On the generalized (edge-)connectivity of graphs*

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Abstract

The generalized k-connectivity $\kappa_k(G)$ of a graph G was introduced by Chartrand et al. in 1984. It is natural to introduce the concept of generalized k-edge-connectivity $\lambda_k(G)$. For general k, the generalized k-edge-connectivity of a complete graph is obtained. For $k \geq 3$, tight upper and lower bounds of $\kappa_k(G)$ and $\lambda_k(G)$ are given for a connected graph G of order n, that is, $1 \leq \kappa_k(G) \leq n - \lceil \frac{k}{2} \rceil$ and $1 \leq \lambda_k(G) \leq n - \lceil \frac{k}{2} \rceil$. Graphs of order n such that $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil$ and $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$ are characterized, respectively. Nordhaus-Gaddum-type results for the generalized k-connectivity are also obtained. For $k = 3$, we study the relation between the edge-connectivity and the generalized 3-edge-connectivity of a graph. Upper and lower bounds of $\lambda_3(G)$ for a graph G in terms of the edge-connectivity λ of G are obtained, that is, $\frac{3\lambda-2}{4} \leq \lambda_3(G) \leq \lambda$, and two graph classes are given showing that the upper and lower bounds are tight. From these bounds, we obtain $\lambda(G) - 1 \leq \lambda_3(G) \leq \lambda(G)$ if G is a connected planar graph, and we also obtain the relation between the generalized 3-connectivity and generalized 3-edge-connectivity of a graph and its line graph.

Keywords: (edge-)connectivity, internally (edge-)disjoint trees, generalized (edge-)connectivity, planar graph, line graph, complementary graph.

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1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to book [2] for graph theoretical notation and terminology not described here. The generalized connectivity of a graph G, introduced by Chartrand et al. in [4], is a natural and nice generalization of the concept of (vertex-)connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a such subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Two Steiner trees T and T' connecting S are *internally disjoint* if $E(T) \cap E(T') = \varnothing$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$, the *generalized local connectivity* $\kappa(S)$ of S is the maximum number of internally disjoint trees connecting S in G. The *generalized k-connectivity* of G, denoted by $\kappa_k(G)$, is then defined as $\kappa_k(G) = min\{\kappa(S) | S \subseteq V(G)\ and\ |S| =$

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$k\}$. Thus, $\kappa_2(G) = \kappa(G)$. Set $\kappa_k(G) = 0$ when G is disconnected. Results on the generalized connectivity can be found in [5, 14, 15, 16, 17, 18, 19, 21].

A natural idea is to introduce the concept of generalized edge-connectivity. For $S \subseteq V(G)$, the *generalized local connectivity* $\lambda(S)$ of S is the maximum number of edge-disjoint Steiner trees connecting S in G. Then the *generalized k-edge-connectivity* $\lambda_k(G)$ of G is defined as $\lambda_k(G) = min\{\lambda(S)|S \subseteq V(G)\ and\ |S| = k\}$. Thus $\lambda_2(G) = \lambda(G)$. Set $\lambda_k(G) = 0$ when G is disconnected. In general, the parameters κ_k and λ_k are different. Take for example, let G be a graph obtained from two copies of the complete graph K_4 by identifying one vertex in each of them. One can easily check that $\lambda_3(G) = 2$ but $\kappa_3(G) = 1$.

The generalized edge-connectivity is related to an important problem, which is called the *Steiner Tree Packing Problem*. For a given graph G and $S \subseteq V(G)$, this problem asks to find a set of maximum number of edge-disjoint Steiner trees connecting S in G. The difference between the Steiner Tree Packing Problem and the generalized edge-connectivity is as follows: The Steiner Tree Packing Problem studies local properties of graphs since S is given beforehand, but the generalized edge-connectivity focuses on global properties of graphs since it first needs to find the maximum number $\lambda(S)$ of edge-disjoint trees connecting S and then S runs over all k-subsets of $V(G)$ to get the minimum value of $\lambda(S)$.

The problem for $S = V(G)$ is called the *Spanning Tree Packing Problem* (Note that the Steiner Tree Packing Problem is a generalization of the Spanning Tree Packing Problem). For any graph G of order n, the *spanning tree packing number* or *STP number*, is the maximum number of edge-disjoint spanning trees contained in G. For the spanning tree packing number, Palmer gave a good survey (see [22]). One can see that the STP number of a graph G is just $\kappa_n(G)$ or $\lambda_n(G)$.

In addition to being natural combinatorial measures, the generalized connectivity and generalized edge-connectivity can be motivated by their interesting interpretation in practice as well as theoretical consideration.

From a theoretical perspective, both extremes of this problem are fundamental theorems in combinatorics. One extreme of the problem is when we have two terminals. In this case internally (edge-)disjoint trees are just internally (edge-)disjoint paths between the two terminals, and so the problem becomes the well-known Menger theorem. The other extreme is when all the vertices are terminals. In this case internally disjoint trees and edge-disjoint trees are just spanning trees of the graph, and so the problem becomes the classical Nash-Williams-Tutte theorem (for short proofs, see [9]).

Theorem 1. *(Nash-Williams [20],Tutte [24]) A multigraph G contains a system of k edge-disjoint spanning trees if and only if*

$$\|G/\mathscr{P}\| \geq k(|\mathscr{P}| - 1)$$

holds for every partition $\mathscr{P}$ of $V(G)$, where $\|G/\mathscr{P}\|$ denotes the number of edges in G between distinct blocks of $\mathscr{P}$.

The following corollary is immediate from Theorem 1.

Corollary 1. *Every 2ℓ-edge-connected graph contains a system of ℓ edge-disjoint spanning trees.*

Kriesell [11] conjectured that this corollary can be generalized for Steiner trees.

Conjecture 1. *(Kriesell [11]) If a set S of vertices of G is $2k$-edge-connected (See Section 2 for the definition), then there is a set of k edge-disjoint Steiner trees in G.*

Motivated by this conjecture, the Steiner Tree Packing Problem has obtained wide attention and many results have been worked out, see [10, 11, 12, 13, 25].

The generalized edge-connectivity and the Steiner Tree Packing Problem have applications in $VLSI$ circuit design, see [7, 8, 23]. In this application, a Steiner tree is needed to share an electronic signal by a set of terminal nodes. Another application, which is our primary focus, arises in the Internet Domain. Imagine that a given graph G represents a network. We choose arbitrary k vertices as nodes. Suppose one of the nodes in G is a *broadcaster*, and all the other nodes are either *users* or *routers* (also called *switches*). The broadcaster wants to broadcast as many streams of movies as possible, so that the users have the maximum number of choices. Each stream of movie is broadcasted via a tree connecting all the users and the broadcaster. So, in essence we need to find the maximum number Steiner trees connecting all the users and the broadcaster, namely, we want to get $\lambda(S)$, where S is the set of the k nodes. Clearly, it is a Steiner tree packing problem. Furthermore, if we want to know whether for any k nodes the network G has the above properties, then we need to compute $\lambda_k(G) = min\{\lambda(S)\}$ in order to prescribe the reliability and the security of the network.

For general k, the generalized k-edge-connectivity of a complete graph is obtained. Tight upper and lower bounds of $\kappa_k(G)$ and $\lambda_k(G)$ are given for a connected graph G of order n, that is, $1 \leq \kappa_k(G) \leq n - \lceil \frac{k}{2} \rceil$ and $1 \leq \lambda_k(G) \leq n - \lceil \frac{k}{2} \rceil$.

By Nash-Williams-Tutte theorem, graphs of order n such that $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil$ and $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$ are characterized, respectively. Nordhaus-Gaddum-type results for the generalized k-connectivity are also obtained in Section 3. For $k = 3$, we study the relation between the edge-connectivity and the generalized 3-edge-connectivity of a graph. Kriesell in [11] showed that for any two natural numbers t, ℓ there exists a smallest natural number $f_\ell(t)$ ($g_\ell(t)$) such that for any $f_\ell(t)$-edge-connected ($g_\ell(t)$-edge-connected) vertex set S of a graph G with $|S| \leq \ell$ ($|V(G) - S| \leq \ell$) there exists a system $\mathscr{T}$ of t edge-disjoint trees such that $S \subseteq V(T)$ for each $T \in \mathscr{T}$. He determined $f_3(t) = \lfloor \frac{8t+3}{6} \rfloor$. In Section 4, we use his result to derive a tight lower bound of $\lambda_3(G)$. We also give a tight upper bound of $\lambda_k(G)$. Altogether we get that $\frac{3\lambda-2}{4} \leq \lambda_3(G) \leq \lambda$. Two graph classes are given showing that the upper and lower bounds are tight. From these bounds, we obtain two results: one is $\lambda(G) - 1 \leq \lambda_3(G) \leq \lambda(G)$ if G is a connected planar graph, the other is the relation between the generalized 3-connectivity and generalized 3-edge-connectivity of a graph and its line graph.

2 Preliminaries

For a graph G, let $V(G)$, $E(G)$, $|G|$, $\|G\|$, $L(G)$ and $\overline{G}$ denote the set of vertices, the set of edges, the order, the size, the line graph and the complement graph of G, respectively. As usual, the *union* of two graphs G and H is the graph, denoted by $G \cup H$, with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For $S \subseteq V(G)$, we denote by $G \setminus S$ the subgraph obtained by deleting from G the vertices of S together with the edges incident with them. If $S = \{v\}$, we simply write $G \setminus v$ for $G \setminus \{v\}$. If S is a subset of vertices of a graph G, the subgraph of G induced by S is denoted by $G[S]$. If M is the edge subset of G, then $G \setminus M$ denotes the subgraph obtained by deleting the edges of M from G. $G \setminus \{e\}$ is abbreviated to $G \setminus e$. If M is a subset of edges of a graph G, the subgraph of G induced by M is denoted by $G[M]$. We denote by $E_G[X, Y]$ the set of edges of G with one end in X and the other in Y. If $X = \{x\}$, we simply write $E_G[x, Y]$ for $E_G[\{x\}, Y]$.

Chartrand et al. in [5] obtained the first result in the generalized connectivity.

Theorem 2. *[5] For every two integers n and k with $2 \leq k \leq n$,*

$$\kappa_k(K_n) = n - \lceil k/2 \rceil.$$

For distinct vertices x, y in G, let $\lambda(x, y; G)$ denote the local edge-connectivity of x and y. $S \subseteq V(G)$ is called *n-edge-connected*, if $\lambda(x, y; G) \geq n$ for all $x \neq y$ in S. In [11], Kriesell gave the following result.

Lemma 1. *[11] Let $t \geq 1$ be a natural number, and G be a graph, and let $\{a, b, c\} \subseteq V(G)$ be $\lfloor \frac{8t+3}{6} \rfloor$-edge-connected in G. Then there exists a system of t edge-disjoint $\{a, b, c\}$-trees.*

Chartrand et al. [6] investigated the relation between the connectivity and edge-connectivity of a graph and its line graph.

Lemma 2. *[6] If G is a connected graph, then*

(1) $\kappa(L(G)) \geq \lambda(G)$ *if* $\lambda(G) \geq 2$.

(2) $\lambda(L(G)) \geq 2\lambda(G) - 2$.

(3) $\kappa(L(L(G))) \geq 2\kappa(G) - 2$.

Palmer [22] gave the STP number of a complete bipartite graph.

Lemma 3. *[22] The STP number of a complete bipartite graph $K_{a,b}$ is $\lfloor \frac{ab}{a+b-1} \rfloor$.*

3 Results of $\kappa_k(G)$ and $\lambda_k(G)$ for general k

After the preparation of the above section, we start to give our main results of this paper.

3.1 Results for complete graphs

The following two observations are easily seen.

Observation 1. *If G be a connected graph, then $\kappa_k(G) \leq \lambda_k(G) \leq \delta(G)$.*

Observation 2. *If H is a spanning subgraph of G, then $\kappa_k(H) \leq \kappa_k(G)$ and $\lambda_k(H) \leq \lambda_k(G)$.*

For a general k and the complete graph K_n, $\kappa_k(K_n)$ was determined by Chartrand et al.; see Theorem 2. Now we give the result for $\lambda_k(K_n)$.

Choose $S \subseteq V(G)$ with $|S| = k$. Let $\mathscr{T}$ be a maximum set of edge-disjoint trees in G connecting S. Let $\mathscr{T}_1$ be the set of trees in $\mathscr{T}$ whose edges belong to $E(G[S])$, and $\mathscr{T}_2$ be the set of trees containing at least one edge of $E_G[S, \bar{S}]$, where $\bar{S} = V(G) \setminus S$. Thus, $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$ (Throughout this paper, $\mathscr{T}$, $\mathscr{T}_1$, $\mathscr{T}_2$ are always defined as this).

Lemma 4. *Let $S \subseteq V(G)$, $|S| = k$ and T be a tree connecting S. If $T \in \mathscr{T}_1$, then T uses $k-1$ edges of $E(G[S]) \cup E_G[S, \bar{S}]$. If $T \in \mathscr{T}_2$, then T uses at least k edges of $E(G[S]) \cup E_G[S, \bar{S}]$.*

Proof. It is easy to see that for each tree T in $\mathscr{T}_1$, T uses $k-1$ edges in $E(G[S])$, namely, T uses $k-1$ edges of $E(G[S]) \cup E_G[S, \bar{S}]$.

For $T \in \mathscr{T}_2$, by deleting all the vertices of T from $\bar{S}$, we obtain some components of T in S, denoted by $C_1, C_2, \cdots, C_s$. Let $|C_i| = c_i$. Then $|E(C_i)| = c_i - 1$ and $\sum_{i=1}^{s}(c_i - 1) = k - s$. Since there exists one edge of T between each C_i and $\bar{S}$, where $1 \leq i \leq s$, T uses $(k-s)+s = k$ edges in $E(G[S]) \cup E_G[S, \bar{S}]$. □

Theorem 3. *For every two integers n and k with $2 \leq k \leq n$,*

$$\lambda_k(K_n) = n - \lceil k/2 \rceil.$$

Proof. Let $G = K_n$. We choose $S \subseteq V(G)$ such that $|S| = k$. Let $|\mathscr{T}| = y$ and $|\mathscr{T}_1| = x$. From Lemma 4, each tree $T \in \mathscr{T}_1$ uses $k-1$ edges in $E(G[S]) \cup E_G[S, \bar{S}]$, $|\mathscr{T}_1| = x \leq \lfloor \binom{k}{2}/(k-1) \rfloor = \lfloor \frac{k}{2} \rfloor$. Since each tree $T \in \mathscr{T}_2$ uses k edges in $E(G[S]) \cup E_G[S, \bar{S}]$, we have $|\mathscr{T}_1|(k-1) + |\mathscr{T}_2|k \leq |E_G[S, \bar{S}]| + |E(G[S])|$, that is, $x(k-1) + (y-x)k \leq \binom{k}{2} + k(n-k)$. So $\lambda_k(G) \leq y \leq \frac{k-1}{2} + n - k + \frac{x}{k} = n - \lceil \frac{k}{2} \rceil + \frac{x}{k}$ since $x \leq \lfloor \frac{k}{2} \rfloor$ and y is an integer.

From the above arguments, we conclude that $\lambda_k(K_n) \leq n - \lceil \frac{k}{2} \rceil$. Combining this with Theorem 2 and Observation 1, we have $\lambda_k(K_n) = n - \lceil \frac{k}{2} \rceil$. □

From Theorems 2 and 3, we get that $\lambda_k(G) = \kappa_k(G)$ for a complete graph $G = K_n$. However, this is a very special case. Actually, $\lambda_k(G) - \kappa_k(G)$ could be very large. For example, let G be a graph obtained from two copies of the complete graph K_n by identifying one vertex in each of them. Then for $k < n$, $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$, but $\kappa_k(G) = 1$.

3.2 Graphs with $\kappa_k(G) = n - \lceil k/2 \rceil$ and $\lambda_k(G) = n - \lceil k/2 \rceil$, respectively

At first, we give the tight bounds for $\kappa_k(G)$ and $\lambda_k(G)$:

Proposition 1. *For a connected graph G of order n and $3 \leq k \leq n$, $1 \leq \kappa_k(G) \leq n - \lceil k/2 \rceil$. Moreover, the upper and lower bounds are tight.*

Proof. From Observation 2 and Theorem 2, we have $\kappa_k(G) \leq \kappa_k(K_n) = n - \lceil \frac{k}{2} \rceil$. Since G is connected, then $\kappa_k(G) \geq 1$. The result holds.

One can easily check that the complete graph K_n attains the upper bound and any tree T_n on n vertices attains the lower bound. □

The same upper and lower bounds can be established for the generalized k-edge-connectivity.

Proposition 2. *For a connected graph G of order n and $3 \leq k \leq n$, $1 \leq \lambda_k(G) \leq n - \lceil k/2 \rceil$. Moreover, the upper and lower bounds are tight.*

Next, we will characterize graphs with $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil$ and $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$, respectively. Let us start with some lemmas, which will be used later.

Lemma 5. *For an even k with $4 \leq k \leq n$, $\lambda_k(K_n \setminus e) < n - \frac{k}{2}$ for any $e \in E(K_n)$.*

Proof. Let $G = K_n \setminus e$. We choose $S \subseteq V(G)$ such that $|S| = k$ and $K_n[S]$ containing e. Let $|\mathscr{T}| = y$ and $|\mathscr{T}_1| = x$. Since every tree $T \in \mathscr{T}_1$ uses $k-1$ edges in $E(G[S]) \cup E_G[S, \bar{S}]$, $|\mathscr{T}_1| = x \leq \left(\binom{k}{2} - 1\right)/(k-1) = \frac{k}{2} - \frac{1}{k-1}$. From Lemma 4, each tree $T \in \mathscr{T}_2$ uses k edges of $E(G[S]) \cup E_G[S, \bar{S}]$. Thus $|\mathscr{T}_1|(k-1) + |\mathscr{T}_2|k \leq |E_G[S, \bar{S}]| + |E(G[S])|$, that is, $x(k-1) + (y-x)k \leq \binom{k}{2} + k(n-k) - 1$. So $\lambda_k(G) = y \leq \frac{k-1}{2} + n - k + \frac{x-1}{k} \leq n - \frac{k}{2} - \frac{1}{k-1} < n - \frac{k}{2}$. □

Lemma 6. *If k is odd with $3 \leq k \leq n$, and M is an edge set of the complete graph K_n such that $|M| \geq \frac{k+1}{2}$, then $\lambda_k(K_n \setminus M) < n - \frac{k+1}{2}$.*

Proof. Let $G = K_n \setminus M$. We can choose $S \subseteq V(G)$ such that $|S| = k$ and $|M \cap \left(E(K_n[S]) \cup E_{K_n}[S, \bar{S}]\right)| \geq \frac{k+1}{2}$. Let $|\mathscr{T}| = y$ and $|\mathscr{T}_1| = x$. Since each tree $T \in \mathscr{T}_1$ uses $k-1$ edges in $E(G[S]) \cup E_G[S, \bar{S}]$, $|\mathscr{T}_1| = x \leq \binom{k}{2}/(k-1) = \frac{k-1}{2}$. From Lemma 4, each tree $T \in \mathscr{T}_2$ uses k edges of $E(G[S]) \cup E_G[S, \bar{S}]$. Thus $|\mathscr{T}_1|(k-1) + |\mathscr{T}_2|k \leq |E_G[S, \bar{S}]| + |E(G[S])|$, that is, $x(k-1) + (y-x)k \leq \binom{k}{2} + k(n-k) - \frac{k+1}{2}$. So $\lambda_k(G) = y \leq \frac{k-1}{2} + n - k + \frac{x}{k} - \frac{k+1}{2k} \leq n - \frac{k+1}{2} - \frac{1}{2k} < n - \frac{k+1}{2}$. □

Lemma 7. *If n is odd and M is an edge set of the complete graph K_n such that $0 \leq |M| \leq \frac{n-1}{2}$, then $G = K_n \setminus M$ contains $\frac{n-1}{2}$ edge-disjoint spanning trees.*

Proof. Let $\mathscr{P} = \bigcup_{i=1}^{p} V_i$ be a partition of $V(G)$ with $|V_i| = n_i$ $(1 \leq i \leq p)$, and $\mathcal{E}_p$ be the set of edges between distinct blocks of $\mathscr{P}$ in G. The case $p = 1$ is trivial, thus we assume $p \geq 2$. Then $|\mathcal{E}_p| \geq \binom{n}{2} - \sum_{i=1}^{p} \binom{n_i}{2} - |M| \geq \binom{n}{2} - \sum_{i=1}^{p} \binom{n_i}{2} - \frac{n-1}{2}$. We will show that $\binom{n}{2} - \sum_{i=1}^{p} \binom{n_i}{2} - \frac{n-1}{2} \geq \frac{n-1}{2}(p-1)$, that is, $(n-p)\frac{n-1}{2} \geq \sum_{i=1}^{p} \binom{n_i}{2}$. We only need to prove that $(n-p)\frac{n-1}{2} \geq max\{\sum_{i=1}^{p} \binom{n_i}{2}\}$. Since $f(n_1, n_2, \cdots, n_p) = \sum_{i=1}^{p} \binom{n_i}{2}$ obtains its maximum value when $n_1 = n_2 = \cdots = n_{p-1} = 1$ and $n_p = n - p + 1$, we need to show the inequality $(n-p)\frac{n-1}{2} \geq \binom{1}{2}(p-1) + \binom{n-p+1}{2}$, that is $(n-p)\frac{p-2}{2} \geq 0$. It is easy to see that the inequality holds. Thus, $|\mathcal{E}_p| \geq \binom{n}{2} - \sum_{i=1}^{p} \binom{n_i}{2} - |M| \geq \frac{n-1}{2}(p-1)$. From Theorem 1, we know that there exist $\frac{n-1}{2}$ edge-disjoint spanning trees (Note that we can use the result of Theorem 1, although Nash-Williams and Tutte considered multigraphs but here we are concerned with the generalized connectivity and generalized edge-connectivity for simple graphs). □

Theorem 4. *Let G be a connected graph of order n and k be an integer such that $3 \leq k \leq n$. Then $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil$ if and only if $G = K_n$ for k even; $G = K_n \setminus M$ for k odd, where M is an edge set such that $0 \leq |M| \leq \frac{k-1}{2}$.*

Proof. First we consider the case that k is even. From Theorem 2, we have $\kappa_k(K_n) = n - \frac{k}{2}$. Actually, the complete graph K_n is the unique graph with this property. We only need to show that $\kappa_k(K_n \setminus e) < n - \frac{k}{2}$ for any $e \in E(K_n)$. From Lemma 5 and Observation 1, we know that $\kappa_k(K_n \setminus e) \leq \lambda_k(K_n \setminus e) < n - \frac{k}{2}$ for $e \in E(K_n)$. Thus, the result holds for k even.

Next we consider the case that k is odd.

Necessity: Let G be a graph of order n such that $\kappa_k(G) = n - \frac{k+1}{2}$. Since G is connected, we can consider G as a graph obtained by deleting some edges from the complete graph K_n. If $G = K_n \setminus M$ such that $|M| \geq \frac{k+1}{2}$, then $\kappa_k(K_n \setminus M) \leq \lambda_k(K_n \setminus M) < n - \frac{k+1}{2}$ by Observation 1 and Lemma 6, a contradiction. Thus, $G = K_n \setminus M$, where $0 \leq |M| \leq \frac{k-1}{2}$.

Sufficiency: We will show that $\kappa_k(G) \geq n - \frac{k+1}{2}$ if $G = K_n \setminus M$ such that $0 \leq |M| \leq \frac{k-1}{2}$. It suffices to prove that $\kappa_k(G) \geq n - \frac{k+1}{2}$ for $|M| = \frac{k-1}{2}$.

Let $S = \{u_1, u_2, \cdots, u_k\} \subseteq V(G)$ and $\bar{S} = \{w_1, w_2, \cdots, w_{n-k}\}$. We have the following two cases to consider:

Case 1. $M \subseteq E(K_n[S]) \cup E(K_n[\bar{S}])$.

Let $M' = M \cap E(K_n[S])$ and $M'' = M \cap E(K_n[\bar{S}])$. Then $|M'| + |M''| = |M| = \frac{k-1}{2}$ and $0 \leq |M'|, |M''| \leq \frac{k-1}{2}$. We can consider $G[S]$ as a graph obtained by deleting $|M'|$ edges from the complete graph K_k. From Lemma 7, there exist $\frac{k-1}{2}$ edge-disjoint spanning trees in $G[S]$. Actually, these $\frac{k-1}{2}$ edge-disjoint trees are all trees connecting S in $G[S]$. All these trees together with the trees $T_i = w_iu_1 \cup w_iu_2 \cup \cdots \cup w_iu_k$ $(1 \leq i \leq n-k)$ form $n - \frac{k+1}{2}$ internally disjoint trees connecting S, namely, $\kappa(S) \geq n - \frac{k+1}{2}$ (Note that the trees connecting S can be edge-disjoint in $G[S]$, but must be internally disjoint in $G \setminus S$).

Case 2. $M \nsubseteq E(K_n[S]) \cup E(K_n[\bar{S}])$.

In this case, there exist some edges of M in $E_{K_n}[S, \bar{S}]$. Let $M' = M \cap E(K_n[S])$ and $M'' = M \cap E(K_n[\bar{S}])$, and let $|M'| = m_1$ and $|M''| = m_2$. Clearly, $0 \leq m_i \leq \frac{k-3}{2}$ $(i = 1, 2)$.

For $w_i \in \bar{S}$, we let $|E_{K_n[M]}[w_i, S]| = x_i$, where $1 \leq i \leq n-k$. Without loss of generality, let $x_1 \geq x_2 \geq \cdots \geq x_{n-k}$. Thus $\sum_{i=1}^{n-k} x_i + m_1 + m_2 = \frac{k-1}{2}$ and $|E_G[w_i, S]| = k - x_i$.

Our basic idea is to seek for some edges in $G[S]$, and let them together with the edges of $E_G[S, \bar{S}]$ form $n-k$ internally disjoint trees connecting S.

For $w_1 \in \bar{S}$, without loss of generality, let $S_1 = \{u_1, u_2, \cdots, u_{x_1}\}$ such that $u_jw_1 \in M$ $(1 \leq j \leq x_1)$ and $S_2 = S \setminus S_1 = \{u_{x_1+1}, u_{x_1+2}, \cdots, u_k\}$. Clearly, $S = S_1 \cup S_2$ and $u_jw_1 \in E(G)$ $(x_1 + 1 \leq j \leq k)$, namely, $S_2 = N_G(w_1) \cap S$. One can see that the tree $T_1' = w_1u_{x_1+1} \cup w_1u_{x_1+2} \cup \cdots \cup w_1u_k$ is a Steiner tree connecting S_2. Our idea is to seek for x_1 edges in $E_G[S_1, S_2]$ and add them to T_1' to form a Steiner tree connecting S. For each $u_j \in S_1$ $(1 \leq j \leq x_1)$, we claim that $|E_G[u_j, S_2]| \geq 1$. Otherwise, let $|E_G[u_j, S_2]| = 0$. Then $|E_{K_n[M]}[u_j, S_2]| = k - x_1$ and $|M| \geq |E_{K_n[M]}[u_j, S_2]| + d_{K_n[M]}(w_1) \geq (k - x_1) + x_1 = k$, which contradicts to $|M| = \frac{k-1}{2}$. Since $|E_G[u_j, S_2]| \geq 1$ for each u_j $(1 \leq j \leq x_1)$, we can find a vertex u_r $(x_1 + 1 \leq r \leq k)$ such that $e_{1j} = u_ju_r \in E(G[S])$. Let $M_1 = \{e_{11}, e_{12}, \cdots, e_{1x_1}\}$ and $G_1 = G \setminus M_1$. Thus the tree $T_1 = w_1u_{x_1+1} \cup w_1u_{x_1+2} \cup \cdots \cup w_1u_k \cup e_{11} \cup e_{12} \cup \cdots \cup e_{1x_1}$ is our desired one.

For $w_2 \in \bar{S}$, without loss of generality, let $S_1 = \{u_1, u_2, \cdots, u_{x_2}\}$ such that $u_jw_2 \in M$ $(1 \leq j \leq x_2)$ and $S_2 = S \setminus S_1 = \{u_{x_2+1}, u_{x_2+2}, \cdots, u_k\}$. Clearly, $S = S_1 \cup S_2$ and $u_jw_2 \in E(G)$ $(x_2 +$

$1 \le j \le k$), namely, $S_2 = N_G(w_2) \cap S$. One can see that the tree $T_2' = w_2u_{x_2+1} \cup w_2u_{x_2+2} \cup \cdots \cup w_2u_k$ is a Steiner tree connecting S_2. Our idea is to seek for x_2 edges in $E_{G_1}[S_1, S_2]$ and add them to T_2' to form a Steiner tree connecting S. For each $u_j \in S_1$ $(1 \le j \le x_2)$, we claim that $|E_{G_1}[u_j, S_2]| \ge 1$. Otherwise, we let $|E_{G_1}[u_j, S_2]| = 0$. For $e \notin E_{G_1}[u_j, S_2]$, $e \in M$ or $e \in M_1 = \{e_{11}, e_{12}, \cdots, e_{1x_1}\}$. Then $|E_{K_n[M]}[u_j, S_2]| \ge k - x_2 - x_1$ and $|M| \ge |E_{K_n[M]}[u_j, S_2]| + d_{K_n[M]}(w_1) + d_{K_n[M]}(w_2) \ge (k - x_2 - x_1) + x_1 + x_2 = k$, which contradicts to $|M| = \frac{k-1}{2}$. Since $|E_{G_1}[u_j, S_2]| \ge 1$ for each u_j $(1 \le j \le x_2)$, we can find a vertex u_r $(x_2 + 1 \le r \le k)$ such that $e_{2j} = u_ju_r \in E(G_1[S])$. Let $M_2 = \{e_{21}, e_{22}, \cdots, e_{2x_2}\}$ and $G_2 = G_1 \setminus M_2$. Thus the tree $T_2 = w_2u_{x_2+1} \cup w_2u_{x_2+2} \cup \cdots \cup w_2u_k \cup e_{21} \cup e_{22} \cup \cdots \cup e_{2x_2}$ is our desired tree. Clearly, T_2 and T_1 are two edge-disjoint trees connecting S.

For $w_i \in \bar{S}$ $(3 \le i \le n-k)$, without loss of generality, let $S_1 = \{u_1, u_2, \cdots, u_{x_i}\}$ such that $u_jw_i \in M$ $(1 \le j \le x_i)$ and $S_2 = S \setminus S_1 = \{u_{x_i+1}, u_{x_i+2}, \cdots, u_k\}$. Clearly, $S = S_1 \cup S_2$ and $w_iu_j \in E(G)$ $(x_i + 1 \le j \le k)$, namely, $S_2 = N_G(w_i) \cap S$. One can see the tree $T_i' = w_iu_{x_i+1} \cup w_iu_{x_i+2} \cup \cdots \cup w_iu_k$ is a Steiner tree connecting S_2. Our idea is to seek for x_i edges in $E_{G_{i-1}}[S_1, S_2]$ and add them to T_i' to form a Steiner tree connecting S. For each $u_j \in S_1$ $(1 \le j \le x_i)$, we claim that $|E_{G_{i-1}}[u_j, S_2]| \ge 1$. Otherwise, let $|E_{G_{i-1}}[u_j, S_2]| = 0$. For $e \notin E_{G_{i-1}}[u_j, S_2]$, we have that $e \in M$ or $e \in \bigcup_{r=1}^{i-1} M_r$. Then $|E_{K_n[M]}[u_j, S_2]| \ge k - x_i - \sum_r^{i-1} x_r$ and $|M| \ge |E_{K_n[M]}[u_j, S_2]| + \sum_r^i d_{K_n[M]}(w_r) \ge (k - \sum_r^i x_r) + \sum_r^i x_r = k$, which contradicts to $|M| = \frac{k-1}{2}$. Since $|E_{G_{i-1}}[u_j, S_2]| \ge 1$ for each u_j $(1 \le j \le x_i)$, we can find a vertex u_r $(x_i + 1 \le r \le k)$ such that $e_{ij} = u_ju_r \in E(G_{i-1}[S])$. Let $M_i = \{e_{i1}, e_{i2}, \cdots, e_{ix_i}\}$ and $G_i = G_{i-1} \setminus M_i$. Thus the tree $T_i = w_iu_{x_i+1} \cup w_iu_{x_i+2} \cup \cdots \cup w_iu_k \cup e_{i1} \cup e_{i2} \cup \cdots \cup e_{ix_i}$ is our desired one (Note that if $x_i = 0$ then we do not need to search for some edges of $E(G_{i-1}[S])$ and $T_i = w_iu_1 \cup w_iu_2 \cup \cdots \cup w_iu_k$ is our desired tree). Clearly, T_i and T_j $(1 \le j \le i-1)$ are two edge-disjoint trees connecting S.

We continue this procedure until we find out $n-k$ trees connecting S, say $T_1, T_2, \cdots, T_{n-k}$. Now we terminate this procedure. Clearly, we can consider $G_{n-k}[S] = G[S] \setminus \bigcup_{i=1}^{n-k} M_i$ as a graph obtained by deleting $|M'| + \sum_{i=1}^{n-k} |M_i|$ edges from the complete graph K_k. Since $\sum_{i=1}^{n-k} x_i + m_1 + m_2 = \frac{k-1}{2}$, we have $1 \le \sum_{i=1}^{n-k} |M_i| + m_1 \le \frac{k-1}{2}$. From Lemma 7, there exist $\frac{k-1}{2}$ edge-disjoint trees connecting S in $G[S]$ (Note that these trees can be edge-disjoint by the definition of generalized k-connectivity). These trees together with $T_1, T_2, \cdots, T_{n-k}$ form $n - \frac{k+1}{2}$ internally disjoint trees connecting S, namely, $\kappa(S) \ge n - \frac{k+1}{2}$.

From the above discussion, we get that $\kappa(S) \ge n - \frac{k+1}{2}$ for $S \subseteq V(G)$, which implies that $\kappa_k(G) \ge n - \frac{k+1}{2}$. From this together with Proposition 1, we have $\kappa_k(G) = n - \frac{k+1}{2}$. □

Theorem 5. *For a connected graph G of order n and $n \ge k \ge 3$, $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$ if and only if $G = K_n$ for k even; $G = K_n \setminus M$ for k odd, where M is an edge set such that* $0 \le |M| \le \frac{k-1}{2}$.

Proof. First we consider the case that k is even. From Proposition 2 and Lemma 5, we have that $\lambda_k(K_n) = n - \frac{k}{2}$ if and only if $G = K_n$.

Next we consider the case that k is odd. If $G = K_n \setminus M$ $(0 \le |M| \le \frac{k-1}{2})$, then $\lambda_k(G) \ge \kappa_k(G) = n - \frac{k+1}{2}$ by Observation 1 and Theorem 4. From this together with Proposition 2, we know that $\lambda_k(G) = n - \frac{k+1}{2}$. Conversely, assume that $\lambda_k(G) = n - \frac{k+1}{2}$. Since G is connected, we can consider G as a graph obtained by deleting some edges from the complete graph K_n. If $G = K_n \setminus M$ such that $|M| \geq \frac{k+1}{2}$, then $\lambda_k(G) < n - \frac{k+1}{2}$ by Lemma 6, a contradiction. So $G = K_n \setminus M$, where $0 \leq |M| \leq \frac{k-1}{2}$. □

Remark 1. The graphs with $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil$ or $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$ has been characterized by Theorems 4 and 5. A natural question is, for the lower bounds, whether we can characterize the graphs with $\kappa_k(G) = 1$ or $\lambda_k(G) = 1$. It seems not easy to solve such a problem. Note that the minimal graphs with $\kappa_k(G) = 1$ or $\lambda_k(G) = 1$ are the trees of order n. So, an interesting problem could be what is the maximal graphs with $\kappa_k(G) = 1$ or $\lambda_k(G) = 1$? Actually, one can check that a connected graph G obtained from the complete graph K_{n-1} by attaching a pendant edge is a such graph, which is obviously a unique maximum such graph. However, to characterize all the maximal graphs is left unsolved. Here maximal (minimal) means that adding (deleting) any edge with destroy $\kappa_k(G) = 1$ or $\lambda_k(G) = 1$, whereas maximum means a such graph that has the largest number of edges.

3.3 Nordhaus-Gaddum-type results

Alavi and Mitchem in [1] considered the Nordhaus-Gaddum-type results for the connectivity and edge-connectivity. We are concerned with analogous inequalities involving the generalized k-connectivity.

Theorem 6. *For any graph G of order n, we have*

(1) $1 \leq \kappa_k(G) + \kappa_k(\overline{G}) \leq n - \lceil k/2 \rceil$;

(2) $0 \leq \kappa_k(G) \cdot \kappa_k(\overline{G}) \leq [\frac{n-\lceil k/2 \rceil}{2}]^2$.

Moreover, the upper and lower bounds are tight.

Proof. (1) To avoid confusion, we denote the generalized local connectivity of a k-subset S in a graph G by $\kappa(G; S)$. Since $G \cup \overline{G} = K_n$, for any k-subset S we have $\kappa(G; S) + \kappa(\overline{G}; S) \leq \kappa(K_n; S)$. Suppose that $\kappa_k(K_n) = \kappa(K_n; S_0)$ for some k-subset S_0. Then we have $\kappa_k(K_n) = \kappa(K_n; S_0) \geq \kappa(G; S_0) + \kappa(\overline{G}; S_0) \geq \kappa_k(G) + \kappa_k(\overline{G})$. This together with $\kappa_k(K_n) = n - \lceil \frac{k}{2} \rceil$ results in $\kappa_k(G) + \kappa_k(\overline{G}) \leq n - \lceil \frac{k}{2} \rceil$. If $\kappa_k(G) + \kappa_k(\overline{G}) = 0$, then $\kappa_k(G) = \kappa_k(\overline{G}) = 0$. Thus G and $\overline{G}$ are all disconnected, which is impossible. So $\kappa_k(G) + \kappa_k(\overline{G}) \geq 1$.

(2) It follows immediately from (1). □

To see that the lower bound of (1) is tight, it suffices to take G as the complete bipartite graph $K_{1,n-1}$ since $\kappa_k(K_{1,n-1}) + \kappa_k(\overline{K_{1,n-1}}) = 1 + 0 = 1$.

The following observation indicates the graphs attaining the lower bound of (2).

Observation 3. $\kappa_k(G) \cdot \kappa_k(\overline{G}) = 0$ *if and only if G or $\overline{G}$ is disconnected.*

We construct a graph class to show that the two upper bounds of are tight for $k = n$.

Example 3. Let n, r be two positive integers such that $n = 4r + 1$. From Lemma 3, we know that $\kappa_n(K_{2r,2r+1}) = \lambda_n(K_{2r,2r+1}) = r$. Let $\mathcal{E}$ be the set of the edges of these r spanning trees in $K_{2r,2r+1}$. Then there exist $2r(2r+1) - 4r^2 = 2r$ remaining edges in $K_{2r,2r+1}$ except the edges in $\mathcal{E}$. Let M be the set of these $2r$ edges. Set $G = K_{2r,2r+1} \setminus M$. Then $\kappa_n(G) = r$, $M \subseteq E(\overline{G})$ and $\overline{G}$ is a graph obtained from two cliques K_{2r} and K_{2r+1} by adding $2r$ edges in M between them, that is, one end of each edge belongs to K_{2r} and the other belongs to K_{2r+1}. Note that $E(\overline{G}) = E(K_{2r}) \cup M \cup E(K_{2r+1})$. Now we show that $\kappa_n(\overline{G}) \geq r$. As we know, K_{2r} contains r Hamiltonian paths, say $P_1, P_2, \cdots, P_r$, and so does K_{2r+1}, say $P'_1, P'_2, \cdots, P'_r$. Pick up r edges from M, say $e_1, e_2, \cdots, e_r$, let $T_i = P_i \cup P'_i \cup e_i (1 \leq i \leq r)$. Then $T_1, T_2, \cdots, T_r$ are r spanning trees in $\overline{G}$, namely, $\kappa_n(\overline{G}) \geq r$. Since $|E(\overline{G})| = \binom{2r}{2} + \binom{2r+1}{2} + 2r = 4r^2 + 2r$ and each spanning tree uses $4r$ edges, these edges can form at most $\lfloor \frac{4r^2+2r}{4r} \rfloor = r$ spanning trees, that is, $\kappa_n(\overline{G}) \leq r$. So $\kappa_n(\overline{G}) = r$. Clearly, $\kappa_n(G) + \kappa_n(\overline{G}) = 2r = \frac{n-1}{2} = n - \lceil \frac{n}{2} \rceil$ and $\kappa_n(G) \cdot \kappa_n(\overline{G}) = r^2 = \lceil \frac{n - \lceil n/2 \rceil}{2} \rceil^2$.

Remark 2. The above example only shows that the upper bound of (2) in Theorem 6 is tight for the case $k = n$. A natural question is to find examples showing that the upper bounds of Theorem 6 are tight for each k with $3 \leq k < n$. Note that the complete graph $G = K_n$ can attain the upper bound of (1), but clearly $\overline{G}$ is disconnected. Therefore, when we require that both G and $\overline{G}$ are connected, is there a graph which can attain the upper bounds of Theorem 6 respectively or simultaneously for each k with $3 \leq k \leq n$?

4 Results for $\lambda_3(G)$ and $\kappa_3(G)$

4.1 Upper and lower bounds for $\lambda_3(G)$

From now on, we focus our attention on the generalized 3-edge-connectivity. From Proposition 2, we obtained tight upper and lower bounds of $\lambda_3(G)$, that is, $1 \leq \lambda_3(G) \leq n-2$. Now we give another tight upper and lower bounds of $\lambda_3(G)$ by the edge-connectivity, that is, $\frac{3\lambda-2}{4} \leq \lambda_3(G) \leq \lambda$, which will be used in planar graph and line graph. At first we give a tight upper bound for $\lambda_k(G)$.

Proposition 3. *For any graph G of order n, $\lambda_k(G) \leq \lambda(G)$. Moreover, the upper bound is tight.*

Proof. Let M be a $\lambda(G)$-edge-cut of G, where $1 \leq \lambda(G) \leq n-1$. Then $G \setminus M$ has exactly two components. We can choose $S = \{v_1, v_2, \cdots, v_k\}$ so that $S \subseteq V(G)$ and at least two of the k vertices are in different components. Thus any tree connecting S must contain an edge in M. By the definition of $\lambda(S)$, we get $\lambda(S) \leq |M|$. So $\lambda_k(G) \leq \lambda(S) \leq |M| = \lambda(G)$.

Furthermore, we will show that the graph $G = K_k \vee (n-k)K_1$ $(n \geq 3k)$ satisfies that $\kappa_k(G) = \lambda_k(G) = \kappa(G) = \lambda(G) = \delta(G) = k$ (see Figure 1).

Let $W = \{w_1, w_2, \cdots, w_k\}$, $U = K_k \setminus W = \{u_1, u_2, \cdots, u_{n-k}\}$, and S be a k-subset of vertices of G. Without loss of generality, let $|S \cap V(U)| = s$ $(s \leq k)$. Then $|S \cap V(W)| = k - s$. Without loss of generality, let $u_i \in S$ $(1 \leq i \leq s)$ and $w_j \in S$ $(1 \leq j \leq k-s)$. Then the trees $T_i = w_i u_1 \cup w_i u_2 \cup \cdots \cup w_i u_s \cup u_{k+i} w_1 \cup u_{k+i} w_2 \cup \cdots \cup u_{k+i} w_{k-s} (i = 1, 2, \cdots, k-s)$ and $T_j = w_j u_1 \cup w_j u_2 \cup \cdots \cup w_j u_s \cup w_j w_1 \cup w_j w_2 \cup \cdots \cup w_j w_{k-s}$ $(j = k-s+1, k-s+2, \cdots, k)$ form k pairwise edge-disjoint trees connecting S, namely $\lambda(S) \geq k$. Combining this with $\lambda_k(G) \leq$

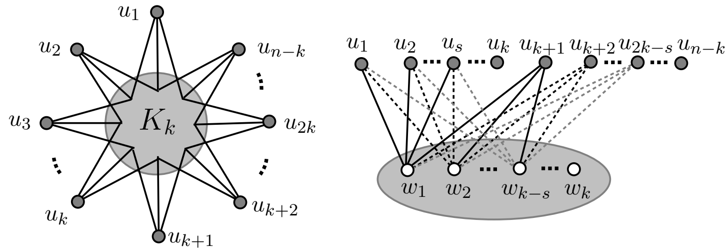

Figure 1: Graph G with $\kappa_k(G) = \lambda_k(G) = \kappa(G) = \lambda(G) = \delta(G) = k$.

$\lambda(G) = k$, we get $\lambda_k(G) = k$. Since the above k trees are also internally disjoint trees connecting S, we have $\kappa_k(G) = k$. So $\kappa_k(G) = \lambda_k(G) = \kappa(G) = \lambda(G) = \delta(G) = k$. Clearly, the upper bound of Proposition 3 is tight. $\square$

Next we give a tight lower bound for $\lambda_3(G)$.

Proposition 4. *Let G be a connected graph with n vertices. For every two integers s and r with $s \geq 0$ and $r \in \{0, 1, 2, 3\}$, if $\lambda(G) = 4s + r$, then $\lambda_3(G) \geq 3s + \lceil \frac{r}{2} \rceil$. Moreover, the lower bound is tight. We simply write $\lambda_3(G) \geq \frac{3\lambda - 2}{4}$.*

Proof. Let $\lambda = \lfloor \frac{8t+3}{6} \rfloor$. From Lemma 1, we have $\lambda_3(G) \geq t$ (Note that we can use the result of Lemma 1, although Kriesell [11] considered graphs containing multiple edges but here we are concerned with the generalized edge-connectivity for simple graphs).

If $\lambda = 4s$, since $\frac{8t+3}{6}$ is not an integer, then $4s < \frac{8t+3}{6}$. Thus $\lambda_3(G) \geq t > 3s - \frac{3}{8}$, which implies $\lambda_3(G) \geq 3s$. With a similar method, we can obtain that $\lambda_3(G) \geq 3s + 1$ if $\lambda = 4s + 1$, and $\lambda_3(G) \geq 3s + 2$ if $\lambda = 4s + 3$.

Note that there exists no integer t such that $4s + 2 = \lfloor \frac{8t+3}{6} \rfloor$ if $\lambda = 4s + 2$. But a graph G with $\lambda(G) = 4s + 2$ is also $(4s + 1)$-edge-connected, and so we have $\lambda_3(G) \geq 3s + 1$.

$$\lambda_3(G) \geq \begin{cases} 3s & if\ \lambda = 4s, \\ 3s + 2 & if\ \lambda = 4s + 3, \\ 3s + 1 & if\ \lambda = 4s + 1\ or\ \lambda = 4s + 2. \end{cases}$$

So the result holds. Simply, we write $\lambda_3(G) \geq \frac{3\lambda - 2}{4}$.

Now we give graphs attaining the lower bound.

For $\lambda = 4s$ with $s \geq 1$, we construct a graph G as follows (see Figure 2 (a)): Let $P = X_1 \cup X_2$ and $Q = Y_1 \cup Y_2$ be two cliques with $|X_1| = |Y_1| = 2s$ and $|X_2| = |Y_2| = 2s$. Let u, v be adjacent to every vertex in P, Q, respectively, and w be adjacent to every vertex in X_1 and Y_1. Finally, we finish the construction of the graph G by adding a perfect matching between X_2 and Y_2. It can be easily checked that $\lambda = 4s$.

We consider the case $S = \{u, v, w\}$. There exist two kinds of edge-disjoint trees connecting S (see Figure 2 (b)): the tree of Type I is a path u-v_1-w-v_2-v; the tree of Type II is T_1 or T_2, where $T_1 = uv_5 \cup v_3v_5 \cup wv_3 \cup v_5v_7 \cup v_7v$ and $T_2 = uv_6 \cup v_6v_8 \cup v_8v_4 \cup v_4w \cup v_8v$, respectively. We denote the numbers of trees of Type I and Type II by x and y, respectively. Note that

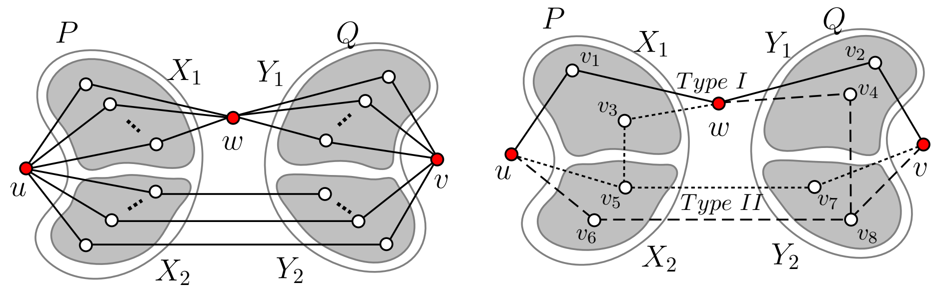

Figure 2 (a): The graph with $\lambda(G) = 4s$ and $\lambda_3(G) = 3s$.
Figure 2 (b): Two types of trees connecting $\{u, v, w\}$.

$|E_G[w, X_1 \cup Y_1]| = 4s$ and each tree of Type I uses two edges of $E_G[w, X_1 \cup Y_1]$, we have $x \leq 2s$. Although each tree of Type II uses one edge of $E_G[w, X_1 \cup Y_1]$, we have $y \leq 2s$ since each tree of Type II uses one edge of $E_G[X_2, Y_2]$ and $|E_G[X_2, Y_2]| = 2s$. Combining these with $2x + y \leq 4s$, we can derive the optimal solution $x = s$ and $y = 2s$ by solving the following integer linear programming:

$$\begin{cases} Maximize: \ x + y \\ Subject\ to: \ x \leq 2s, \ y \leq 2s, \ 2x + y \leq 4s, \\ and \qquad\qquad x, y \geq 0. \end{cases}$$

Thus $\lambda(S) \geq 3s$. We can check that for any other three vertices of G the number of edge-disjoint trees connecting them is not less than $3s$. So $\lambda_3(G) = 3s$ and the graph G attaining the lower bound.

For $\lambda = 4s + 1$, let $|X_1| = |Y_1| = 2s + 1$ and $|X_2| = |Y_2| = 2s$; for $\lambda = 4s + 2$, let $|X_1| = |Y_1| = 2s + 1$ and $|X_2| = |Y_2| = 2s + 1$; for $\lambda = 4s + 3$, let $|X_1| = |Y_1| = 2s + 2$ and $|X_2| = |Y_2| = 2s + 1$, where $s \geq 1$. Similarly, we can check that $\lambda_3(G) = 3s + 1$ for $\lambda = 4s + 1$; $\lambda_3(G) = 3s + 1$ for $\lambda = 4s + 2$; $\lambda_3(G) = 3s + 2$ for $\lambda = 4s + 3$.

For the case $s = 0$, we have $G = P_n$ such that $\lambda(G) = \lambda_3(G) = 1$; $G = C_n$ such that $\lambda(G) = 2$ and $\lambda_3(G) = 1$; $G = H_t$ such that $\lambda(G) = 3$ and $\lambda_3(G) = 2$, where H_t denotes the graph obtained from t copies of K_4 by identifying a vertex from each of them in the way shown in Figure 3.

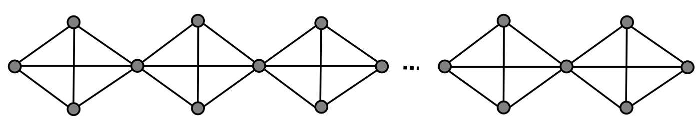

Figure 3: $\lambda(H_t) = 3, \lambda_3(H_t) = 2$.

□

As we know, every planar graph G has a vertex of degree at most 5, i.e., $\delta(G) \leq 5$. Since $\lambda(G) \leq \delta$, we only need to consider a planar graph G with edge-connectivity $\lambda(G)$ at most 5. From Proposition 4, it can be deduced that for any graph (not necessarily planar) if $\lambda(G) = 1$,

$\lambda_3(G)=1$; if $\lambda(G)=2$, $\lambda_3(G)\geq 1$; if $\lambda(G)=3$, $\lambda_3(G)\geq 2$; if $\lambda(G)=4$, $\lambda_3(G)\geq 3$; and if $\lambda(G)=5$, $\lambda_3(G)\geq 4$. Therefore, the following corollary is obvious.

Corollary 2. *If G is a connected planar graph, then $\lambda(G)-1\leq \lambda_3(G)\leq \lambda(G)$.*

4.2 Results for line graphs

This section investigate the relation between the generalized 3-connectivity and generalized 3-edge-connectivity of a graph and its line graph.

Proposition 5. *If G is a connected graph, then*

(1) $\lambda_3(G)\leq \kappa_3(L(G))$.

(2) $\lambda_3(L(G))\geq \frac{3}{2}\lambda_3(G)-2$.

(3) $\kappa_3(L(L(G))\geq \frac{3}{2}\kappa_3(G)-2$.

Proof. For (1), let e_1,e_2,e_3 be three arbitrary distinct vertices of the line graph of G such that $\lambda_3(G)=t$ with $t\geq 1$. Let $e_1=v_1v_1'$, $e_2=v_2v_2'$ and $e_3=v_3v_3'$ be those edges of G corresponding to the vertices e_1,e_2,e_3 in $L(G)$, respectively.

Consider three distinct vertices of the six end-vertices of e_1,e_2,e_3. Without loss of generality, let $S=\{v_1,v_2,v_3\}$ be three distinct vertices. Since $\lambda_3(G)=t$, there exist t edge-disjoint trees $T_1,T_2,\cdots,T_t$ connecting S in G. We define a minimal tree T connecting S as a tree connecting S whose subtree obtained by deleting any edge of T does not connect S.

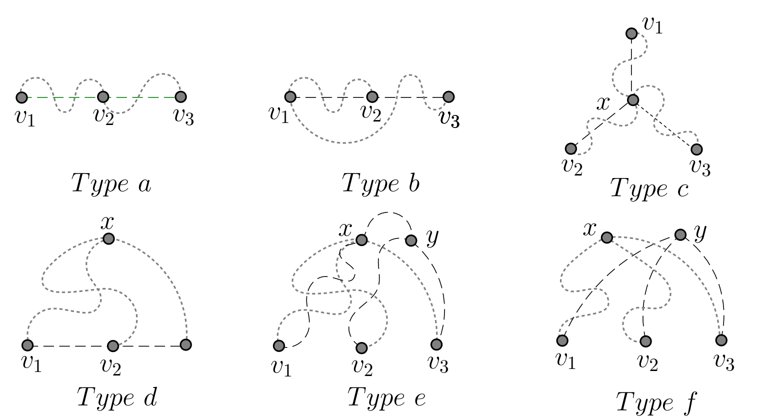

Figure 4: Six possible types of $T_i\cup T_j$.

Choosing any two edge-disjoint minimal trees T_i and T_j $(1\leq i,j\leq t)$ connecting S in G, we will show that the trees T_i' and T_j' corresponding to T_i and T_j in $L(G)$ are internally disjoint trees. It is easy to see that $T_i\cup T_j$ has six possible types, as shown in Figure 4. Since T_i and T_j are edge-disjoint in G, we can find internally disjoint trees T_i' and T_j' connecting e_1,e_2,e_3 in $L(G)$. We give an example of Type c, see Figure 5. So $\kappa_3(L(G))\geq t$ and we know that the result holds.

For (2), from Propositions 3 and 4 and (2) of Lemma 2 we have that $\lambda_3(L(G))\geq \frac{3}{4}\lambda(L(G))-\frac{1}{2}\geq \frac{3}{4}(2\lambda(G)-2)-\frac{1}{2}=\frac{3}{2}\lambda(G)-2\geq \frac{3}{2}\lambda_3(G)-2$.

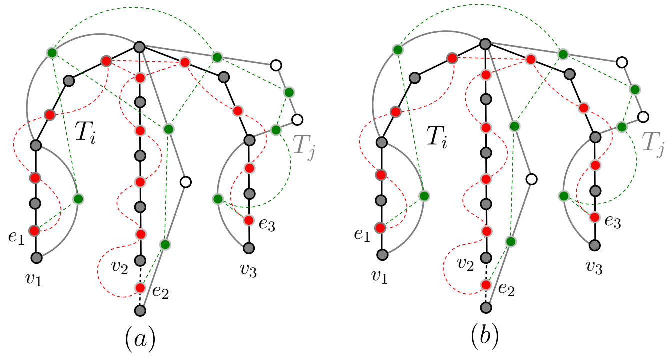

Figure 5 (a): An example for T_i and T_j connecting S and their line graphs.
Figure 5 (b): An example for T_i' and T_j' corresponding to T_i and T_j.

For (3), from (1) and (2) of this proposition and Observation 1 we have that $\kappa_3(L(L(G))) \geq \lambda_3(L(G)) \geq \frac{3}{2}\lambda_3(G) - 2 \geq \frac{3}{2}\kappa_3(G) - 2$.

One can check that (1) of this proposition is tight since $G = C_n$ can attain this bound. $\square$

Let $L^0(G) = G$ and $L^1(G) = L(G)$. Then for $k \geq 2$, the *k-th* iterated line graph $L^k(G)$ is defined by $L(L^{k-1}(G))$. The next statement follows immediately from Proposition 5 and a routine application of recursions.

Corollary 3. *$\lambda_3(L^k(G)) \geq (\frac{3}{2})^k(\kappa_3(G) - 4) + 4$, and $\kappa_3(L^k(G)) \geq (\frac{3}{2})^{\lfloor \frac{k}{2} \rfloor}(\kappa_3(G) - 4) + 4$.*

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