A sharp upper bound for the rainbow 2-connection number of a 2-connected graph*

Xueliang Li, Sujuan Liu

Center for Combinatorics and LPMC-TJKLC

Nankai University, Tianjin 300071, China

lxl@nankai.edu.cn; sjliu0529@126.com

Abstract

A path in an edge-colored graph is called *rainbow* if no two edges of it are colored the same. For an ℓ-connected graph G and an integer k with $1 \leq k \leq \ell$, the *rainbow k-connection number* $rc_k(G)$ of G is defined to be the minimum number of colors required to color the edges of G such that every two distinct vertices of G are connected by at least k internally disjoint rainbow paths. Fujita et. al. proposed a problem: What is the minimum constant $\alpha > 0$ such that for every 2-connected graph G on n vertices, we have $rc_2(G) \leq \alpha n$? In this paper, we prove that the minimum constant $\alpha = 1$ and $rc_2(G) = n$ if and only if G is a cycle of order n, which solves the problem of Fujita et. al.

Keywords: rainbow edge-coloring, rainbow k-connection number, 2-connected graph, ear decomposition.

AMS subject classification 2010: 05C40, 05C15.

1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [2]. A path in an edge-colored graph is called *rainbow* if every two edges on it have distinct colors. Let G be an edge-colored ℓ-connected graph, where ℓ is a positive integer. For $1 \leq k \leq \ell$, G is *rainbow k-connected* if every pair of distinct vertices of G are connected by at least k internally disjoint rainbow

*Supported by NSFC No.11071130 and the "973" project.

paths. The minimum number of colors required to color the edges of G to make G rainbow k-connected is called the *rainbow k-connection number* of G, denoted by $rc_k(G)$. Particularly, $rc_1(G)$ is equal to $rc(G)$, the rainbow connection number. For more results on this topic, see a recent book by Li and Sun [12] and a survey paper [11].

A graph G is *minimally k-connected* if G is k-connected but $G-e$ is not k-connected for every $e \in E(G)$. Let G' be a subgraph of a graph G. An *ear* of G' in G is a nontrivial path in G whose end vertices lie in G' but whose internal vertices are not. An *ear decomposition* of a 2-connected graph G is a sequence $G_0, G_1, \cdots, G_k$ of 2-connected subgraphs of G such that (1) G_0 is a cycle of G; (2) $G_i = G_{i-1} \bigcup P_{i-1}$ $(1 \leq i \leq k)$, where P_{i-1} is an ear of G_{i-1} in G; (3) G_{i-1} $(1 \leq i \leq k)$ is a proper subgraph of G_i; (4) $G_k = G$. It is obvious that every graph G_i in an ear decomposition is 2-connected. Two paths P' and P'' from v_i to v_j are internally disjoint if $V(P') \bigcap V(P'') = \{v_i, v_j\}$. For three distinct vertices v', v''_1, v''_2, the paths P' and P'' from v' to v''_1 and v''_2, respectively, are internally disjoint if $V(P') \bigcap V(P'') = \{v'\}$. Two paths P' and P'' are disjoint if $V(P') \bigcap V(P'') = \emptyset$.

The concept of rainbow k-connection number $rc_k(G)$ was introduced by Chartrand et. al. [5, 6]. It was shown in [7] that computing the rainbow connection number of a graph is NP-hard. Hence, bounds on rainbow connection number for graphs have been a subject of investigation. There are some results in this direction. For a connected graph G, $rc(G) \leq n-1$ in [3]. An upper bound for the rainbow connection number of a connected graph with minimum degree δ is $3n/(\delta+1)+3$ in [4]. If G is a 2-connected graph of order n, then $rc(G) \leq \lceil \frac{n}{2} \rceil$ and $rc(C_n) = \lceil \frac{n}{2} \rceil$, where C_n is an n-vertex cycle in [10]. An easy observation is that $rc_2(C_n) = n$. In [8], the authors proved the following theorem and proposed a problem.

Theorem 1.1. *[8] If $\ell \geq 2$ and G is an ℓ-connected graph of order $n \geq \ell+1$, then $rc_2(G) \leq (\ell+1)n/\ell$.*

Problem 1.1. *[8] What is the minimum constant $\alpha > 0$ such that for all 2-connected graphs G on n vertices, we have $rc_2(G) \leq \alpha n$?*

In a published version of [8], they stated the following theorem and problem.

Theorem 1.2. *[9] If G is a 2-connected graph of order $n \geq 3$, then $rc_2(G) \leq 3n/2$.*

Problem 1.2. *[9] For $1 \leq k \leq \ell$, derive a sharp upper bound for $rc_k(G)$, if G is an ℓ-connected graph on n vertices. Is there a constant $\alpha = \alpha(k, \ell)$ such that $rc_k(G) \leq \alpha n$?*

Problem 1.1 is restated in [12]. From Theorem 1.2 and $rc_2(C_n) = n$, it is obvious that $1 \leq \alpha \leq 3/2$. For a 2-connected series-parallel graph G, the authors of [8, 9] showed the following result.

Theorem 1.3. *[8, 9] If G is a 2-connected series-parallel graph on n vertices, then $rc_2(G) \leq n$.*

In this paper, we will show that the above result holds for general 2-connected graphs.

Theorem 1.4. *If G is a 2-connected graph on n vertices, then $rc_2(G) \leq n$ with equality if and only if G is a cycle of order n.*

Therefore, the minimum constant $\alpha = 1$ in Problem 1.1. The following classic results on minimally 2-connected graphs are needed in the sequel.

Theorem 1.5. *[1] Let G be a minimally 2-connected graph that is not a cycle. Let $D \subset V(G)$ be the set of vertices of degree two. Then $F = G - D$ is a forest with at least two components. A component P of $G[D]$ is a path and the end vertices of P are not joined to the same tree of the forest F.*

Theorem 1.6. *[1] Every 2-connected subgraph of a minimally 2-connected graph is minimally 2-connected.*

2 Main results

We first give a lemma, which will be used later.

Lemma 2.1. *Let G be a minimally 2-connected graph, but not a cycle. Then G has an ear decomposition $G_0, G_1, \cdots, G_t$ $(t \geq 1)$ satisfying the following conditions:*

(1) $G_i = G_{i-1} \bigcup P_{i-1}$ $(1 \leq i \leq t)$, where P_{i-1} is an ear of G_{i-1} in G and at least one vertex of P_{i-1} has degree two in G;

(2) each of the two internally disjoint paths in G_0 between the end vertices of P_0 has at least one vertex of degree two in G.

Proof. We first construct a sequence of 2-connected subgraphs of G. Let $D \subset V(G)$ be the set of vertices of degree two in G. Let G_0 be a cycle of G which contains as many vertices of D as possible. If $D \backslash V(G_0) \neq \emptyset$, then choose a vertex $v_0 \in D \backslash V(G_0)$. Since G is 2-connected, from Menger's Theorem there exist two internally disjoint paths P' and P'' from v_0 to two distinct vertices of G_0 such that the internal vertices of P' and P'' do not belong to G_0. Hence $P_0 = P' \bigcup P''$ is an ear of G_0 which contains a vertex v_0 in D. Let $G_1 = G_0 \bigcup P_0$. If $D \backslash V(G_1) \neq \emptyset$, then we continue the procedure. After a finite number of steps, we get a sequence $G_0, G_1, \cdots, G_t$ $(t \geq 1)$ of 2-connected subgraphs of G such that $D \backslash V(G_t) = \emptyset$ and $G_i = G_{i-1} \bigcup P_{i-1}$ $(1 \leq i \leq t)$, where P_{i-1} is an ear of G_{i-1}

containing at least one vertex in D. If $G_t = G$, then from the procedures of construction, the sequence $G_0, G_1, \cdots, G_t$ $(t \geq 1)$ is an ear decomposition of G satisfying condition (1).

We first show that $G_t = G$. Suppose on the contrary that $G_t \neq G$, i.e., G_t is a proper 2-connected subgraph of G. Since G is minimally 2-connected, we have $V(G) \backslash V(G_t) \neq \emptyset$. From Theorem 1.5, $G - D$ is a forest. Since $D \subseteq V(G_t)$, $F = G - V(G_t) \subseteq G - D$ is also a forest with $|F| \geq 1$. Let T be a component of F with $|T| \geq 1$. Then T is a tree. If $|T| = 1$ and $V(T) = \{v\}$, then there exist three distinct vertices v_1, v_2, v_3 in G_t such that $vv_j \in E(G)$ $(1 \leq j \leq 3)$. Let $G' = (V', E')$, where $V' = V(G_t) \bigcup \{v\}$ and $E' = E(G_t) \bigcup \{vv_j : 1 \leq j \leq 3\}$. So G' is a 2-connected subgraph of G. Since $G' - vv_3$ is also 2-connected, G' is not minimally 2-connected which contradicts to Theorem 1.6. Suppose $|T| \geq 2$. Then T has at least two leaves, say v' and v''. Since $v', v'' \notin D$ and $d_T(v') = d_T(v'') = 1$, there exist four vertices v_i $(1 \leq i \leq 4, v_1 \neq v_2,\ v_3 \neq v_4)$ in G_t such that $v'v_1, v'v_2, v''v_3, v''v_4 \in E(G)$. Let P be the path from v' to v'' in T. Then $G' = (V', E')$, where $V' = V(G_t) \bigcup V(P)$ and $E' = E(G_t) \bigcup E(P) \bigcup \{v'v_1, v'v_2, v''v_3, v''v_4\}$ is a 2-connected subgraph of G. Since $G' - v'v_1$ is also 2-connected, G' is not minimally 2-connected which contradicts to Theorem 1.6. Therefore, $G_t = G$.

Now we show that the ear decomposition $G_0, G_1, \cdots, G_t$ $(t \geq 1)$ of G satisfies condition (2). Denote by P' and P'' the two internally disjoint paths in G_0 between the two end vertices of P_0. Suppose on the contrary that one of P' and P'', say P', has no vertex of degree two in G, i.e., $V(P') \bigcap D = \emptyset$. From the procedure of construction, $V(P_0) \bigcap D \neq \emptyset$. Hence, $P'' \bigcup P_0$ is a cycle of G, which contains more vertices in D than G_0, a contradiction. Therefore, the ear decomposition $G_0, G_1, \cdots, G_t$ $(t \geq 1)$ of G satisfies condition (2). $\square$

For convenience, we give some more notations. If c is an edge-coloring of a graph G, then $c(G)$ denotes the set of colors appearing in G. Write $|G|$ for the order of a graph G. If P is a path and $v_i, v_j \in V(P)$, then v_iPv_j denotes the segment of P from v_i to v_j.

Lemma 2.2. *Let G ba a minimally 2-connected graph of order $n \geq 3$. If G is not a cycle, then $rc_2(G) \leq n - 1$.*

Proof. Let G be a minimally 2-connected graph of order n, but not a cycle. We will prove the result by giving an edge-coloring of G with $n-1$ colors which makes G rainbow 2-connected. From Lemma 2.1, G has an ear decomposition $G_0, G_1, \cdots, G_t$ $(t \geq 1)$ satisfying the two conditions in Lemma 2.1. Let $D \subseteq V(G)$ be the set of vertices of degree two in G and $\overline{D} = V(G) \backslash D$. In the following, for every graph G_i $(1 \leq i \leq t)$ we will define an edge-coloring c_i of G_i with $|G_i| - 1$ colors and a map f_i from $\overline{D} \bigcap V(G_i)$ to $c_i(G_i)$ satisfying the following conditions:

(A1) G_i is rainbow 2-connected;

(A2) for any three distinct vertices $v', v_1'', v_2'' \in V(G_i)$, G_i has two internally disjoint rainbow paths P' and P'' from v' to v_1'' and v_2'', respectively;

(A3) for any four distinct vertices $v_1', v_2', v_1'', v_2'' \in V(G_i)$, G_i has two disjoint rainbow paths P' from v_1' to one of v_1'' and v_2'', say v_1'', and P'' from v_2' to the other vertex v_2'';

(A4) f_i is injective, i.e., for any two distinct vertices $v', v'' \in \overline{D} \bigcap V(G_i)$, $f_i(v') \neq f_i(v'')$;

(A5) for any vertex $v \in \overline{D} \bigcap V(G_i)$, the color $f_i(v)$ appears exactly once in c_i and the edge colored by $f_i(v)$ in G_i is incident with v.

We define c_i and f_i of G_i $(1 \leq i \leq t)$ by induction. First, consider the graph $G_1 = G_0 \bigcup P_0$. Without loss of generality, suppose that $G_0 = v_1 v_2 \cdots v_s v_1$ and $P_0 = v_1 v_{s+1} v_{s+2} \cdots v_\ell v_p$ $(\ell > s)$, where G_0 is a cycle, P_0 is a path and $V(G_0) \bigcap V(P_0) = \{v_1, v_p\}$ $(3 \leq p \leq s-1)$. Since the ear decomposition $G_0, G_1, \cdots, G_t$ $(t \geq 1)$ of G satisfies the two conditions in Lemma 2.1, there exist three vertices $v_{p_1}, v_{p_2}, v_{p_3} \in D$ $(1 < p_1 < p < p_2 \leq s < p_3 \leq \ell)$ in G_1. Define an edge-coloring c_1 of G_1 by $c_1(v_j v_{j+1}) = x_j$ if $1 \leq j \leq s-1$ or $s+1 \leq j \leq \ell - 1$; $c_1(v_s v_1) = c_1(v_\ell v_p) = x_s$ and $c_1(v_1 v_{s+1}) = x_p$, where $x_1, x_2, \cdots, x_{\ell-1}$ are distinct colors. It is obvious that c_1 uses $|G_1| - 1$ colors. Define a map $f_1 : \overline{D} \bigcap V(G_1) \to c_1(G_1)$ by $f_1(v_j) = x_j$ if $v_j \in \overline{D} \bigcap V(G_1)$ and $1 \leq j < p_1, p+1 \leq j < p_2$ or $s+1 \leq j < p_3$ and $f_1(v_j) = x_{j-1}$ if $v_j \in \overline{D} \bigcap V(G_1)$ and $p_1 < j \leq p$, $p_2 < j \leq s$ or $p_3 < j \leq \ell$. It can be checked that c_1 and f_1 satisfy the above conditions (A1)-(A5).

If $t = 1$, then c_1 is the rainbow 2-connected edge-coloring of G with $n-1$ colors. Consider the case $t \geq 2$. Assume that we have defined c_{i-1} and f_{i-1} of G_{i-1} $(2 \leq i \leq t)$ satisfying conditions (A1)-(A5) and the edge-coloring c_{i-1} of G_{i-1} uses $|G_{i-1}| - 1$ colors. Now consider the graph $G_i = G_{i-1} \bigcup P_{i-1}$. Suppose that $P_{i-1} = v_1 v_2 \cdots v_q$ $(q \geq 3)$, where $V(G_{i-1}) \bigcap V(P_{i-1}) = \{v_1, v_q\}$. It is obvious that $v_1, v_q \in \overline{D} \bigcap V(G_{i-1})$. Define an edge-coloring c_i of G_i by $c_i(e) = c_{i-1}(e)$ for $e \in E(G_{i-1})$, $c_i(v_{q-1} v_q) = f_{i-1}(v_1)$ and $c_i(v_j v_{j+1}) = y_j$ $(1 \leq j \leq q-2)$, where $y_1, y_2, \cdots, y_{q-2}$ are distinct new colors. It is clear that c_i uses $|G_i| - 1$ colors. From condition (1) of the Lemma 2.1, there exists a vertex $v_{q_0} \in D$ $(2 \leq q_0 \leq q-1)$ in P_{i-1}. Define a map $f_i : \overline{D} \bigcap V(G_i) \to c_i(G_i)$ as follows: $f_i(v) = f_{i-1}(v)$ for $v \in [\overline{D} \bigcap V(G_{i-1})] \backslash \{v_1\}$, $f_i(v_j) = y_j$ for $v_j \in \overline{D} \bigcap V(v_1 P_{i-1} v_{q_0 - 1})$ and $f_i(v_j) = y_{j-1}$ for $v_j \in \overline{D} \bigcap V(v_{q_0+1} P_{i-1} v_{q-1})$. The edge-coloring c_i of G_i has the following two properties.

(B1) There exists a rainbow path P_{i-1}' from v_1 to v_q in G_{i-1} such that the color $f_{i-1}(v_1)$ does not appear on it. In fact, since G_{i-1} is rainbow 2-connected, there are two internally disjoint rainbow paths in G_{i-1} connecting v_1, v_q. Since the map f_{i-1} satisfies condition (A5), the color $f_{i-1}(v_1)$ appears exactly once in G_{i-1}. So $f_{i-1}(v_1)$ does not appear on one of the two rainbow paths, denoted by P_{i-1}', from v_1 to v_q.

(B2) Since c_{i-1} and f_{i-1} satisfy condition (A5), the color $f_{i-1}(v_1)$ does not appear on any path in G_{i-1} which does not contain v_1.

We will distinguish some cases to show that c_i and f_i satisfy conditions (A1)-(A5).

(I) For any two distinct vertices $v', v'' \in V(G_i)$, we distinguish the following three cases to show that (A1) is satisfied:

If $v', v'' \in V(G_{i-1})$, there exist two internally disjoint rainbow paths connecting them in G_{i-1}, which are also rainbow paths in G_i according to the definition of c_i.

If $v', v'' \in V(P_{i-1})$, from property (B1) we have that $P'_{i-1} \bigcup P_{i-1}$ is a cycle whose colors are distinct, and hence there are two internally disjoint rainbow paths from v' to v'' on the cycle $P'_{i-1} \bigcup P_{i-1}$.

If $v' \in V(G_{i-1}) \backslash \{v_1, v_q\}$ and $v'' \in V(P_{i-1}) \backslash \{v_1, v_q\}$, since c_{i-1} satisfies condition (A2) there exist two internally disjoint rainbow paths P' and P'' in G_{i-1} from v' to v_1 and v_q, respectively. From property (B2) and $v_1 \notin V(P'')$, we have $f_{i-1}(v_1) \notin c_i(P'')$. So $v'P'v_1P_{i-1}v''$ and $v'P''v_qP_{i-1}v''$ are two internally disjoint rainbow paths from v' to v'' in G_i.

Therefore, G_i is rainbow 2-connected.

(II) For any three distinct vertices $v', v''_1, v''_2 \in V(G_i)$, we distinguish the following six cases to show that (A2) is satisfied:

If $v', v''_1, v''_2 \in V(G_{i-1})$, then from condition (A2) of c_{i-1} and the definition of c_i, there exist two internally disjoint rainbow paths P' and P'' in G_{i-1} from v' to v''_1 and v''_2, respectively.

If $v', v''_1, v''_2 \in V(P_{i-1})$, from property (B1) there exist two internally disjoint rainbow paths on the cycle $P'_{i-1} \bigcup P_{i-1}$ from v' to v''_1 and v''_2, respectively.

If $v', v''_1 \in V(G_{i-1}) \backslash \{v_1\}$ and $v''_2 \in V(P_{i-1}) \backslash \{v_q\}$, from condition (A2) of c_{i-1} there exist two internally disjoint rainbow paths P' and P'' in G_{i-1} from v' to v''_1 and v_1, respectively. So P' and $v'P''v_1P_{i-1}v''_2$ are two internally disjoint rainbow paths in G_i from v' to v''_1 and v''_2, respectively.

If $v' \in V(G_{i-1}) \backslash \{v_1, v_q\}$ and $v''_1, v''_2 \in V(P_{i-1})$, without loss of generality, v''_1, v''_2 appear on P_{i-1} in this order. From condition (A2) of c_{i-1}, there exist two internally disjoint rainbow paths P' and P'' in G_{i-1} from v' to v_1 and v_q, respectively. From property (B2) and $v_1 \notin V(P'')$, we have $f_{i-1}(v_1) \notin c_i(P'')$. So $v'P'v_1P_{i-1}v''_1$ and $v'P''v_qP_{i-1}v''_2$ are two internally disjoint rainbow paths in G_i from v' to v''_1 and v''_2, respectively.

If $v''_1, v''_2 \in V(G_{i-1}) \backslash \{v_1\}$ and $v' \in V(P_{i-1}) \backslash \{v_q\}$, from condition (A3) of c_{i-1} there exist two disjoint rainbow paths P' from v''_1 to one of v_1 and v_q, say v_1, and P'' from v''_2 to the other vertex v_q. If $v''_2 = v_q$, then $P'' = v''_2$. From property (B2), $v'P_{i-1}v_1P'v''_1$ and $v'P_{i-1}v_qP''v''_2$ are two internally disjoint rainbow paths from v' to v''_1 and v''_2, respectively.

If $v_2'' \in V(G_{i-1})\backslash\{v_1\}$ and $v_1'', v' \in V(P_{i-1})$, without loss of generality, v_1'', v' appear on P_{i-1} in this order. Since the color $f_{i-1}(v_1)$ appears exactly once in G_{i-1}, one of the two internally disjoint rainbow paths in G_{i-1} from v_q to v_2'', denoted by P', does not contain the edge colored by $f_{i-1}(v_1)$, i.e., $f_{i-1}(v_1) \notin c_i(P')$. So $v'P_{i-1}v_1''$ and $v'P_{i-1}v_qP'v_2''$ are two internally disjoint rainbow paths in G_i from v' to v_1'' and v_2'', respectively.

Therefore, c_i satisfies condition (A2).

(III) For any four distinct vertices $v_1', v_2', v_1'', v_2'' \in V(G_i)$, we distinguish the following six cases to show that (A3) is satisfied:

If $v_1', v_2', v_1'', v_2'' \in V(G_{i-1})$, then there exist two required disjoint rainbow paths in G_{i-1} from condition (A3) of c_{i-1} and the definition of c_i.

If $v_1', v_2', v_1'', v_2'' \in V(P_{i-1})$, then there exist two required disjoint rainbow paths on the cycle $P_{i-1}' \bigcup P_{i-1}$ from property (B1).

If $v_1', v_2', v_1'' \in V(G_{i-1})\backslash\{v_1\}$ and $v_2'' \in V(P_{i-1})\backslash\{v_q\}$, from condition (A3) of c_{i-1} there exist two disjoint rainbow paths P' from v_1' to one of v_1 and v_1'', say v_1'', and P'' from v_2' to the other vertex v_1 in G_{i-1}. Then P' and $v_2'P''v_1P_{i-1}v_2''$ are two required disjoint rainbow paths in G_i.

If $v_1', v_2' \in V(G_{i-1})\backslash\{v_1\}$ and $v_1'', v_2'' \in V(P_{i-1})\backslash\{v_q\}$, without loss of generality, v_1'', v_2'' appear on P_{i-1} in this order. From condition (A3) of c_{i-1}, there exist two disjoint rainbow paths P' from v_1' to v_1 and v_q, say v_1, and P'' from v_2' to the other vertex v_q in G_{i-1}. If $v_2' = v_q$, then $P'' = v_2'$. Hence $v_1'P'v_1P_{i-1}v_1''$ and $v_2'P''v_qP_{i-1}v_2''$ are two required disjoint rainbow paths in G_i.

If $v_1', v_1'' \in V(G_{i-1})\backslash\{v_1\}$ and $v_2', v_2'' \in V(P_{i-1})\backslash\{v_q\}$, from condition (A1) of c_{i-1} let P' be a rainbow path from v_1' to v_1'' in G_{i-1}. Then P' and $v_2'P_{i-1}v_2''$ are two required disjoint rainbow paths in G_i.

If $v_1' \in V(G_{i-1})\backslash\{v_1\}$ and $v_2', v_1'', v_2'' \in V(P_{i-1})\backslash\{v_q\}$, without loss of generality, v_2', v_2'', v_1'' appear on P_{i-1} in this order. From conditions (A1) and (A5) of c_{i-1} and f_{i-1}, there exists one rainbow path P' in G_{i-1} from v_1' to v_q such that $f_{i-1}(v_1) \notin c_i(P')$. Then $v_1'P'v_qP_{i-1}v_1''$ and $v_2'P_{i-1}v_2''$ are two required disjoint rainbow paths in G_i.

Therefore, c_i satisfies condition (A3).

(VI) From condition (A4) of f_{i-1} and the definition of f_i, f_i is injective. Hence, f_i satisfies condition (A4).

(V) From condition (A4) of f_{i-1} and the definition of f_i, f_i satisfies condition (A5).

Therefore, we can get an edge-coloring c_t of G $(= G_t)$ with $n-1$ $(= |G_t| - 1)$ colors which makes G rainbow 2-connected. So $rc_2(G) \leq n-1$. □

An easy observation is that if G' is a spanning subgraph of a graph G and $rc_k(G)$ and

$rc_k(G')$ indeed exist, then we have $rc_k(G) \leq rc_k(G')$ $(k \geq 1)$.

Now we are ready to prove our main result Theorem 1.4.

Proof of Theorem 1.4: If G is an n-vertex cycle, then we have $rc_2(G) = n$. Hence, to prove the result we only need to show that $rc_2(G) \leq n-1$ for any 2-connected graph G of order n but not a cycle. Let G be such a graph. Consider the following two cases.

Case 1. G is Hamiltonian.

Let $C = v_1v_2\cdots v_nv_1$ be a Hamiltonian cycle of G. Since G is not a cycle, C must have a chord, say $v_1v_j \in E(G)$ $(3 \leq j \leq n-1)$. Then $G' = (V(G), E(C)\bigcup\{v_1v_j\})$ is a spanning 2-connected subgraph of G. Let $x_1, x_2, \cdots, x_{n-1}$ be $n-1$ distinct colors. Define an edge-coloring c of G' with $n-1$ colors as follows: $c(v_1v_2) = c(v_jv_{j+1}) = x_1$, $c(v_1v_n) = c(v_{j-1}v_j) = x_2$ and the other $n-3$ edges of G' are colored by colors $x_3, \cdots, x_{n-1}$. It can be checked that G' is rainbow 2-connected. From the above observation, $rc_2(G) \leq rc_2(G') \leq n-1$.

Case 2. G is not Hamiltonian.

Let G' be a spanning minimally 2-connected subgraph of G. Since G is not Hamiltonian, G' is not a cycle. From Lemma 2.2 and the above observation, we have $rc_2(G) \leq rc_2(G') \leq n-1$.

The proof is now complete. □

Acknowledgment. The authors would like to thank the referees for valuable comments and suggestions, which helped to improve the presentation of the paper.

References

[1] B. Bollobás, Extremal Graph Theory, *Academic Press*, London, 1978.

[2] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM 244, *Springer*, New York, 2008.

[3] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, R. Yuster, On rainbow connection, *Electron. J. Combin.* 15(1)(2008), R57.

[4] L.S. Chandran, A. Das, D. Rajendraprasad, N.M. Varma, Rainbow connection nuumber and connected dominating sets, *Electronic Notes in Discrete Math.* 38(2011), 239-244. Also see J. Graph Theory 71(2012), 206-218.

[5] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, Rainbow connection in graphs, *Mathematica Bohemica* 133(2008), 85-98.

[6] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, The rainbow connectivity of a graph, Networks 54(2009), 75-81.

[7] S. Chakraborty, E. Fischer, A. Matsliah, R. Yuster, Hardness and algorithms for rainbow connection, *J. Combin. Optim.* 21(2010), 330-347.

[8] S. Fujita, H. Liu, C. Magnant, Rainbow k-connection in dense graphs, preprint, availabel at *www.cantab.net/users/henry.liu/maths.htm*.

[9] S. Fujita, H. Liu, C. Magnant, Rainbow k-connection in dense graphs (Extended Abstract), *Electronic Notes in Discrete Math.* 38(2011), 361-366.

[10] X. Li, S. Liu, L.S. Chandran, R. Mathew, D. Rajendraprasad, Rainbow connection number and connectivity, *Electron. J. Combin.* 19(2012), ♯P20.

[11] X. Li, Y. Shi, Y. Sun, Rainbow connections of graphs: A survey, Graphs Combin. 29(1)(2013), 1-38.

[12] X. Li, Y. Sun, Rainbow Connections of Graphs, SpringerBriefs in Math., *Springer*, New York, 2012.