

Zeta Functions and the Log-behavior of Combinatorial Sequences

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Abstract. In this paper, we use the Riemann zeta function $\zeta(x)$ and the Bessel zeta function $\zeta_\mu(x)$ to study the log-behavior of combinatorial sequences. We prove that $\zeta(x)$ is log-convex for $x > 1$. As a consequence, we deduce that the sequence $\{|B_{2n}|/(2n)!\}_{n \geq 1}$ is log-convex, where B_n is the n -th Bernoulli number. We introduce the function $\theta(x) = (2\zeta(x)\Gamma(x+1))^{\frac{1}{x}}$, where $\Gamma(x)$ is the gamma function, and we show that $\log \theta(x)$ is strictly increasing for $x \geq 6$. This confirms a conjecture of Sun stating that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n \geq 1}$ is strictly increasing. Amdeberhan, Moll and Vignat defined the numbers $a_n(\mu) = 2^{2n+1}(n+1)!(\mu+1)_n \zeta_\mu(2n)$ and conjectured that the sequence $\{a_n(\mu)\}_{n \geq 1}$ is log-convex for $\mu = 0$ and $\mu = 1$. By proving that $\zeta_\mu(x)$ is log-convex for $x > 1$ and $\mu > -1$, we show that the sequence $\{a_n(\mu)\}_{n \geq 1}$ is log-convex for any $\mu > -1$. We introduce another function $\theta_\mu(x)$ involving $\zeta_\mu(x)$ and the gamma function $\Gamma(x)$ and we show that $\log \theta_\mu(x)$ is strictly increasing for $x > 8e(\mu+2)^2$. This implies that $\sqrt[n]{a_n(\mu)} < \sqrt[n+1]{a_{n+1}(\mu)}$ for $n > 4e(\mu+2)^2$. Based on Dobinski's formula, we prove that $\sqrt[n]{B_n} < \sqrt[n+1]{B_{n+1}}$ for $n \geq 1$, where B_n is the n -th Bell number. This confirms another conjecture of Sun. We also establish a connection between the increasing property of $\{\sqrt[n]{B_n}\}_{n \geq 1}$ and Hölder's inequality in probability theory.

Keywords: log-convexity, Riemann zeta function, Bernoulli number, Bell number, Bessel zeta function, Narayana number, Hölder's inequality

AMS Classification: 05A20, 11B68

1 Introduction

The objective of this paper is to present an analytic approach to the log-behavior of combinatorial sequences.

Let B_n denote the n -th Bernoulli number, see [11] and [14]. Recall that $B_{2n+1} = 0$ for $n \geq 1$ and B_{2n} alternate in sign for $n \geq 1$. We consider the log-behavior of the sequence $\{|B_{2n}|\}_{n \geq 1}$. A sequence $\{a_n\}_{n \geq 1}$ of real numbers is said to be log-convex if for $n \geq 2$,

$$a_n^2 \leq a_{n-1}a_{n+1}.$$

It is well-known that

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}|, \quad (1.1)$$

where

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

is the Riemann zeta function. By proving that $\zeta(x)$ is log-convex for $x > 1$, we establish the log-convexity of the sequence $\{|B_{2n}|/(2n)!\}_{n \geq 1}$. Consequently, the sequence $\{|B_{2n}|\}_{n \geq 1}$ is log-convex. Moreover, we introduce the function

$$\theta(x) = (2\zeta(x)\Gamma(x+1))^{\frac{1}{x}}, \quad (1.2)$$

where $\Gamma(x)$ is the gamma function. We show that $\log \theta(x)$ is strictly increasing for $x \geq 6$. From relation (1.1), it can be seen that

$$\sqrt[n]{|B_{2n}|} = \frac{1}{4\pi^2}\theta^2(2n).$$

So we reach the assertion that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n \geq 1}$ is strictly increasing. This confirms a conjecture of Sun [15], which has been independently proved by Luca and Stănică [9]. We conjecture that $(\log \theta(x))'' < 0$ for $x \geq 6$.

Our approach also applies to the sequence of generalized Lasalle numbers. Let C_n denote the n th Catalan number, that is,

$$C_n = \frac{1}{n+1} \binom{2n}{n},$$

and let $N_r(z)$ denote the r -th Narayana polynomial as given by

$$N_r(z) = \sum_{k=1}^r \frac{1}{r} \binom{r}{k-1} \binom{r}{k} z^k.$$

Lasalle [8] derived the recurrence relation

$$(z+1)N_r(z) - N_{r+1}(z) = \sum_{n \geq 1} (-z)^n \binom{r-1}{2n-1} A_n N_{r-2n+1}(z),$$

where the numbers A_n satisfy the recurrence relation

$$(-1)^{n-1}A_n = C_n + \sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-1} A_j C_{n-j}. \quad (1.3)$$

Let

$$a_n = \frac{2A_n}{C_n}.$$

Lasalle [8] showed that $\{a_n\}_{n \geq 1}$ is an increasing sequence of positive integers. Amdeberhan, Moll and Vignat [2] established a connection between a_n and the Bessel zeta functions $\zeta_\mu(x)$. Recall that for a real number μ , the Bessel function $J_\mu(z)$ of the first kind of order μ is defined by

$$J_\mu(z) = \left(\frac{z}{2}\right)^\mu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\mu+k+1)k!} \left(\frac{z}{2}\right)^{2k}.$$

For $\mu \geq -1$, $J_\mu(z)$ has infinitely many positive real zeros $j_{\mu,n}$, where we assume that

$$0 < j_{\mu,1} < j_{\mu,2} < j_{\mu,3} < \cdots,$$

see [3, Sect. 4.14]. The Bessel zeta functions $\zeta_\mu(x)$ are defined by

$$\zeta_\mu(x) = \sum_{n=1}^{\infty} \frac{1}{j_{\mu,n}^x}. \quad (1.4)$$

Amdeberhan, Moll and Vignat [2] found the following relation

$$a_n = 2^{2n+1}(n+1)!(n-1)!\zeta_1(2n). \quad (1.5)$$

They also gave the following generalization of a_n for $\mu \geq -1$,

$$a_n(\mu) = 2^{2n+1}(n-1)!(\mu+1)_n \zeta_\mu(2n), \quad (1.6)$$

where $(\mu+1)_n = (\mu+1)(\mu+2)\cdots(\mu+n)$.

It is easily seen that $a_n = a_n(1)$. Setting $\mu = 0$ in (1.6), Amdeberhan, Moll and Vignat defined the sequence $\{b_n\}_{n \geq 1}$ as given by

$$b_n = \frac{1}{2}a_n(0) = 2^{2n}n!(n-1)!\zeta_0(2n). \quad (1.7)$$

Note that this sequence has been studied by Carlitz [6]. It is listed as Sequence A002190 in [10].

Amdeberhan, Moll and Vignat conjectured that the sequences $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ are log-convex. We show that $\zeta_\mu(x)$ is log-convex for $x > 1$. This implies that the sequence $\{a_n(\mu)\}_{n \geq 1}$ is log-convex for any $\mu > -1$.

This confirms the above conjectures, which have been independently proved by Wang and Zhu [16].

Moreover, we define the following function

$$\theta_\mu(x) = \left(\frac{2}{\mu!} \Gamma\left(\frac{x}{2}\right) \Gamma\left(\frac{x}{2} + \mu + 1\right) \zeta_\mu(x) \right)^{\frac{1}{x}}. \quad (1.8)$$

It can be easily checked that

$$4\theta_\mu^2(2n) = \sqrt[n]{a_n(\mu)}. \quad (1.9)$$

We show that $\log \theta_\mu(x)$ is strictly increasing for $x > 8e(\mu + 2)^2$. This leads to the increasing property that

$$\sqrt[n]{a_n(\mu)} < \sqrt[n+1]{a_{n+1}(\mu)}, \quad (1.10)$$

for $n > 4e(\mu + 2)^2$. We note that for $\mu = 0$ and $\mu = 1$ the above relation (1.10) has been independently proved by Wang and Zhu [16].

Owing to the formula of Dobinski, we may use our analytic approach to study the log-behavior of Bell numbers. Let B_n be the n -th Bell number, that is, the number of partitions of $\{1, 2, \dots, n\}$, see [5] and [12]. Notice that we have adopted the same notation B_n for both Bell numbers and Bernoulli numbers. Recall that Dobinski's formula for the Bell numbers states that

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

For $x > 0$, we define

$$B(x) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^x}{k!}, \quad (1.11)$$

so that we have $B_n = B(n)$ whenever n is a nonnegative integer.

We show that $\log B(x)^{1/x}$ is increasing for $x \geq 1$. This implies that the sequence $\{\sqrt[n]{B_n}\}_{n \geq 1}$ is increasing, as conjectured by Sun [15]. We conjecture that $(\log B(x)^{1/x})'' < 0$ for $x \geq 1$. In the last section, we give a probabilistic proof of the increasing property of the sequence $\{\sqrt[n]{B_n}\}_{n \geq 1}$ by using Hölder's inequality.

2 The log-convexity of Bernoulli numbers

To prove the log-convexity of Bernoulli numbers, we consider the log-behavior of the Riemann zeta function $\zeta(x)$ for $x > 1$. Recall that a positive function

f is called log-convex on a real interval $I = [a, b]$, if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq f(x)^\lambda f(y)^{1-\lambda}, \quad (2.12)$$

see, for example, Artin [4]. It is known that a positive function f is log-convex if and only if $(\log f(x))'' \geq 0$. So, if

$$(\log \zeta(x))'' > 0, \quad (2.13)$$

for $x > 1$, then we can deduce that $\zeta(x)$ is log-convex for $x > 1$.

Lemma 2.1. *The Riemann zeta function $\zeta(x)$ is log-convex for $x > 1$.*

Proof. Clearly, condition (2.13) is equivalent to

$$\zeta(x) \cdot \zeta''(x) - (\zeta'(x))^2 > 0. \quad (2.14)$$

Since $\zeta(x)$ converges for $x > 1$, we find that for $x > 1$,

$$\begin{aligned} & \zeta(x)\zeta''(x) - (\zeta'(x))^2 \\ &= \sum_{m=1}^{\infty} \frac{1}{m^x} \sum_{n=1}^{\infty} \frac{(\log n)^2}{n^x} - \sum_{m=1}^{\infty} \frac{\log m}{m^x} \sum_{n=1}^{\infty} \frac{\log n}{n^x} \\ &= \sum_{n>m \geq 1} \frac{(\log n)^2 + (\log m)^2 - 2 \log m \log n}{(mn)^x} \\ &= \sum_{n>m \geq 1} \frac{(\log n - \log m)^2}{(mn)^x}, \end{aligned}$$

which is positive. This completes the proof. ■

The log-convexity of $\zeta(x)$ enables us to deduce the following property of Bernoulli numbers.

Theorem 2.2. *The sequence $\left\{ \frac{|B_{2n}|}{(2n)!} \right\}_{n \geq 1}$ is log-convex.*

Proof. Since $\zeta(x)$ is log-convex, setting $x = 2n - 2$, $y = 2n + 2$ and $\lambda = 1/2$ in the defining relation (2.12), we find that

$$\zeta(2n - 2)\zeta(2n + 2) \geq \zeta(2n)^2. \quad (2.15)$$

Invoking relation (1.1) between $\zeta(x)$ and B_n , we obtain that

$$\left(\frac{|B_{2n}|}{(2n)!} \right)^2 \leq \frac{|B_{2n-2}|}{(2n-2)!} \cdot \frac{|B_{2n+2}|}{(2n+2)!}.$$

This completes the proof. ■

Since $((2n)!)^2 < (2n-2)! \cdot (2n+2)!$ for $n \geq 1$, the above theorem implies the following property.

Corollary 2.3. *The sequence $\{|B_{2n}|\}_{n \geq 1}$ is log-convex.*

3 The log-behavior of $\theta(x)$

In this section, we consider the log-behavior of the function

$$\theta(x) = (2\zeta(x)\Gamma(x+1))^{\frac{1}{x}}.$$

We begin with the following monotone property of $\log \theta(x)$.

Theorem 3.1. $\log \theta(x)$ is strictly increasing for $x \geq 6$.

Proof. To prove that $\log \theta(x)$ is increasing for $x \geq 6$, we aim to show that

$$(\log \theta(x))' > 0, \quad (3.16)$$

for $x \geq 6$. Let

$$g(x) = 2\zeta(x)\Gamma(x+1).$$

Then we have

$$\theta(x) = g(x)^{1/x}$$

and

$$(\log \theta(x))' = \frac{1}{x} \left(\frac{g'(x)}{g(x)} - \frac{\log g(x)}{x} \right).$$

Thus (3.16) can be rewritten as

$$\frac{g'(x)}{g(x)} > \frac{\log g(x)}{x},$$

for $x \geq 6$. Since $\zeta(x)$ and $\Gamma(x)$ are continuous and differentiable on $(1, \infty)$, so is $g(x)$ on $(1, \infty)$. Applying the mean value theorem to $\log g(x)/x$, it can be shown that there exists t in $(2, x)$ such that

$$\frac{g(t)'}{g(t)} > \frac{\log g(x)}{x}. \quad (3.17)$$

Since $\zeta(2) = \frac{\pi^2}{6}$ and $\Gamma(3) = 2$, we find that

$$\log g(2) = \log(2\zeta(2)\Gamma(3)) = \log \frac{2\pi^2}{3} < 2. \quad (3.18)$$

On the other hand, for $x \geq 6$, it is easily seen that $\zeta(x) > 1$ and $\Gamma(x+1) > e^x$. It follows that

$$\log g(x) = \log 2 + \log \zeta(x) + \log \Gamma(x+1) > x. \quad (3.19)$$

In view of (3.18) and (3.19), we deduce that for $x \geq 6$,

$$\frac{\log g(x)}{x} = \frac{(1 - 2/x) \log g(x)}{(1 - 2/x)x} < \frac{\log g(x) - 2}{x - 2} < \frac{\log g(x) - \log g(2)}{x - 2}. \quad (3.20)$$

Applying the mean value theorem to $\log g(x)$, we see that there exists $t \in (2, x)$ such that

$$(\log g(t))' = \frac{\log g(x) - \log g(2)}{x - 2}, \quad (3.21)$$

that is,

$$\frac{g'(t)}{g(t)} = \frac{\log g(x) - \log g(2)}{x - 2}. \quad (3.22)$$

Combining (3.20) and (3.22), we get (3.17).

Now we proceed to show that

$$\frac{g(x)'}{g(x)} > \frac{g(t)'}{g(t)}. \quad (3.23)$$

Clearly, (3.23) is equivalent to

$$\left(\frac{g'(y)}{g(y)} \right)' > 0. \quad (3.24)$$

By the definition of $g(x)$, we have

$$\left(\frac{g'(y)}{g(y)} \right)' = (\log g(y))'' = (\log \Gamma(y + 1))'' + (\log \zeta(y))''.$$

It is known that $(\log \Gamma(y + 1))'' > 0$ for $y > 1$, see Andrews, Askey and Roy [3, Theorem. 1.2.5]. On the other hand, in the proof of Lemma 2.1, we have shown that $(\log \zeta(y))'' > 0$. This proves (3.24). In other words, $\frac{g'(y)}{g(y)}$ is strictly increasing for $y > 1$. Thus for $2 < t < x$, inequality (3.23) holds.

Combining (3.17) and (3.23), we deduce that for $x \geq 6$,

$$\frac{g'(x)}{g(x)} - \frac{\log g(x)}{x} > \frac{g'(x)}{g(x)} - \frac{g'(t)}{g(t)} > 0.$$

Thus $(\log \theta(x))' > 0$ for $x \geq 6$. This completes the proof. \blacksquare

From the log-behavior of $\theta(x)$, we are led to an affirmative answer to a conjecture of Sun [15].

Corollary 3.2. *The sequence $\{\sqrt[n]{|B_{2n}|}\}_{n \geq 1}$ is strictly increasing.*

Proof. From relation (1.1), we see that for $n \geq 1$,

$$\sqrt[n]{|B_{2n}|} = \frac{1}{4\pi^2} \sqrt[n]{2\zeta(2n)(2n)!} = \frac{1}{4\pi^2} \theta^2(2n). \quad (3.25)$$

Since $\log \theta(x)$ is strictly increasing for $x \geq 6$, we find that $\theta(x)$ is also strictly increasing for $x \geq 6$. It follows from (3.25) that $\sqrt[n]{|B_{2n}|}$ is strictly increasing for $n \geq 3$. On the other hand, it is easily checked that

$$|B_2| < \sqrt{|B_4|} < \sqrt[3]{|B_6|}.$$

This completes the proof. ■

The conjecture of Sun was independently proved by Luca and Stănică [9]. In fact, they proved that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n \geq 1}$ is log-concave, which was also conjectured by Sun [15].

We pose the following conjecture concerning the function $\theta(x)$. If it is true, then it implies that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n \geq 1}$ is log-concave.

Conjecture 3.3. *The function $\theta(x)$ is log-concave for $x \geq 6$, that is, for $x \geq 6$, $(\log f(x))'' < 0$.*

4 The log-behavior of the sequence $\{a_n(\mu)\}_{n \geq 1}$

In this section, we study the log-behavior of the sequence $\{a_n(\mu)\}_{n \geq 1}$. We begin with the log-behavior of the Bessel zeta functions $\zeta_\mu(x)$.

Lemma 4.1. *For $\mu > -1$, the Bessel zeta function $\zeta_\mu(x)$ is log-convex for $x > 1$.*

Proof. We proceed to show that for $x > 1$,

$$(\log \zeta_\mu(x))'' > 0,$$

or equivalently,

$$\zeta_\mu(x)\zeta_\mu''(x) - (\zeta_\mu'(x))^2 > 0. \quad (4.26)$$

By the convergence of $\zeta_\mu(x)$, it is easily seen that

$$\zeta_\mu'(x) = - \sum_{n=1}^{\infty} \frac{\log j_{\mu,n}}{j_{\mu,n}^x}$$

and

$$\zeta_\mu''(x) = \sum_{n=1}^{\infty} \frac{(\log j_{\mu,n})^2}{j_{\mu,n}^x}.$$

Hence

$$\begin{aligned} & \zeta_\mu(x)\zeta_\mu''(x) - (\zeta_\mu'(x))^2 \\ &= \sum_{m=1}^{\infty} \frac{1}{j_{\mu,m}^x} \sum_{n=1}^{\infty} \frac{(\log j_{\mu,n})^2}{j_{\mu,n}^x} - \sum_{m=1}^{\infty} \frac{\log j_{\mu,m}}{j_{\mu,m}^x} \sum_{n=1}^{\infty} \frac{\log j_{\mu,n}}{j_{\mu,n}^x} \\ &= \sum_{n>m \geq 1} \frac{(\log j_{\mu,m})^2 + (\log j_{\mu,n})^2 - 2(\log j_{\mu,m})(\log j_{\mu,n})}{j_{\mu,m}^x j_{\mu,n}^x} \\ &= \sum_{n>m \geq 1} \frac{(\log j_{\mu,m} - \log j_{\mu,n})^2}{j_{\mu,m}^x j_{\mu,n}^x}, \end{aligned}$$

which is positive. This completes the proof. \blacksquare

Setting $f(x) = \zeta_\mu(x)$, $x = 2n - 2$, $y = 2n + 2$ and $\lambda = 1/2$ in the defining relation (2.12) of a log-convex function, we obtain that for $\mu > -1$,

$$\zeta_\mu(2n - 2)\zeta_\mu(2n + 2) > \zeta_\mu(2n)^2. \quad (4.27)$$

This yields that the sequence $\{\zeta_\mu(2n)\}_{n \geq 1}$ is log-convex for $\mu > -1$. On the other hand, it is easily checked that the sequence $\{2^{2n+1}(n+1)!(\mu+1)_n\}_{n \geq 1}$ is log-convex for $\mu > -1$. It is well-known that for two positive log-convex sequences $\{u_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$, the sequence $\{u_n v_n\}_{n \geq 1}$ is also log-convex. So we arrive at the following property.

Theorem 4.2. *The sequence $\{a_n(\mu)\}_{n \geq 1}$ is log-convex for $\mu > -1$.*

For $\mu = 0$ and $\mu = 1$, Theorem 4.2 gives affirmative answers to the two conjectures of Amdeberhan, Moll and Vignat [2] on the log-convexity of the sequences $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$, where $a_n = a_n(1)$ and $b_n = \frac{1}{2}a_n(0)$.

Next we consider the monotone property of the sequence $\{\sqrt[n]{a_n(\mu)}\}_{n \geq 1}$ for $\mu > 0$.

Theorem 4.3. *For $\mu > 0$, the sequence $\{\sqrt[n]{a_n(\mu)}\}_{n \geq 1}$ is increasing for $n > 4e(\mu + 2)^2$*

To prove the above theorem, we introduce the function

$$\theta_\mu(x) = \left(\frac{2}{\mu!} \Gamma\left(\frac{x}{2}\right) \Gamma\left(\frac{x}{2} + \mu + 1\right) \zeta_\mu(x) \right)^{\frac{1}{x}},$$

which has the following monotone property.

Theorem 4.4. *For $\mu \geq 0$, the function $\log \theta_\mu(x)$ is strictly increasing for $x > 8e(\mu + 2)^2$.*

Proof. Assume that $\mu \geq 0$. To prove the monotone property in the theorem, we aim to show that for $x > 8e(\mu + 2)^2$,

$$(\log \theta_\mu(x))' > 0. \quad (4.28)$$

Let

$$h(x) = \frac{2}{\mu!} \Gamma(x/2) \Gamma(x/2 + \mu + 1) \zeta_\mu(x). \quad (4.29)$$

Recalling the definition of $\theta_\mu(x)$ as given by (1.8), we have

$$\theta_\mu(x) = h(x)^{\frac{1}{x}}$$

and

$$\log \theta_\mu(x) = \frac{1}{x} \log h(x).$$

It follows that

$$(\log \theta_\mu(x))' = \frac{1}{x} \left(\frac{h'(x)}{h(x)} - \frac{\log h(x)}{x} \right). \quad (4.30)$$

Since $\zeta_\mu(x)$ and $\Gamma(x)$ are continuous and differentiable on $(1, \infty)$, so is $h(x)$. We shall apply the mean value theorem to $\log h(x)$ on $[2, x]$, where $x > 8e(\mu+2)^2$ and $\mu > -1$. To this end, we need to show that $h(2) < 1$ and $h(x) > 1$ for $\mu > -1$ and $x > 8e(\mu+2)^2$.

Recalling the definition of $h(x)$ as given by (4.29), we get

$$h(2) = \frac{2}{\mu!} \Gamma(1) \Gamma(\mu+2) \zeta_\mu(2),$$

where

$$\zeta_\mu(2) = \frac{1}{4(\mu+1)},$$

$\Gamma(1) = 1$ and $\Gamma(\mu+2) = (\mu+1)!$. Then

$$h(2) = \frac{2}{\mu!} \cdot (\mu+1)! \cdot \frac{1}{4(\mu+1)}, \quad (4.31)$$

so $h(2) < 1$.

It remains to show that $h(x) > 1$ for $\mu > -1$ and $x > 8e(\mu+2)^2$. Recall that

$$j_{\mu,1} < (\mu+1)^{\frac{1}{2}} \left((\mu+2)^{\frac{1}{2}} + 1 \right), \quad (4.32)$$

for $\mu > -1$, see Chamber [7]. It follows that for $\mu > -1$,

$$j_{\mu,1} < 2(\mu+2). \quad (4.33)$$

Therefore, we obtain that for $\mu > -1$,

$$\zeta_\mu(x) = \sum_{n=1}^{\infty} \frac{1}{j_{\mu,n}^x} > \frac{1}{j_{\mu,1}^x} > \frac{1}{2^x(\mu+2)^x}. \quad (4.34)$$

On the other hand, it is known that for $x \geq 0$,

$$\Gamma(x) > \sqrt{2\pi x} \left(\frac{x}{e} \right)^x, \quad (4.35)$$

see Alzer [1]. Combining (4.34) and (4.35), we deduce that for $x > 2$ and $\mu > -1$,

$$2\Gamma\left(\frac{x}{2}\right) \zeta_\mu(x) > 2\sqrt{\pi x} \left(\frac{x}{8e(\mu+2)^2} \right)^{\frac{x}{2}}.$$

Consequently, for $\mu > -1$ and $x > 8e(\mu + 2)^2$, we obtain that

$$2\Gamma\left(\frac{x}{2}\right)\zeta_\mu(x) > 2\sqrt{\pi x} > 1. \quad (4.36)$$

Clearly, for $x > 0$ we have

$$\frac{\Gamma(x/2 + \mu + 1)}{\mu!} > 1. \quad (4.37)$$

In view of (4.36) and (4.37), we find that for $\mu > -1$ and $x > 8e(\mu + 2)^2$,

$$h(x) = \frac{2}{\mu!}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x}{2} + \mu + 1\right)\zeta_\mu(x) > 1, \quad (4.38)$$

as claimed.

Next we proceed to prove that there exists t in $(2, x)$ such that

$$\frac{h'(t)}{h(t)} > \frac{\log h(x)}{x}. \quad (4.39)$$

By the mean value theorem applied to $\log h(x)$ on $[2, x]$, there exists $t \in (2, x)$ such that

$$\frac{h'(t)}{h(t)} = (\log h(t))' = \frac{\log h(x) - \log h(2)}{x - 2}. \quad (4.40)$$

On the other hand, we have shown that $h(2) < 1$ and $h(x) > 1$ for $\mu > -1$ and $x > 8e(\mu + 2)^2$. Consequently, we have $\log h(2) < 0$ and $\log h(x) > 0$. Note that for $\mu > -1$ and $x > 8e(\mu + 2)^2$, we have $x > 2$. Hence

$$\frac{\log h(x)}{x} < \frac{\log h(x) - \log h(2)}{x - 2}. \quad (4.41)$$

Combining (4.40) and (4.41), we obtain (4.39).

Moreover, it can be shown that

$$\frac{h'(x)}{h(x)} > \frac{h'(t)}{h(t)}. \quad (4.42)$$

We claim that for $y > 2$,

$$\left(\frac{h'(y)}{h(y)}\right)' > 0. \quad (4.43)$$

By the definition of $h(x)$ as given by (4.29), we have

$$\begin{aligned} \left(\frac{h'(y)}{h(y)}\right)' &= (\log h(y))'' \\ &= (\log \Gamma(y/2))'' + (\log \Gamma(y/2 + \mu + 1))'' + (\log \zeta_\mu(x))''. \end{aligned}$$

It is known that $(\log \Gamma(y))'' > 0$ for $y > 1$, see [3, Theorem 1.2.5]. Thus, $(\log \Gamma(y/2))'' > 0$ and $(\log \Gamma(y/2 + \mu + 1))'' > 0$ for $y > 2$. Moreover, in the proof of Lemma 4.1, we have shown that $(\log \zeta_\mu(y))'' > 0$. This proves (4.43). In other words, $\frac{h'(y)}{h(y)}$ is strictly increasing for $y > 2$. Thus for $2 < t < x$, (4.42) holds.

Combining (4.39) and (4.42), for $\mu > -1$ and $x > 8e(\mu+2)^2$, we find that

$$\frac{h'(x)}{h(x)} - \frac{\log h(x)}{x} > \frac{h'(x)}{h(x)} - \frac{h'(t)}{h(t)} > 0.$$

Hence (4.28) follows from (4.30). This completes the proof. \blacksquare

In view of relation (1.6), it can be checked that

$$\sqrt[n]{a_n(\mu)} = 4\theta_\mu(2n)^2. \quad (4.44)$$

Thus Theorem 4.4 implies that for any $\mu \geq 0$ and $n > 4e(\mu+2)^2$, we have $\sqrt[n]{a_n(\mu)} < \sqrt[n+1]{a_{n+1}(\mu)}$.

For $\mu = 1$, it can be verified that $\sqrt[n]{a_n(1)} < \sqrt[n+1]{a_{n+1}(1)}$ for $2 \leq n \leq 108$. In the meantime, for $\mu = 1$, Theorem 4.4 states that $\sqrt[n]{a_n(1)} < \sqrt[n+1]{a_{n+1}(1)}$ for $n > 101$. Thus we have the following assertion.

Theorem 4.5. *The sequence $\{\sqrt[n]{a_n}\}_{n \geq 2}$ is strictly increasing.*

For $\mu = 0$, it can be verified that $\sqrt[n]{a_n(0)} < \sqrt[n+1]{a_{n+1}(0)}$ for $2 \leq n \leq 48$. Meanwhile, for $\mu = 0$, Theorem 4.4 states that $\sqrt[n]{a_n(0)} < \sqrt[n+1]{a_{n+1}(0)}$ for $n > 45$. So we have $\sqrt[n]{a_n(0)} < \sqrt[n+1]{a_{n+1}(0)}$ for $n \geq 2$. Since $b_n = \frac{1}{2}a_n(0)$, we have for $n \geq 2$,

$$\sqrt[n]{b_n} = \frac{\sqrt[n]{a_n(0)}}{\sqrt[n]{2}} < \frac{\sqrt[n+1]{a_{n+1}(0)}}{\sqrt[n+1]{2}} = \sqrt[n+1]{b_{n+1}}.$$

Thus we have the following monotone property.

Theorem 4.6. *The sequence $\{\sqrt[n]{b_n}\}_{n \geq 2}$ is strictly increasing.*

Note that Wang and Zhu [16] independently proved the log-convexity of $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ and the increasing properties of $\{\sqrt[n]{a_n}\}_{n \geq 1}$ and $\{\sqrt[n]{b_n}\}_{n \geq 1}$.

5 The log-behavior of Bell numbers

In this section, we consider the log-behavior of Bell numbers, which are also denoted by B_n . Recall that the function $B(x)$ is defined by

$$B(x) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^x}{k!}.$$

Lemma 5.1. *The function $B(x)$ is log-convex for $x > 1$.*

Proof. We proceed to show that

$$(\log B(x))'' > 0,$$

that is,

$$B(x)B''(x) - (B'(x))^2 > 0. \quad (5.45)$$

For $x \geq 1$, by the convergence of $B(x)$, we have

$$B'(x) = \frac{1}{e} \sum_{n=0}^{\infty} \frac{n^x \log n}{n!}$$

and

$$B''(x) = \frac{1}{e} \sum_{n=0}^{\infty} \frac{n^x (\log n)^2}{n!}.$$

Thus, for $x > 1$, we have

$$\begin{aligned} & B(x)B''(x) - (B'(x))^2 \\ &= \frac{1}{e^2} \sum_{m=0}^{\infty} \frac{m^x}{m!} \sum_{n=0}^{\infty} \frac{n^x (\log n)^2}{n!} - \frac{1}{e^2} \sum_{m=0}^{\infty} \frac{m^x \log m}{m!} \sum_{n=0}^{\infty} \frac{n^x \log n}{n!} \\ &= \frac{1}{e^2} \sum_{n>m \geq 0} \frac{m^x n^x}{m!n!} ((\log m)^2 + (\log n)^2 - 2 \log m \log n) \\ &= \frac{1}{e^2} \sum_{n>m \geq 0} \frac{m^x n^x}{m!n!} (\log n - \log m)^2, \end{aligned}$$

which is positive. This completes the proof. ■

We now turn to the log-behavior of the function $B(x)^{1/x}$.

Theorem 5.2. *$\log B(x)^{1/x}$ is strictly increasing for $x \geq 1$.*

Proof. To prove that $\log B(x)^{1/x}$ is strictly increasing, we aim to show that

$$(\log B(x)^{1/x})' > 0. \quad (5.46)$$

Since

$$(\log B(x)^{1/x})' = \frac{1}{x} \left(\frac{B'(x)}{B(x)} - \frac{\log B(x)}{x} \right),$$

(5.46) can be rewritten as

$$\frac{B'(x)}{B(x)} > \frac{\log B(x)}{x}. \quad (5.47)$$

We claim that there exists t in $(1, x)$ such that

$$\frac{B'(t)}{B(t)} > \frac{\log B(x)}{x}. \quad (5.48)$$

Since $B(1) = 1$ and $B(x) > 1$ for $x > 1$, by the mean value theorem with respect to $\log B(x)$ on $[1, x]$, there exists $t \in (1, x)$ such that

$$\frac{B'(t)}{B(t)} = \frac{\log B(x) - \log B(1)}{x - 1} = \frac{\log B(x)}{x - 1}. \quad (5.49)$$

Since $x > 1$, we have

$$\frac{\log B(x)}{x - 1} > \frac{\log B(x)}{x}. \quad (5.50)$$

Combining (5.49) and (5.50), we obtain (5.48).

Next we show that for $x > t > 1$,

$$\frac{B'(x)}{B(x)} > \frac{B'(t)}{B(t)}. \quad (5.51)$$

In fact, by Lemma 5.1, we see that for $y \geq 1$,

$$\left(\frac{B'(y)}{B(y)} \right)' = (\log B(y))'' > 0.$$

This implies that $\frac{B'(y)}{B(y)}$ is strictly increasing for $y > 1$. This proves (5.51).

Combining (5.48) and (5.51), we obtain (5.47). This completes the proof. \blacksquare

Since $B(n) = B_n$ whenever n is a positive integer, Theorem 5.2 implies the following monotone property conjectured by Sun [15].

Corollary 5.3. *The sequence $\{\sqrt[n]{B_n}\}_{n \geq 1}$ is strictly increasing.*

The above property was independently obtained by Wang and Zhu [16] via a different approach. Furthermore, we pose the following conjecture which implies the conjecture of Sun [15] stating that the sequence $\{\sqrt[n]{B_n}\}_{n \geq 1}$ is log-concave.

Conjecture 5.4. *The function $B(x)^{1/x}$ is log-concave for $x \geq 1$, that is, $(\log B(x)^{1/x})'' < 0$ for $x > 1$.*

6 A connection to Hölder's inequality

In this section, we give a derivation of the monotone property of the function $B(x)^{1/x}$ as given in Theorem 5.2 by applying Hölder's inequality in probability theory. In fact, it can be shown the condition $1 < x < y$ in Theorem 5.2 can be relaxed to $0 < x < y$.

Let Z be the discrete random variable with Poisson distribution as given by

$$P(Z = k) = \frac{1}{e} \frac{1}{k!}.$$

From Dobinski's formula, it is easily checked that $B(x) = E[Z^x]$. Hölder's inequality states that for real-valued random variables U , V and positive numbers p and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$E[|UV|] \leq E[|U|^p]^{1/p} E[|V|^q]^{1/q},$$

and the equality holds if and only if either there exist constants $\alpha, \beta > 0$ such that $\alpha|U|^p = \beta|V|^q$ or $E[|U|^p] = 0$ or $E[|V|^q] = 0$, see, for example, Sachkov [13]. For $0 < x < y$, we set $p = y/x$, and set $U = Z^x$ and $V = 1$. It is not hard to see that in this case Hölder inequality is strict. Hence we obtain that

$$E[Z^x]^{1/x} < E[Z^y]^{1/y},$$

which can be restated as follows.

Theorem 6.1. *For $0 < x < y$, we have $B(x)^{1/x} < B(y)^{1/y}$.*

Acknowledgments. We wish to thank the referee for valuable suggestions. This work was supported by the 973 Project, the PCSIRT Project, the Doctoral Program Fund of the Ministry of Education, and the National Science Foundation of China.

References

- [1] H. Alzer, Sharp upper and lower bounds for the gamma function, Proc. Royal Soc. Edinburgh A 139 (2009) 709–718.
- [2] T. Amdeberhan, V.H. Moll and C. Vignat, A probabilistic interpretation of a sequence related to Narayana polynomials, arXiv:1202.1203.
- [3] G.E. Andrews, R. Askey and R. Roy, Special Functions, Cambridge University Press, 1999.

- [4] E. Artin, *The Gamma Function*, Holt, Rinehart and Winston, New York, 1964.
- [5] E.T. Bell, Exponential Numbers, *Amer. Math. Monthly* 41 (1934) 411–419.
- [6] L. Carlitz, A sequence of integers related to the Bessel function, *Proc. Amer. Math. Soc.* 14 (1963) 1–9.
- [7] L. G. Chamber, An upper bound for the first zero of Bessel functions, *Math. Comp.* 38 (1982) 589–591.
- [8] M. Lasalle, Two integer sequences related to Catalan numbers, *J. Combin. Theory Ser. A* 119 (2012) 923–935.
- [9] F. Luca and P. Stănică, On some conjectures on the monotonicity of some combinatorial sequences, *J. Combin. Number Theory* 4 (2012) 1–10.
- [10] The OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, <http://oeis.org>.
- [11] S.M. Roman and Gian-Carlo Rota, The umbral calculus, *Adv. in Math.* 27 (1978) 95–188.
- [12] G.-C. Rota, The number of partitions of a set, *Amer. Math. Monthly* 71 (1964) 498–504.
- [13] V.N. Sachkov, *Probabilistic Methods in Combinatorial Analysis*, Cambridge University Press, New York, 1997.
- [14] D.E. Smith, *Source Book in Mathematics*, Vols. 1 and 2, New York: Dover, 1959.
- [15] Z. Sun, Conjectures involving arithmetical sequences, *Numbers Theory: Arithmetic in Shangri-La* (eds., S. Kanemitsu, H. Li and J. Liu), *Proc. 6th China-Japan Seminar* (Shang-hai, August 15-17, 2011), World Sci., Singapore, 2013, pp. 244-258.
- [16] Y. Wang and B. Zhu, Proofs of some conjectures on monotonicity of number-theoretic and combinatorial sequences, arXiv:1303.5595.