Proof of a Conjecture of Hirschhorn and Sellers on Overpartitions

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Abstract

Let $\bar{p}(n)$ denote the number of overpartitions of n. It was conjectured by Hirschhorn and Sellers that $\bar{p}(40n+35) \equiv 0 \pmod{40}$ for $n \geq 0$. Employing 2-dissection formulas of theta functions due to Ramanujan, and Hirschhorn and Sellers, we obtain a generating function for $\bar{p}(40n+35)$ modulo 5. Using the (p,k)-parametrization of theta functions given by Alaca, Alaca and Williams, we prove the congruence $\bar{p}(40n+35) \equiv 0 \pmod{5}$ for $n \geq 0$. Combining this congruence and the congruence $\bar{p}(4n+3) \equiv 0 \pmod{8}$ for $n \geq 0$ obtained by Hirschhorn and Sellers, and Fortin, Jacob and Mathieu, we confirm the conjecture of Hirschhorn and Sellers.

1 Introduction

The objective of this paper is to give a proof of a conjecture of Hirschhorn and Sellers on the number of overpartitons. We shall use the technique of dissections of theta functions.

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Let us begin with some notation and terminology on q-series and partitions. We adopt the common notation

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \tag{1.1}$$

where $|q|<1$, and we write

$$(a_1,a_2,\ldots,a_n;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty. \tag{1.2}$$

Recall that the Ramanujan theta function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \tag{1.3}$$

where $|ab|<1$. The Jacobi triple product identity can be restated as

$$f(a,b) = (-a,-b,ab;ab)_\infty. \tag{1.4}$$

Here is a special case of (1.3), namely,

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.5}$$

For any positive integer n, we use f_n to denote $f(-q^n)$, that is,

$$f_n = (q^n;q^n)_\infty = \prod_{k=1}^{\infty}(1-q^{nk}). \tag{1.6}$$

The function f_n is related to the generating function of overpartitions. A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. An overpartition of n is a partition in which the first occurrence of a number may be overlined, see Corteel and Lovejoy [CL]. For $n\geq 1$, let $\bar{p}(n)$ denote the number of overpartitions of n, and we set $\bar{p}(0)=1$. Corteel and Lovejoy [CL] showed that the generating function for $\bar{p}(n)$ is given by

$$\sum_{n=0}^{\infty}\bar{p}(n)q^n = \frac{f_2}{f_1^2}. \tag{1.7}$$

Hirschhorn and Sellers [HS-1], and Fortin, Jacob and Mathieu [FJM] obtained the following Ramanujan-type generating function formulas for $\bar{p}(2n+1)$, $\bar{p}(4n+3)$, and $\bar{p}(8n+7)$:

$$\sum_{n=0}^{\infty}\bar{p}(2n+1)q^n = 2\frac{f_2^2 f_8^2}{f_1^4 f_4}, \tag{1.8}$$

$$\sum_{n=0}^{\infty} \bar{p}(4n+3)q^n = 8\frac{f_2 f_4^6}{f_1^8}, \tag{1.9}$$

$$\sum_{n=0}^{\infty} \bar{p}(8n+7)q^n = 64\frac{f_2^{22}}{f_1^{23}}. \tag{1.10}$$

The above identities lead to congruences modulo 2, 8 and 64 for the overpartition function. Mahlburg [M] proved that $\bar{p}(n)$ is divisible by 64 for almost all n by using relations between $\bar{p}(n)$ and the number of representations of n as a sum of squares. Using the theory of modular forms, Treneer [T] showed that the coefficients of a wide class of weakly holomorphic modular forms have infinitely many congruence relations for powers of every prime p other than 2 and 3. In particular, Treneer [T] proved that $\bar{p}(5m^3n) \equiv 0 \pmod 5$ for any n that is coprime to m, where m is a prime satisfying $m \equiv -1 \pmod 5$.

The following conjecture was posed by Hirschhorn and Sellers [HS-1].

Conjecture 1.1. *For $n \geq 0$,*

$$\bar{p}(40n+35) \equiv 0 \pmod{40}. \tag{1.11}$$

To prove the above conjecture, we derive a generating function for $\bar{p}(40n+35)$ modulo 5 by using 2-dissection formulas for quotients of theta functions given by Ramanujan [B], and Hirschhorn and Sellers [HS-2]. Then we use the (p,k)-parametrization of theta functions due to Alaca, Alaca and Williams [AAW, AW, W] to show that $\bar{p}(40n+35) \equiv 0 \pmod 5$ for $n \geq 0$. Combining this congruence and the congruence $\bar{p}(4n+3) \equiv 0 \pmod 8$ for $n \geq 0$, we confirm the conjecture.

2 The generating function

In this Section, we derive a generating function of $\bar{p}(40n+35)$ modulo 5. We first recall several 2-dissection formulas for quotients of theta functions due to Ramanujan [B], Hirschhorn and Sellers [HS-2].

The following relations are consequences of dissection formulas of Ramanujan collected in Entry 25 in Berndt's book [B, p. 40]. Recall that $f_n = (q^n; q^n)_\infty$ as given by (1.6).

Lemma 2.1. *We have*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{2.1}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.2}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2} \tag{2.3}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.4}$$

Hirschhorn and Sellers [HS-2] established the following 2-dissection formula.

Lemma 2.2. *We have*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \tag{2.5}$$

By Lemmas 2.1 and 2.2, we are led to a generating function of $\bar{p}(40n+35)$ modulo 5.

Theorem 2.3. *We have*

$$\sum_{n=0}^{\infty} \bar{p}(40n+35)q^n \equiv 2\frac{f_2^{122}}{f_1^{63} f_4^{40}} + 3\frac{f_1 f_2^{26}}{f_4^8} + 4q\frac{f_2^{98}}{f_1^{55} f_4^{24}} + 3q f_1^9 f_2^2 f_4^8 + 4q^2\frac{f_2^{74}}{f_1^{47} f_4^8}$$

$$+ 4q^3\frac{f_2^{50} f_4^8}{f_1^{39}} + 4q^4\frac{f_2^{26} f_4^{24}}{f_1^{31}} + 2q^5\frac{f_2^2 f_4^{40}}{f_1^{23}} \pmod 5. \tag{2.6}$$

Proof. Recall that the theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{2.7}$$

and

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}. \tag{2.8}$$

By the Jacobi triple product identity, we find

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2} \tag{2.9}$$

and

$$\psi(q) = \frac{f_2^2}{f_1}. \tag{2.10}$$

Replacing q by $-q$ in (2.9) and (2.10), and using the fact that

$$f(q) = (-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}, \tag{2.11}$$

we deduce that

$$\varphi(-q) = \frac{f_1^2}{f_2} \tag{2.12}$$

and

$$\psi(-q) = \frac{f_1 f_4}{f_2}. \tag{2.13}$$

In view of (1.3), (1.4) and (2.7), we see that

$$\begin{aligned} \varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} &= \sum_{n=-\infty}^{\infty} q^{25n^2} + 2q \sum_{n=-\infty}^{\infty} q^{25n^2+10n} + 2q^4 \sum_{n=-\infty}^{\infty} q^{25n^2+20n} \\ &= \varphi(q^{25}) + 2qD(q^5) + 2q^4 E(q^5), \end{aligned} \tag{2.14}$$

where

$$D(q) = \sum_{n=-\infty}^{\infty} q^{5n^2+2n} = (-q^3, -q^7, q^{10}; q^{10})_\infty \tag{2.15}$$

and

$$E(q) = \sum_{n=-\infty}^{\infty} q^{5n^2+4n} = (-q, -q^9, q^{10}; q^{10})_\infty. \tag{2.16}$$

It is easily checked that

$$D(q)E(q) = \frac{f_2^2 f_5 f_{20}}{f_1 f_4}. \tag{2.17}$$

By the binomial theorem, we see that for any positive integer k,

$$(1-q^k)^5 \equiv (1-q^{5k}) \pmod 5, \tag{2.18}$$

which implies that

$$\varphi^5(q) \equiv \varphi(q^5) \pmod 5. \tag{2.19}$$

Based on (1.7), (2.9) and (2.19), we have

$$\sum_{n=0}^{\infty} \bar{p}(n)(-q)^n = \frac{1}{\varphi(q)} = \frac{\varphi^4(q)}{\varphi^5(q)}$$

$$\begin{aligned}
&\equiv \frac{\varphi^4(q)}{\varphi(q^5)} \equiv \frac{\left(\varphi(q^{25})+2qD(q^5)+2q^4E(q^5)\right)^4}{\varphi(q^5)}\\
&\equiv \frac{1}{\varphi(q^5)}(\varphi^4(q^{25})+3q\varphi^3(q^{25})D(q^5)+4q^2\varphi^2(q^{25})D^2(q^5)\\
&\quad +2q^3\varphi(q^{25})D^3(q^5)+3q^4\varphi^3(q^{25})E(q^5)+q^4D^4(q^5)\\
&\quad +3q^5\varphi^2(q^{25})D(q^5)E(q^5)+q^6\varphi(q^{25})D^2(q^5)E(q^5)\\
&\quad +4q^7D^3(q^5)E(q^5)+4q^8\varphi^2(q^{25})E^2(q^5)+q^9\varphi(q^{25})D(q^5)E^2(q^5)\\
&\quad +q^{10}D^2(q^5)E^2(q^5)+2q^{12}\varphi(q^{25})E^3(q^5)\\
&\quad +4q^{13}D(q^5)E^3(q^5)+q^{16}E^4(q^5)^4) \pmod 5, \qquad (2.20)
\end{aligned}$$

which yields

$$\sum_{n=0}^{\infty}\bar{p}(5n)(-q)^n \equiv \frac{\varphi^4(q^5)+3q\varphi^2(q^5)D(q)E(q)+q^2D^2(q)E^2(q)}{\varphi(q)} \pmod 5. \qquad (2.21)$$

Plugging (2.9) and (2.17) into (2.21), we get

$$\sum_{n=0}^{\infty}\bar{p}(5n)(-q)^n \equiv \frac{f_1^2f_4^2f_{10}^{20}}{f_2^5f_5^8f_{20}^8}+3q\frac{f_1f_4f_{10}^{10}}{f_2^3f_5^3f_{20}^3}+q^2\frac{f_5^2f_{20}^2}{f_2} \pmod 5. \qquad (2.22)$$

Replacing q by $-q$ in (2.22) and invoking (2.11), we arrive at

$$\sum_{n=0}^{\infty}\bar{p}(5n)q^n \equiv \frac{f_2f_5^8}{f_1^2f_{10}^4}-3q\frac{f_5^3f_{10}}{f_1}+q^2\frac{f_{10}^6}{f_2f_5^2} \pmod 5. \qquad (2.23)$$

According to 2-dissection formulas (2.1), (2.2), (2.3), (2.5) and congruence (2.23), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\bar{p}(5n)q^n &\equiv \frac{f_2}{f_{10}^4}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\left(\frac{f_{20}^{10}}{f_{10}^2f_{40}^4}-4q^5\frac{f_{10}^2f_{40}^4}{f_{20}^2}\right)^2\\
&\quad -3qf_{10}\left(\frac{f_{10}f_{40}^5}{f_{20}^2f_{80}^2}-2q^5\frac{f_{10}f_{80}^2}{f_{40}}\right)\left(\frac{f_8f_{20}^2}{f_2^2f_{40}}+q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}\right)\\
&\quad +q^2\frac{f_{10}^6}{f_2}\left(\frac{f_{40}^5}{f_{10}^5f_{80}^2}+2q^5\frac{f_{20}^2f_{80}^2}{f_{10}^5f_{40}}\right)\\
&\equiv \frac{f_8^5f_{20}^{20}}{f_2^4f_{10}^8f_{16}^2f_{40}^8}+2q\frac{f_4^2f_{16}^2f_{20}^{20}}{f_2^4f_8f_{10}^8f_{40}^8}-3q\frac{f_8f_{10}^2f_{40}^4}{f_2^2f_{80}^2}-3q^2\frac{f_4^3f_{10}^3f_{40}^6}{f_2^3f_8f_{20}^3f_{80}^2}\\
&\quad +q^2\frac{f_{10}f_{40}^5}{f_2f_{80}^2}-3q^5\frac{f_8^5f_{20}^8}{f_2^4f_{10}^4f_{16}^2}+q^6\frac{f_8f_{10}^2f_{20}^2f_{80}^2}{f_2^2f_{40}^2}-q^6\frac{f_4^2f_{16}^2f_{20}^8}{f_2^4f_8f_{10}^4}
\end{aligned}$$

$$+q^7\frac{f_4^3f_{10}^3f_{80}^2}{f_2^3f_8f_{20}}+2q^7\frac{f_{10}f_{20}^2f_{80}^2}{f_2f_{40}}+q^{10}\frac{f_8^5f_{40}^8}{f_2^4f_{16}^2f_{20}^4}$$

$$+2q^{11}\frac{f_4^2f_{16}^2f_{40}^8}{f_2^4f_8f_{20}^4} \pmod 5. \tag{2.24}$$

Extracting the terms of odd powers of q on both sides of (2.24), we have

$$\sum_{n=0}^{\infty}\bar{p}(10n+5)q^{2n+1}\equiv 2q\frac{f_4^2f_{16}^2f_{20}^{20}}{f_2^4f_8f_{10}^8f_{40}^8}-3q\frac{f_8f_{10}^2f_{40}^4}{f_2^2f_{80}^2}-3q^5\frac{f_8^5f_{20}^8}{f_2^4f_{10}^4f_{16}^2}+q^7\frac{f_4^3f_{10}^3f_{80}^2}{f_2^3f_8f_{20}}$$

$$+2q^7\frac{f_{10}f_{20}^2f_{80}^2}{f_2f_{40}}+2q^{11}\frac{f_4^2f_{16}^2f_{40}^8}{f_2^4f_8f_{20}^4} \pmod 5. \tag{2.25}$$

Dividing both sides of (2.25) by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty}\bar{p}(10n+5)q^{n}\equiv 2\frac{f_2^2f_8^2f_{10}^{20}}{f_1^4f_4f_5^8f_{20}^8}-3\frac{f_4f_5^2f_{20}^4}{f_1^2f_{40}^2}-3q^2\frac{f_4^5f_{10}^8}{f_1^4f_5^4f_8^2}$$

$$+q^3\frac{f_2^3f_5^3f_{40}^2}{f_1^3f_4f_{10}}+2q^3\frac{f_5f_{10}^2f_{40}^2}{f_1f_{20}}+2q^5\frac{f_2^2f_8^2f_{20}^8}{f_1^4f_4f_{10}^4} \pmod 5. \tag{2.26}$$

Employing 2-dissection formulas (2.4), (2.5) and congruence (2.26), we deduce that

$$\sum_{n=0}^{\infty}\bar{p}(10n+5)q^{n}\equiv 2\frac{f_2^2f_8^2f_{10}^{20}}{f_4f_{20}^8}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)\left(\frac{f_{20}^{14}}{f_{10}^{14}f_{40}^4}+4q^5\frac{f_{20}^2f_{40}^4}{f_{10}^{10}}\right)^2$$

$$-3\frac{f_4f_{20}^4}{f_{40}^2}\left(\frac{f_8f_{20}^2}{f_2^2f_{40}}+q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}\right)^2$$

$$-3q^2\frac{f_4^5f_{10}^8}{f_8^2}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)\left(\frac{f_{20}^{14}}{f_{10}^{14}f_{40}^4}+4q^5\frac{f_{20}^2f_{40}^4}{f_{10}^{10}}\right)$$

$$+q^3\frac{f_2^3f_{40}^2}{f_4f_{10}}\left(\frac{f_8f_{20}^2}{f_2^2f_{40}}+q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}\right)^3$$

$$+2q^3\frac{f_{10}^2f_{40}^2}{f_{20}}\left(\frac{f_8f_{20}^2}{f_2^2f_{40}}+q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}\right)$$

$$+2q^5\frac{f_2^2f_8^2f_{20}^8}{f_4f_{10}^4}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)$$

$$\equiv -3\frac{f_4f_8^2f_{20}^8}{f_2^4f_{40}^4}+2\frac{f_4^{13}f_{20}^{20}}{f_2^{12}f_8^2f_{10}^8f_{40}^8}+3q\frac{f_4f_8^6f_{20}^{20}}{f_2^8f_{10}^8f_{40}^8}-q\frac{f_4^4f_{10}f_{20}^5}{f_2^5f_{40}^2}$$

$$-3q^2\frac{f_4^{19}f_{20}^{14}}{f_2^{14}f_8^6f_{10}^6f_{40}^4}-3q^2\frac{f_4^7f_{10}^2f_{20}^2}{f_2^6f_8^2}-2q^3\frac{f_4^7f_8^2f_{20}^{14}}{f_2^{10}f_{10}^6f_{40}^4}$$

$$
\begin{aligned}
&+ 2q^3\frac{f_8f_{10}^2f_{20}f_{40}}{f_2^2} + q^3\frac{f_8^3f_{20}^6}{f_2^3f_4f_{10}f_{40}} + 2q^4\frac{f_4^3f_{10}^3f_{40}^3}{f_2^3f_8f_{20}^2}\\
&+ 3q^4\frac{f_4^2f_8f_{20}^3f_{40}}{f_2^4} + 3q^5\frac{f_4^5f_{10}f_{40}^3}{f_2^5f_8} + 3q^5\frac{f_4^{13}f_{20}^8}{f_2^{12}f_8^2f_{10}^4}\\
&+ 2q^6\frac{f_4f_8^6f_{20}^8}{f_2^8f_{10}^4} + q^6\frac{f_4^8f_{10}^2f_{40}^5}{f_2^6f_8^3f_{20}^3} - 2q^7\frac{f_4^{19}f_{20}^2f_{40}^4}{f_2^{14}f_8^6f_{10}^2}\\
&- 3q^8\frac{f_4^7f_8^2f_{20}^2f_{40}^4}{f_2^{10}f_{10}^2} + 2q^{10}\frac{f_4^{13}f_{40}^8}{f_2^{12}f_8^2f_{20}^4} + 3q^{11}\frac{f_4f_8^6f_{40}^8}{f_2^8f_{20}^4} \pmod 5. \qquad (2.27)
\end{aligned}
$$

Extracting the terms of odd powers of q on both sides of (2.27), then dividing by q and replacing q^2 by q, we find that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\bar{p}(20n+15)q^n \equiv\, &3\frac{f_2f_4^6f_{10}^{20}}{f_1^8f_5^8f_{20}^8} - \frac{f_2^4f_5f_{10}^5}{f_1^5f_{20}^2} - 2q\frac{f_2^7f_4^2f_{10}^{14}}{f_1^{10}f_5^6f_{20}^4} + 2q\frac{f_4f_5^2f_{10}f_{20}}{f_1^2}\\
&+ q\frac{f_4^3f_{10}^6}{f_1^3f_2f_5f_{20}} + 3q^2\frac{f_2^5f_5f_{20}^3}{f_1^5f_4} + 3q^2\frac{f_2^{13}f_{10}^8}{f_1^{12}f_4^2f_5^4}\\
&- 2q^3\frac{f_2^{19}f_{10}^2f_{20}^4}{f_1^{14}f_4^6f_5^2} + 3q^5\frac{f_2f_4^6f_{20}^8}{f_1^8f_{10}^4} \pmod 5. \qquad (2.28)
\end{aligned}
$$

By (2.18), we see that

$$
f_5 \equiv f_1^5 \pmod 5. \qquad (2.29)
$$

Substituting (2.29) into (2.28) gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\bar{p}(20n+15)q^n \equiv\, &3\frac{f_2^{101}}{f_1^{48}f_4^{34}} - \frac{f_2^{29}}{f_4^{10}} - 2q\frac{f_2^{77}}{f_1^{40}f_4^{18}} + 2qf_1^8f_2^5f_4^6 + q\frac{f_2^{29}}{f_1^8f_4^2}\\
&+ 3q^2f_2^5f_4^{14} + 3q^2\frac{f_2^{53}}{f_1^{32}f_4^2} - 2q^3\frac{f_2^{29}f_4^{14}}{f_1^{24}} + 3q^5\frac{f_4^{46}}{f_1^8f_2^{19}} \pmod 5. \qquad (2.30)
\end{aligned}
$$

Combining 2-dissection formulas (2.3), (2.4) and congruence (2.30), we see that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\bar{p}(20n+15)q^n \equiv\, &3\frac{f_2^{101}}{f_4^{34}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^{12} + 2qf_2^5f_4^6\left(\frac{f_4^{10}}{f_2^2f_8^4} - 4q\frac{f_2^2f_8^4}{f_4^2}\right)^2\\
&- \frac{f_2^{29}}{f_4^{10}} - 2q\frac{f_2^{77}}{f_4^{18}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^{10}\\
&+ q\frac{f_2^{29}}{f_4^2}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2 + 3q^2f_2^5f_4^{14}
\end{aligned}
$$

$$
\begin{aligned}
&+3q^2\frac{f_2^{53}}{f_4^2}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^8\\
&-2q^3f_2^{29}f_4^{14}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^6\\
&+3q^5\frac{f_4^{46}}{f_2^{19}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2\\
\equiv\, &3\frac{f_4^{134}}{f_2^{67}f_8^{48}}-\frac{f_2^{29}}{f_4^{10}}+2q\frac{f_4^{122}}{f_2^{63}f_8^{40}}+3q\frac{f_2f_4^{26}}{f_8^8}+q^2\frac{f_4^{110}}{f_2^{59}f_8^{32}}\\
&+4q^3\frac{f_4^{98}}{f_2^{55}f_8^{24}}+3q^3f_2^9f_4^2f_8^8+q^4\frac{f_4^{86}}{f_2^{51}f_8^{16}}+4q^5\frac{f_4^{74}}{f_2^{47}f_8^8}\\
&+4q^7\frac{f_4^{50}f_8^8}{f_2^{39}}+q^8\frac{f_4^{38}f_8^{16}}{f_2^{35}}+4q^9\frac{f_4^{26}f_8^{24}}{f_2^{31}}+q^{10}\frac{f_4^{14}f_8^{32}}{f_2^{27}}\\
&+2q^{11}\frac{f_4^2f_8^{40}}{f_2^{23}}+3q^{12}\frac{f_8^{48}}{f_2^{19}f_4^{10}} \pmod 5. \qquad (2.31)
\end{aligned}
$$

Extracting the terms of odd powers of q on both sides of (2.31), then dividing by q and replacing q^2 by q, we obtain (2.6). This completes the proof. □

3 Proof of Conjecture 1.1

In this section, we use the (p, k)-parametrization of theta functions given by Alaca, Alaca and Williams [AAW, AW, W] to represent the generating function of $\bar{p}(40n+35)$ modulo 5 as a linear combination of functions in p and k, where p and k are defined in terms of the theta function $\varphi(q)$ as given by

$$p := p(q) = \frac{\varphi^2(q)-\varphi^2(q^3)}{2\varphi^2(q^3)}$$

and

$$k := k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}, \qquad (3.1)$$

see Alaca, Alaca and Williams [AAW]. Williams [W] proved that

$$p = 2\frac{f_2^3f_3^3f_{12}^6}{f_1f_4^2f_9^6}. \qquad (3.2)$$

We have the following congruence.

Theorem 3.1. *For $n \geq 0$,*

$$\bar{p}(40n+35) \equiv 0 \pmod{5}. \tag{3.3}$$

Proof. The following representations of $q^{\frac{1}{24}} f_1,\ q^{\frac{1}{12}} f_2$ and $q^{\frac{1}{6}} f_4$ in terms of p and k are due to Alaca and Williams [AW],

$$q^{\frac{1}{24}} f_1 = 2^{-\frac{1}{6}} p^{\frac{1}{24}} (1-p)^{\frac{1}{2}} (1+p)^{\frac{1}{6}} (1+2p)^{\frac{1}{8}} (2+p)^{\frac{1}{8}} k^{\frac{1}{2}}, \tag{3.4}$$

$$q^{\frac{1}{12}} f_2 = 2^{-\frac{1}{3}} p^{\frac{1}{12}} (1-p)^{\frac{1}{4}} (1+p)^{\frac{1}{12}} (1+2p)^{\frac{1}{4}} (2+p)^{\frac{1}{4}} k^{\frac{1}{2}} \tag{3.5}$$

and

$$q^{\frac{1}{6}} f_4 = 2^{-2/3} p^{\frac{1}{6}} (1-p)^{\frac{1}{8}} (1+p)^{\frac{1}{24}} (1+2p)^{\frac{1}{8}} (2+p)^{\frac{1}{2}} k^{\frac{1}{2}}. \tag{3.6}$$

Substituting (3.4), (3.5) and (3.6) into (2.6), we find that

$$2^{19} \sum_{n=0}^{\infty} \bar{p}(40n+35) q^n \equiv \frac{\sqrt{2} p^{7/8} (1+2p)^{21/8} (2+p)^{21/8}}{16 q^{7/8} (1-p)^6 (1+p)^2 \sqrt{k}} F(p,k) \pmod{5}, \tag{3.7}$$

where $F(p,k)$ is given by

$$\begin{aligned} F(p,k) = 5k^{10}(&524288 + 6029312p + 88735744p^2 + 840761344p^3 \\ &+ 5072977920p^4 + 22470361088p^5 + 75791417344p^6 \\ &+ 196034666496p^7 + 392385622016p^8 + 610286094336p^9 \\ &+ 731633712128p^{10} + 663209854464p^{11} + 441020946176p^{12} \\ &+ 204189055872p^{13} + 59086163776p^{14} + 8129694944p^{15} \\ &+ 138932400p^{16} + 2477318p^{19} - 16585772p^{18} \\ &+ 33708184p^{17} + 19661p^{20}). \end{aligned} \tag{3.8}$$

By (3.4) and (3.5), we have

$$\frac{f_2^{22}}{f_1^{23}} = \frac{\sqrt{2} p^{7/8} (1+2p)^{21/8} (2+p)^{21/8}}{16 q^{7/8} (1-p)^6 (1+p)^2 \sqrt{k}}. \tag{3.9}$$

Hence (3.7) can be rewritten as

$$2^{19} \sum_{n=0}^{\infty} \bar{p}(40n+35) q^n \equiv \frac{f_2^{22}}{f_1^{23}} F(p,k) \pmod{5}, \tag{3.10}$$

where $F(p,k)$ is defined by (3.8). Clearly, $\frac{f_2^{22}}{f_1^{23}}$ is a formal power series in q with integer coefficients. By (3.1) and (3.2), we see that p and k are also formal power series in q with integer coefficients. It can be seen that the coefficients of $F(p,k)$ are divisible by 5. So we reach the assertion that $\bar{p}(40n+35) \equiv 0 \pmod 5$ for $n \geq 0$. □

To complete the proof of Conjecture 1.1, we recall that Hirschhorn and Sellers [HS-1], and Fortin, Jacob and Mathieu [FJM] independently derived the congruence

$$\bar{p}(4n+3) \equiv 0 \pmod 8, \tag{3.11}$$

for $n \geq 0$. This yields

$$\bar{p}(40n+35) \equiv 0 \pmod 8, \tag{3.12}$$

for $n \geq 0$. Combining (3.12) and the congruence $\bar{p}(40n+35) \equiv 0 \pmod 5$ for $n \geq 0$, we come to the conclusion that $\bar{p}(40n+35) \equiv 0 \pmod{40}$ for $n \geq 0$. □

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