oriented if l is odd (resp. even). By hypothesis, $C_{2\ell}$ contains a chord x_1y_1. Suppose that $C_{2\ell} = x_1x_2 \dots x_{\ell_1}y_1y_2 \dots y_{2\ell-\ell_1}x_1$ in clockwise direction. Consider the two cycles $C_1 = x_1x_2 \dots x_{\ell_1}y_1x_1$ and $C_2 = x_1y_1y_2 \dots y_{2\ell-\ell_1}x_1$ with respective lengths $\ell_1 + 1$ and $2\ell - \ell_1 + 1$ in clockwise direction. Note that C_1 and C_2 are also even (G being bipartite). Suppose that C_1 and C_2 contain respectively r_1 and r_2 arcs in the clockwise direction. By the choice of $C_{2\ell}$, C_1 and C_2 possess the parity-linked orientation. Hence

$$\frac{\ell_1 + 1}{2} \equiv r_1 (\text{mod } 2) \text{ and } \frac{2\ell - \ell_1 + 1}{2} \equiv r_2 (\text{mod } 2).$$

It follows that $\ell + 1 \equiv (r_1 + r_2)(\text{mod } 2)$. Observe that if the arc corresponding to x_1y_1 is clockwise in C_1, then it must be anticlockwise in C_2 and vice versa.

This of course means that $C_{2\ell}$ also admits the parity-linked orientation. This contradiction proves the result. $\square$

Combining Theorem 2.1 and Theorem 2.5, we obtain immediately the following corollary.

Corollary 2.6. *Let G be a bipartite graph and ϕ, an orientation of G. Then $Sp_S(G^\phi) = \mathbf{i}Sp(G)$ if and only if ϕ induces a parity-linked orientation on all the chordless cycles of G.*

Remark 2.7. Let $\mathscr{C}$ denote the set of all cycles of a bipartite graph G. A subset $\mathscr{S}$ of $\mathscr{C}$ is called a *generating set* of $\mathscr{C}$ if for any cycle C of $\mathscr{C}$ either $C \in \mathscr{S}$ or there is a sequence of cycles $C_1, C_2, \dots, C_k$ in $\mathscr{S}$ such that $C = ((C_1 \Delta C_2)\Delta C_3)\dots\Delta C_k$ and for $2 \le p \le k-1$, $((C_1\Delta C_2)\Delta C_3)\dots\Delta C_p$ are all cycles of G. With this notation, one can prove that for any oriented bipartite graph G^ϕ, $Sp_S(G^\phi) = \mathbf{i}Sp(G)$ if and only if ϕ induces a parity-linked orientation for every cycle in a generating set $\mathscr{S}$ of $\mathscr{C}$ in G. Actually, the set of all chordless cycles of a graph G is a generating set of the set of all cycles of G.

3 Switching-equivalence in oriented hypercubes

We present below an illustration for Theorem 2.4. In [2], Anuradha and Balakrishnan have constructed an oriented hypercube Q_d^ϕ for which $Sp_S(Q_d^\phi) = \mathbf{i}Sp(Q_d)$, $d \ge 1$.

By Theorem 2.4, ϕ must be switching-equivalent to the canonical orientation σ of Q_d. We now determine a switching set U_d in Q_d^ϕ that takes ϕ to σ.

We first recall the algorithm given in [2] by means of which Q_d^ϕ, $d \ge 1$, is constructed.

Algorithm 3.1. The hypercube Q_d, $d \ge 2$, can be constructed by taking two copies of Q_{d-1} and making the corresponding vertices in the two copies adjacent. Let $V(Q_d) = \{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d) \ : \ \varepsilon_i = 0 \text{ or } 1\}$ be the vertex set of Q_d.

1. For $Q_1 = K_2$, $V(Q_1) = \{(0), (1)\}$. Set $(1, 0) \in \Gamma(Q_1^\phi)$.

2. Assume that for $i = 1, 2, \dots, k(< d)$, the oriented hypercube Q_k^ϕ has been constructed. For $i = k+1$, the oriented hypercube Q_{k+1}^ϕ is formed as follows:

(a). Take two copies $C_0^{(k)}$ and $C_1^{(k)}$ of Q_k^ϕ. Reverse the orientation of all the arcs in $C_1^{(k)}$.

(b). For $j = 0, 1$, relabel the vertices of $C_j^{(k)}$ by adding j as the first coordinate, that is, if $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in Q_k$, then the vertex $(0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in C_0^{(k)}$ and the vertex $(1, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in C_1^{(k)}$.

(c). Let (X_0, Y_0) be the bipartition of $V(Q_k)$ in $C_0^{(k)}$ such that the vertex labeled $(0, 0, \ldots, 0)$ is in X_0. Set the corresponding bipartition in $C_1^{(k)}$ as (X_1, Y_1). (Note that the vertex labeled $(1, 0, 0, \ldots, 0) \in X_1$.) Consequently $X = X_0 \cup Y_1$ and $Y = X_1 \cup Y_0$ form the bipartition of $V(Q_{k+1})$.

(d). Add an edge between the vertices of $C_0^{(k)}$ and $C_1^{(k)}$ that differ in exactly the first coordinate. For each such edge, assign the orientation from X_0 to X_1 and from Y_1 to Y_0 (see Figure 4). This yields the oriented hypercube Q_{k+1}^ϕ. (See Figure 5.)

3. If $k + 1 = d$, stop; else take $k \leftarrow k + 1$, return to Step 2. $\square$

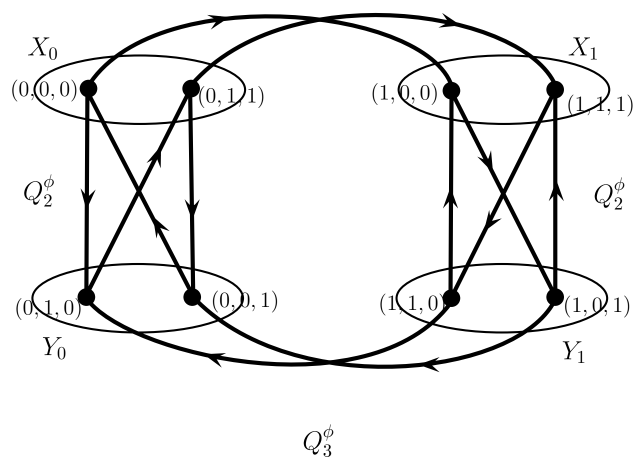


Figure 4: Example for Step 2(d) in Algorithm 3.1

Let (X, Y) be the bipartition of $V(Q_d^\phi)$ for $d \geq 1$ such that the vertex $(0, 0, \ldots, 0)$ is in X. It is then easy to observe from the construction of Q_d^ϕ that the indegree, $\deg^-(u)$, of each vertex $u \in X$ is 1 while the outdegree $\deg^+(v)$ of each vertex $v \in Y$ is 1 in Q_d^ϕ.

For each $d \geq 1$, we now define a set $U_d \subset V(Q_d^\phi)$ recursively as follows: For the oriented hypercube Q_1^ϕ, set $U_1 = \{(0)\}$. For $k \geq 1$ $(k < d)$, assume that the set U_k has been determined. Form the set U_{k+1} of the oriented hypercube

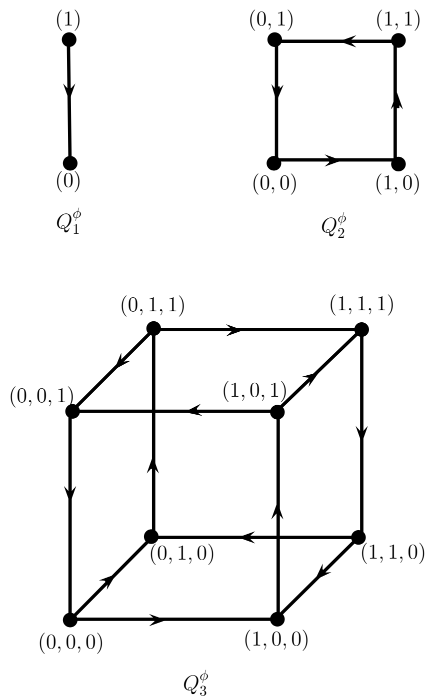


Figure 5: Orientation ϕ of hypercube Q_i, $i = 1, 2, 3$, defined in Algorithm 3.1

Q_{k+1}^ϕ by taking, for each vertex $v = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in U_k$ of the hypercube Q_k^ϕ, the vertices $v_0 = (0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ and $v_1 = (1, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$. Note that in the oriented hypercube Q_{k+1}^ϕ, if U_k^0 and U_k^1 are the two sets corresponding to U_k in the two copies $C_0^{(k)}$ and $C_1^{(k)}$ of Q_k^ϕ then $U_{k+1} = U_k^0 \cup U_k^1$ and $|U_{k+1}| = 2^k$.

We now show that for each $d \geq 1$, a switch with respect to the set U_d in the oriented hypercube Q_d^ϕ results in the canonical orientation σ of Q_d.

Theorem 3.2. *Suppose Q_d^ϕ is the oriented hypercube obtained by Algorithm 3.1. Let $U_d \subset V(Q_d^\phi)$ be determined as above. Then a switch with respect to the set U_d, for $d = 1, 2, \ldots,$ yields the canonical orientation σ of Q_d.*

Proof. Proof by induction on d. It is obvious for $d = 1, 2$. (For $d = 1, 2$, $U_1 = \{(0)\}$ and $U_2 = \{(0, 0), (1, 0)\}$.)

Suppose that a switch with respect to the set U_k $(k < d)$, yields the canonical orientation in the oriented hypercube Q_k^ϕ. Consider the set U_{k+1} of the oriented hypercube Q_{k+1}^ϕ. Clearly Q_{k+1}^ϕ consists of two copies $C_0^{(k)}$ and $C_1^{(k)}$ of Q_k^ϕ. For $i = 0, 1$, let U_k^i be the switch in the corresponding copy $C_i^{(k)}$. It is then easy to observe that

$$U_{k+1} = U_k^0 \cup U_k^1.$$

This shows that the copies $C_0^{(k)}$ and $C_1^{(k)}$ in Q_{k+1}^{ϕ} exhibit canonical orientation after the switch with respect to U_{k+1}. Further any arc between the two copies agrees with the canonical orientation (see Step 2(d) of Algorithm 3.1). Hence the switch with respect to U_{k+1} results in Q_{k+1}^{σ}. Applying induction, the result follows. $\square$

4 The skew spectrum of $H\Box G$ with H bipartite

In [1], Adiga et al. have shown that the skew energy of any oriented graph G^{ϕ} of order n, for which the underlying undirected graph G is k-regular, is bounded above by $n\sqrt{k}$ and posed the following problem:

Problem 4.1. *Which k-regular graphs G on n vertices have orientations ϕ with $\mathcal{E}_S(G^{\phi}) = n\sqrt{k}$, or equivalently, $S(G^{\phi})^T S(G^{\phi}) = kI_n$?*

In this section, we give an orientation of the Cartesian product $H\Box G$, where H is bipartite, by extending the orientation of $P_m\Box G$ in [6], and we calculate its skew spectrum. As an application of this orientation, we construct new families of oriented graphs with maximum skew energy, which generalizes the construction in [6].

Let H and G be graphs with p and n vertices respectively. Recall that the Cartesian product $H\Box G$ of H and G is the graph with vertex set $V(H)\times V(G)$ and the vertices (u_1, v_1) and (u_2, v_2) are adjacent in $H\Box G$ if and only if $u_1 = u_2$ and v_1v_2 is an edge of G, or if $v_1 = v_2$ and u_1u_2 is an edge of H. Assume that τ is any orientation of H and ϕ is any orientation of G. There is a natural way to define the oriented Cartesian product $H^{\tau}\Box G^{\phi}$ of H^{τ} and G^{ϕ} whose underlying undirected graph is $H\Box G$: There is an arc from (u_1, v_1) to (u_2, v_2) if and only if $u_1 = u_2$ and (v_1, v_2) is an arc of G^{ϕ}, or if $v_1 = v_2$ and u_1u_2 is an arc of H^{τ}. The skew spectrum of $H^{\tau}\Box G^{\phi}$ has been determined in [6]. Some interesting results on the skew spectrum of the product $H^{\tau}\Box G^{\phi}$, where H^{τ} is an oriented hypercube are obtained in [2].

When H is a bipartite graph with bipartition X and Y, we modify the above definition of $H^{\tau}\Box G^{\phi}$ to obtain a new product graph $(H^{\tau}\Box G^{\phi})^o$ with the following condition: If $u \in Y$ and $(v_1, v_2) \in \Gamma(G^{\phi})$, then we make $(u, v_2)(u, v_1)$ an arc of $H^{\tau}\Box G^{\phi}$ (instead of $(u, v_1)(u, v_2)$); the other arcs of $H^{\tau}\Box G^{\phi}$ remain unchanged.

Theorem 4.2. *Let H^{τ} be an oriented bipartite graph of order p and let the skew eigenvalues of H^{τ} be the nonzero complex numbers $\pm\mathbf{i}\mu_1, \pm\mathbf{i}\mu_2, \ldots, \pm\mathbf{i}\mu_r$ and $p - 2r$ 0's. Let G^{ϕ} be an oriented graph of order n and let the skew* *eigenvalues of* G^ϕ *be the nonzero complex numbers* $\pm\mathbf{i}\lambda_1, \pm\mathbf{i}\lambda_2, \ldots, \pm\mathbf{i}\lambda_t$ *and* $n-2t$ *0's. Then the skew eigenvalues of the oriented graph* $(H^\tau \Box G^\phi)^o$ *are* $\pm\mathbf{i}\sqrt{\mu_j^2+\lambda_k^2}$, $j=1,\ldots,r$, $k=1,\ldots,t$, *each with multiplicity* 2; $\pm\mathbf{i}\mu_j$, $j=1,\ldots,r$, *each with multiplicity* $n-2t$; $\pm\mathbf{i}\lambda_j$, $k=1,\ldots,t$, *each with multiplicity* $p-2r$ *and* 0 *with multiplicity* $(p-2r)(n-2t)$.

Proof. Let $H = H(X,Y)$ with $|X| = p_1$ and $|Y| = p_2$. With suitable labeling of the vertices of $H \Box G$, the skew adjacency matrix $S = S((H^\tau \Box G^\phi)^o)$ can be chosen as follows:

$$S = I'_{p_1+p_2} \otimes S(G^\phi) + S(H^\tau) \otimes I_n,$$

where $I'_{p_1+p_2} = I'_p = (a_{ij})$, $a_{ii} = 1$ if $1 \le i \le p_1$, $a_{ii} = -1$ if $p_1+1 \le i \le p$ and $a_{ij} = 0$ otherwise; $S(H^\tau)$ is the partitioned matrix $\begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}$, where B is a $p_1 \times p_2$ matrix. Further, $\otimes$ stands for the Kronecker product of two matrices [3].

We first determine the singular values of S. Note that the matrices S, $S(H^\tau)$ and $S(G^\phi)$ are all skew symmetric. By calculation, we have

$$\begin{aligned} SS^T &= [I'_p \otimes S(G^\phi) + S(H^\tau) \otimes I_n][I'_p \otimes (-S(G^\phi)) + (-S(H^\tau)) \otimes I_n] \\ &= -[(I_p \otimes S^2(G^\phi) + S^2(H^\tau) \otimes I_n) + (I'_p \otimes S(G^\phi))(S(H^\tau) \otimes I_n) \\ &\quad +(S(H^\tau) \otimes I_n)(I'_p \otimes S(G^\phi))]. \end{aligned}$$

Define $\omega_i = 1$ for $i = 1, 2, \ldots, p_1$ and $\omega_i = -1$ for $i = p_1+1, p_1+2, \ldots, p$. Denote $P^{(1)} = (I'_{p_1+p_2} \otimes S(G^\phi))(S(H^\tau) \otimes I_n)$ and $P^{(2)} = (S(H^\tau) \otimes I_n)(I'_{p_1+p_2} \otimes S(G^\phi))$. Note that $P^{(1)}$ and $P^{(2)}$ are both partition matrices each of order $p_1 \times p_2$ in which each entry is an $n \times n$ submatrix. The $(i,j)^{\text{th}}$ block in the matrix $P^{(1)} + P^{(2)}$ is given by

$$P^{(1)}_{ij} + P^{(2)}_{ij} = S(H^\tau)_{ij} S(G^\phi)((-1)^{\omega_i} + (-1)^{\omega_j}).$$

For any $1 \le i,\ j \le p$, if $S(H^\tau)_{ij} = 0$, then $P^{(1)}_{ij} + P^{(2)}_{ij} = 0$. Otherwise the vertices corresponding to i and j in H^τ are in different parts of the bipartition. That is, $1 \le i \le p_1$, $p_1+1 \le j \le p$ or $1 \le j \le p_1$, $p_1+1 \le i \le p$. Then $(-1)^{\omega_i} + (-1)^{\omega_j} = 0$. Thus it follows that $P^{(1)} + P^{(2)} = 0$. Hence

$$SS^T = -(I_{p_1+p_2} \otimes S^2(G^\phi) + S^2(H^\tau) \otimes I_n).$$

Therefore, the eigenvalues of SS^T are $\mu(H^\tau)^2 + \lambda(G^\phi)^2$, where $\pm\mathbf{i}\mu(H^\tau) \in Sp_S(H^\tau)$ and $\pm\mathbf{i}\lambda(G^\phi) \in Sp_S(G^\phi)$ and hence the eigenvalues of S are of the form $\pm\mathbf{i}\sqrt{\mu(H^\tau)^2 + \lambda(G^\phi)^2}$. Thus the skew spectrum of $(H^\tau \Box G^\phi)^o$ is as given in the statement of the theorem. The proof is thus complete. $\square$

As an application of Theorem 4.2, we now construct new families of oriented graphs with maximum skew energy.

Theorem 4.3. *Let H^τ be an oriented ℓ-regular bipartite graph on p vertices with maximum skew energy $\mathcal{E}_S(H^\tau) = p\sqrt{\ell}$ and G^ϕ be an oriented k-regular bipartite graph on n vertices with maximum skew energy $\mathcal{E}_S(G^\phi) = n\sqrt{k}$. Then the oriented graph $(H^\tau \Box G^\phi)^o$ of $H\Box G$ has the maximum skew energy $\mathcal{E}_S((H^\tau \Box G^\phi)^o) = np\sqrt{\ell + k}$.*

Proof. Since H^τ and G^ϕ have maximum skew energy, $S(H^\tau)S(H^\tau)^T = \ell I_p$ and $S(G^\phi)S(G^\phi)^T = kI_n$. Then the skew eigenvalues of H^τ are all $\pm\mathbf{i}\sqrt{\ell}$ and the skew eigenvalues of G^ϕ are all $\pm\mathbf{i}\sqrt{k}$. By Theorem 4.2, all the skew eigenvalues of $(H^\tau \Box G^\phi)^o$ are of the form $\pm\mathbf{i}\sqrt{\ell + k}$ and hence its skew energy is $np\sqrt{\ell + k}$, the maximum possible skew energy that an $(\ell+k)$-regular graph on np vertices can have. $\square$

An immediate corollary of Theorem 4.3 is the following result of Cui and hou [6].

Corollary 4.4. *Let G^ϕ be an oriented k-regular graph on n vertices with maximum skew energy $\mathcal{E}_S(G^\phi) = n\sqrt{k}$. Then the oriented graph $(P_2\Box G^\phi)^o$ of $P_2\Box G$ has maximum skew energy $\mathcal{E}_S((P_2\Box G^\phi)^o) = 2n\sqrt{k+1}$.*

Adiga et al. [1] showed that a 1-regular connected graph that has an orientation with maximum skew energy is K_2; while a 2-regular connected graph has an orientation with maximum skew energy if and only if it is an oddly oriented cycle C_4. Tian [14] proved that there exists a k-regular graph with $n = 2^k$ vertices having an orientation ψ with maximum skew energy. Cui and Hou [6] constructed a k-regular graph of order $n = 2^{k-1}$ having an orientation φ with maximum skew energy. The following examples provide new families of oriented graphs with fewer vertices (compared to the last two examples) that have maximum skew energy.

Example 4.5. *Let $G_1 = K_{4,4}$. For each $r \geq 2$, set $G_r = K_{4,4}\Box G_{r-1}$. As there is an orientation of $K_{4,4}$ with maximum skew energy 16 (see [5]), for each $r \geq 1$, there exists an orientation of G_r that yields the maximum skew energy $2^{3r}\sqrt{4r}$. This provides a family of $4r$-regular graphs of order $n = 2^{3r}$ each having an orientation with skew energy $2^{3r}\sqrt{4r}$, $r \geq 1$.*

Example 4.6. *Let $G_1 = K_4$. For each $r \geq 2$, set $G_r = K_{4,4}\Box G_{r-1}$. Since there exist orientations for K_4 with maximum skew energy $4\sqrt{3}$ (see [1, 7]), the skew energy of G_r, $r \geq 1$, is $2^{3r-1}\sqrt{4r-1}$ and it is maximum. This provides a family of $4r-1$-regular graphs of order 2^{3r-1} each having an orientation with maximum skew energy $2^{3r-1}\sqrt{4r-1}$, $r \geq 1$.*

Example 4.7. *A new family of* $4r-2$*-regular oriented graphs of order* 2^{3r-1} *with maximum skew energy* $2^{3r-1}\sqrt{4r-2}$, $r \geq 1$ *is obtained when we set* $G_1 = C_4$ *in place of* K_4 *in Example* 4.6.

Example 4.8. *A new family of* $4r-3$*-regular oriented graphs of order* 2^{3r-2} *with maximum skew energy* $2^{3r-2}\sqrt{4r-3}$, $r \geq 1$ *is obtained when we set* $G_1 = P_2$ *in place of* K_4 *in Example* 4.6.

Acknowledgement

For the first two authors, this research was supported by the Department of Science and Technology, Government of India grant DST:SR/S4/MS:492, dated April 16, 2009. For the third, fourth and fifth authors, this research was supported by NSFC and the "973" program.

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