ARITHMETIC PROPERTIES OF ℓ-REGULAR PARTITIONS

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ABSTRACT. For a given prime p, by studying p-dissection identities for Ramanujan's theta functions $\psi(q)$ and $f(-q)$, we derive infinite families of congruences modulo 2 for some ℓ-regular partition functions, where $\ell = 2, 4, 5, 8, 13, 16$.

1. INTRODUCTION

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. If ℓ is a positive integer, then a partition is called an ℓ-regular partition if there is no part divisible by ℓ. The generating function for the number of ℓ-regular partitions of n, denoted by $b_\ell(n)$, is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}.$$

Recently, arithmetic properties of ℓ-regular partition functions have received a great deal of attention. Andrews et al. [1, Theorem 3.5] showed that for $\alpha \geq 0$ and $n \geq 0$,

$$ped\left(3^{2\alpha+1}n + \frac{17\cdot 3^{2\alpha}-1}{8}\right) \equiv 0 \pmod 2, \tag{1.1}$$

$$ped\left(3^{2\alpha+2}n + \frac{11\cdot 3^{2\alpha+1}-1}{8}\right) \equiv 0 \pmod 2, \tag{1.2}$$

$$ped\left(3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{8}\right) \equiv 0 \pmod 2, \tag{1.3}$$

where $ped(n)$ denotes the number of 4-regular partitions of n. With the aid of the theory of Hecke eigenforms, Chen [4] obtained some more generalized congruences modulo 2 for $b_4(n)$. Calkin et al. [3, Theorem 3] established that

$$b_5(20n+5) \equiv b_5(20n+13) \equiv 0 \pmod 2. \tag{1.4}$$

Later, Hirschhorn and Sellers [9, Theorem 2.5] derived that

$$b_5\left(4p^2m + 4u(pr-7)+1\right) \equiv 0 \pmod 2, \tag{1.5}$$

where p is any prime greater than 3 such that -10 is a quadratic nonresidue modulo p, u is the reciprocal of 24 modulo p^2, and $r \not\equiv 0 \pmod p$. Webb [14] proved the following

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conjecture posed in [3]

$$b_{13}\left(3^{\ell}n+\frac{5\cdot 3^{\ell-1}-1}{2}\right)\equiv 0 \pmod 3, \quad \ell\geq 2. \tag{1.6}$$

Using a modification of the method in [14], Furcy and Penniston [5] provided some infinite families of congruences modulo 3.

In this paper, we follow the standard q-series notation [6]

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k) \quad \text{and} \quad (a_1,a_2,\ldots,a_m;q)_\infty=\prod_{j=1}^{m}(a_j;q)_\infty, \quad |q|<1.$$

Let $f(a,b)$ be Ramanujan's general theta function given by

$$f(a,b):=\sum_{n=-\infty}^{\infty}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}}, \quad |ab|<1.$$

Jacobi's triple product identity can be stated in Ramanujan's notation as follows

$$f(a,b)=(-a,-b,ab;ab)_\infty. \tag{1.7}$$

Thus,

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

and

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}}=(q;q)_\infty.$$

The motivation for this paper is an observation that some generating functions of l-regular partitions are congruent to functions related to the Ramanujan theta functions $\psi(q)$ and $f(-q)$ modulo 2. In view of dissection identities for $\psi(q)$ and $f(-q)$, we obtain many new infinite families of congruences for $b_\ell(n)$, where $\ell=2,4,5,8,13,16$. In particular, we generalize (1.1)-(1.5) and Chen's congruences [4].

2. Preliminaries

In this section, we study p-dissection identities for $\psi(q)$ and $f(-q)$.

Theorem 2.1. *For any odd prime p,*

$$\psi(q)=\sum_{k=0}^{\frac{p-3}{2}}q^{\frac{k^2+k}{2}}f\left(q^{\frac{p^2+(2k+1)p}{2}},q^{\frac{p^2-(2k+1)p}{2}}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2}).$$

Furthermore, we claim that for $0\leq k\leq (p-3)/2$,

$$\frac{k^2+k}{2}\not\equiv\frac{p^2-1}{8} \pmod p.$$

Proof. For any odd prime p, we have

$$\begin{aligned}
\psi(q) &= \frac{1}{2}\sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} \\
&= \frac{1}{2}\sum_{k=0}^{p-1}\sum_{n=-\infty}^{\infty} q^{\frac{(pn+k)(pn+k+1)}{2}} \\
&= \frac{1}{2}\sum_{k=0}^{\frac{p-3}{2}}\sum_{n=-\infty}^{\infty} q^{\frac{p^2n^2+(2k+1)pn+k^2+k}{2}} + \frac{1}{2}q^{\frac{p^2-1}{8}}\sum_{n=-\infty}^{\infty} q^{p^2\frac{n(n+1)}{2}} \\
&\quad + \frac{1}{2}\sum_{k=\frac{p+1}{2}}^{p-1}\sum_{n=-\infty}^{\infty} q^{\frac{p^2n^2+(2k+1)pn+k^2+k}{2}} \\
&= \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}}\sum_{n=-\infty}^{\infty} q^{\frac{p^2n^2+(2k+1)pn}{2}} + q^{\frac{p^2-1}{8}}\psi(q^{p^2}) \\
&= \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}).
\end{aligned}$$

For any integer $0 \le k \le (p-1)/2$, if

$$\frac{k^2+k}{2} \equiv \frac{p^2-1}{8} \pmod{p},$$

namely, $(2k+1)^2 \equiv 0 \pmod{p}$, then it implies that $k = (p-1)/2$. □

Before we state the next theorem, we define that for any prime $p \ge 5$,

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & p \equiv 1 \pmod{6}, \\ \frac{-p-1}{6}, & p \equiv -1 \pmod{6}. \end{cases}$$

Theorem 2.2. *For any prime $p \ge 5$,*

$$f(-q) = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}).$$

Further, we claim that for $-(p-1)/2 \le k \le (p-1)/2$ and $k \neq (\pm p-1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

Proof. We show that

$$\begin{aligned}
f(-q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} \qquad (2.1) \\
&= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}}\sum_{n=-\infty}^{\infty} (-1)^{pn+k} q^{\frac{(pn+k)(3pn+3k+1)}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3p^2n^2+(6k+1)pn}{2}} \\
&= \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3p^2n^2+(6k+1)pn}{2}} \\
&\qquad\qquad + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} \sum_{n=-\infty}^{\infty} (-1)^n q^{p^2\frac{n(3n+1)}{2}} \\
&= \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}).
\end{aligned}
$$

To obtain (2.1), we use Euler's pentagonal number theorem [2, Corollary 1.3.5]. For any integer $-(p-1)/2 \le k \le (p-1)/2$, if

$$\frac{3k^2+k}{2} \equiv \frac{p^2-1}{24} \pmod{p},$$

then we get $(6k+1)^2 \equiv 0 \pmod{p}$, which yields $k = (\pm p - 1)/6$. We complete the proof. □

In particular, we consider a special case of Theorem 2.2 by setting $p = 5$. In light of the quintuple product identity [2, Theorem 1.3.19]

$$\frac{f(-x^2, -\lambda x) f(-\lambda x^3)}{f(-x, -\lambda x^2)} = f(-\lambda^2 x^3, -\lambda x^6) + x f(-\lambda, -\lambda^2 x^9),$$

we can easily derive the identity given by Ramanujan [12, p. 212]

$$(q;q)_\infty = \frac{(q^{10}, q^{15}, q^{25}; q^{25})_\infty}{(q^5, q^{20}; q^{25})_\infty} - q(q^{25};q^{25})_\infty - q^2 \frac{(q^5, q^{20}, q^{25}; q^{25})_\infty}{(q^{10}, q^{15}; q^{25})_\infty}. \tag{2.2}$$

Recently, Hirschhorn [8] gave a simple proof of the above identity by using Jacobi's triple product identity (1.7).

3. Arithmetic of ℓ-regular partition functions

In this section, with the aid of Theorem 2.1 and Theorem 2.2, we study infinite families of congruences for ℓ-regular partition functions, where $\ell = 2, 4, 5, 8, 13, 16$.

3.1. **2-regular partitions.** Arithmetic of 2-regular partitions which are usually called distinct partitions was widely studied in the literature, see, for example [7, 10, 11]. Combining the fact

$$\sum_{n=0}^{\infty} b_2(n) q^n = (-q;q)_\infty \equiv f(-q) \pmod{2} \tag{3.1}$$

with Theorem 2.2, we find the following lemma.

Lemma 3.1. *For any prime $p \geq 5$, $\alpha \geq 0$, and $n \geq 0$,*

$$\sum_{n=0}^{\infty} b_2\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{24}\right)q^n \equiv f(-q) \pmod 2.$$

Proof. We prove the lemma by induction on α. Note that (3.1) is the case when $\alpha = 0$. Suppose that the lemma holds for α. Then using Theorem 2.2, we get

$$\begin{aligned}\sum_{n=0}^{\infty} b_2\left(p^{2\alpha}\left(pn + \frac{p^2-1}{24}\right) + \frac{p^{2\alpha}-1}{24}\right)q^n &= \sum_{n=0}^{\infty} b_2\left(p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{24}\right)q^n \\ &\equiv f(-q^p) \pmod 2. \end{aligned} \tag{3.2}$$

This implies that

$$\begin{aligned}\sum_{n=0}^{\infty} b_2\left(p^{2\alpha}\left(p^2n + \frac{p^2-1}{24}\right) + \frac{p^{2\alpha}-1}{24}\right)q^n &= \sum_{n=0}^{\infty} b_2\left(p^{2\alpha+2}n + \frac{p^{2\alpha+2}-1}{24}\right)q^n \\ &\equiv f(-q) \pmod 2.\end{aligned}$$

Therefore, the lemma holds for $\alpha + 1$. This completes the proof. □

Theorem 3.2. *For any prime $p \geq 5$, $\alpha \geq 1$, and $n \geq 0$,*

$$b_2\left(p^{2\alpha}n + \frac{(24i+p)p^{2\alpha-1}-1}{24}\right) \equiv 0 \pmod 2, \quad i = 1, 2, \ldots, p-1.$$

Proof. Applying (3.2) yields that for $i = 1, 2, \ldots, p-1$,

$$b_2\left(p^{2\alpha+1}(pn+i) + \frac{p^{2\alpha+2}-1}{24}\right) \equiv 0 \pmod 2.$$

□

For convenience, we introduce the Legendre symbol

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod p, \\ -1, & \text{if } a \text{ is a quadratic non-residue modulo } p, \\ 0, & \text{if } a \equiv 0 \pmod p. \end{cases}$$

Theorem 3.3. *For any prime $p \geq 5$, $\alpha \geq 0$, and $n \geq 0$,*

$$b_2\left(p^{2\alpha+1}n + \frac{(24j+1)p^{2\alpha}-1}{24}\right) \equiv 0 \pmod 2,$$

where j is an integer with $0 \leq j \leq p-1$ such that $\left(\frac{24j+1}{p}\right) = -1$.

Proof. According to Theorem 2.2 and Lemma 3.1, if we have

$$j \not\equiv \frac{3k^2+k}{2} \pmod p$$

for $|k| \le (p-1)/2$, namely, $\left(\frac{24j+1}{p}\right) = -1$, then it can be shown that

$$b_2\left(p^{2\alpha}(pn+j) + \frac{p^{2\alpha}-1}{24}\right) \equiv 0 \pmod 2.$$

□

In addition, we obtain the following generalizations.

Lemma 3.4. *For primes $p_1, p_2, \ldots, p_r \ge 5$, $r \ge 0$, and $n \ge 0$,*

$$\sum_{n=0}^{\infty} b_2\left(\prod_{s=1}^{r} p_s^2 n + \frac{\prod_{s=1}^{r} p_s^2 - 1}{24}\right) q^n \equiv f(-q) \pmod 2.$$

By convention, we set $\prod_{s=1}^{0} p_s^2 = 1$.

Proof. We prove it by induction on r. First, we derive the case $r = 0$ from Lemma 3.1. Then suppose that the lemma holds for r. Based on Theorem 2.2, we obtain that for a given prime $p_{r+1} \ge 5$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} b_2\left(\prod_{s=1}^{r} p_s^2 \left(p_{r+1}^2 n + \frac{p_{r+1}^2 - 1}{24}\right) + \frac{\prod_{s=1}^{r} p_s^2 - 1}{24}\right) q^n \\
&= \sum_{n=0}^{\infty} b_2\left(\prod_{s=1}^{r+1} p_s^2 n + \frac{\prod_{s=1}^{r+1} p_s^2 - 1}{24}\right) q^n \\
&\equiv f(-q) \pmod 2.
\end{aligned}$$

Therefore, the lemma holds for $r+1$. This completes the proof. □

Theorem 3.5. *For primes $p_1, p_2, \ldots, p_r \ge 5$, $r \ge 1$, and $n \ge 0$, we have*

$$b_2\left(\prod_{s=1}^{r} p_s^2 n + \frac{(24i + p_r)\prod_{s=1}^{r-1} p_s^2 p_r - 1}{24}\right) \equiv 0 \pmod 2, \tag{3.3}$$

where $i = 1, 2, \ldots, p_r - 1$. We also have

$$b_2\left(\prod_{s=1}^{r-1} p_s^2 p_r n + \frac{(24j+1)\prod_{s=1}^{r-1} p_s^2 - 1}{24}\right) \equiv 0 \pmod 2, \tag{3.4}$$

where j is an integer with $0 \le j \le p_r - 1$ such that $\left(\frac{24j+1}{p_r}\right) = -1$.

Proof. From Theorem 2.2 and Lemma 3.4, it follows that

$$\sum_{n=0}^{\infty} b_2\left(\prod_{s=1}^{r-1} p_s^2 \left(p_r n + \frac{p_r^2 - 1}{24}\right) + \frac{\prod_{s=1}^{r-1} p_s^2 - 1}{24}\right) q^n \equiv f(-q^{p_r}) \pmod 2.$$

Therefore, we deduce that for $i = 1, 2, \ldots, p_r - 1$,

$$b_2\left(\prod_{s=1}^{r-1} p_s^2 \left(p_r(p_r n + i) + \frac{p_r^2 - 1}{24}\right) + \frac{\prod_{s=1}^{r-1} p_s^2 - 1}{24}\right) \equiv 0 \pmod 2,$$

which implies (3.3). The proof of (3.4) is analogous to that of Theorem 3.3, and hence is omitted. □

3.2. **4-regular partitions and 13-regular partitions.** In light of Theorem 2.1, we find the following results.

Lemma 3.6. *For any odd prime p, $\alpha \geq 0$, and $n \geq 0$,*

$$\sum_{n=0}^{\infty} b_4\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{8}\right)q^n \equiv \psi(q) \pmod{2}.$$

Proof. We see that

$$\begin{aligned}\sum_{n=0}^{\infty} b_4(n)q^n &\equiv (q;q)_\infty^3 \pmod{2} \\ &= \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}} \qquad (3.5) \\ &\equiv \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \pmod{2} \\ &= \psi(q).\end{aligned}$$

To obtain (3.5), we use Jacobi's identity [2, Theorem 1.3.9]. Invoking Theorem 2.1, it can be seen that

$$\sum_{n=0}^{\infty} b_4\left(pn + \frac{p^2-1}{8}\right)q^n \equiv \psi(q^p) \pmod{2}.$$

Then we have

$$\sum_{n=0}^{\infty} b_4\left(p^2n + \frac{p^2-1}{8}\right)q^n \equiv \psi(q) \pmod{2}.$$

Therefore,

$$b_4(n) \equiv b_4\left(p^2n + \frac{p^2-1}{8}\right) \pmod{2}.$$

By induction on α, we conclude the lemma based on the above relation. □

Theorem 3.7. *For any odd prime p, $\alpha \geq 1$, and $n \geq 0$,*

$$b_4\left(p^{2\alpha}n + \frac{(8i+p)p^{2\alpha-1}-1}{8}\right) \equiv 0 \pmod{2}, \quad i = 1, 2, \ldots, p-1.$$

Proof. The combination of Theorem 2.1 and Lemma 3.6 gives that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} b_4\left(p^{2\alpha}\left(pn + \frac{p^2-1}{8}\right) + \frac{p^{2\alpha}-1}{8}\right)q^n \equiv \psi(q^p) \pmod{2}.$$

Thus, we see that for $i = 1, 2, \ldots, p-1$,

$$b_4\left(p^{2\alpha+1}(pn+i) + \frac{p^{2\alpha+2}-1}{8}\right) \equiv 0 \pmod{2}.$$

□

Theorem 3.8. *For any odd prime p, $\alpha \geq 0$, and $n \geq 0$,*

$$b_4\left(p^{2\alpha+1}n + \frac{(8j+1)p^{2\alpha}-1}{8}\right) \equiv 0 \pmod 2,$$

where j is an integer with $0 \leq j \leq p-1$ such that $\left(\frac{8j+1}{p}\right) = -1$.

Proof. By the expansion of $\psi(q)$ in Theorem 2.1 and Lemma 3.6, we consider that for $0 \leq k \leq (p-1)/2$,

$$j \not\equiv \frac{k^2+k}{2} \pmod p,$$

that is, $\left(\frac{8j+1}{p}\right) = -1$. Then it follows that

$$b_4\left(p^{2\alpha}(pn+j) + \frac{p^{2\alpha}-1}{8}\right) \equiv 0 \pmod 2.$$

□

Notice that (1.1)-(1.3) are special cases of Theorem 3.7 and Theorem 3.8. In addition, we derive the following results which are generalizations of congruences given by Chen [4].

Theorem 3.9. *For odd primes $p_1, p_2, \ldots, p_r$, $r \geq 1$, and $n \geq 0$, we have*

$$b_4\left(\prod_{s=1}^{r} p_s^2 n + \frac{(8i+p_r)\prod_{s=1}^{r-1} p_s^2 p_r - 1}{8}\right) \equiv 0 \pmod 2,$$

where $i = 1, 2, \ldots, p_r - 1$. We also have

$$b_4\left(\prod_{s=1}^{r-1} p_s^2 p_r n + \frac{(8j+1)\prod_{s=1}^{r-1} p_s^2 - 1}{8}\right) \equiv 0 \pmod 2,$$

where j is an integer with $0 \leq j \leq p_r - 1$ such that $\left(\frac{8j+1}{p_r}\right) = -1$.

Since the proof is analogous to that of Theorem 3.5, we omit it.

In the following, we can also obtain the above kind of congruences for different primes. However, in this paper, we focus on illustrating a type of congruences for the same prime.

Calkin et al. [3, Theorem 2] showed that

$$\sum_{n=0}^{\infty} b_{13}(2n)q^n \equiv (q^2;q^2)_\infty^3 + q^3(q^{26};q^{26})_\infty^3 \pmod 2.$$

This implies that

$$\sum_{n=0}^{\infty} b_{13}(2(2n))q^n = \sum_{n=0}^{\infty} b_{13}(4n)q^n \equiv (q;q)_\infty^3 \equiv \sum_{n=0}^{\infty} b_4(n)q^n \pmod 2. \tag{3.6}$$

Therefore, applying Theorem 3.7 and Theorem 3.8 to (3.6), we get the following corollary.

Corollary 3.10. *For any odd prime p, $\alpha \geq 0$, and $n \geq 0$, we have*

$$b_{13}\left(4p^{2\alpha+2}n + \frac{(8i+p)p^{2\alpha+1}-1}{2}\right) \equiv 0 \pmod 2,$$

where $i = 1, 2, \ldots, p-1$. We also have

$$b_{13}\left(4p^{2\alpha+1}n + \frac{(8j+1)p^{2\alpha}-1}{2}\right) \equiv 0 \pmod 2,$$

where j is an integer with $0 \leq j \leq p-1$ such that $\left(\frac{8j+1}{p}\right) = -1$.

Substituting (1.1) and (1.2) into (3.6) yields

$$b_{13}\left(4\cdot 3^{2\alpha+1}n + \frac{17\cdot 3^{2\alpha}-1}{2}\right) \equiv b_{13}\left(4\cdot 3^{2\alpha}n + \frac{11\cdot 3^{2\alpha-1}-1}{2}\right) \equiv 0 \pmod 2.$$

Combining the above congruences with (1.6), we obtain the following congruences.

Corollary 3.11. *For $\alpha \geq 1$ and $n \geq 0$,*

$$b_{13}\left(4\cdot 3^{2\alpha+1}n + \frac{17\cdot 3^{2\alpha}-1}{2}\right) \equiv b_{13}\left(4\cdot 3^{2\alpha}n + \frac{11\cdot 3^{2\alpha-1}-1}{2}\right) \equiv 0 \pmod 6.$$

3.3. **5-regular partitions.** Hirschhorn and Sellers [9] found that

$$\sum_{n=0}^{\infty} b_5(2n)q^n \equiv f(-q^2) \pmod 2 \tag{3.7}$$

and

$$\sum_{n=0}^{\infty} b_5(2n+1)q^n \equiv f(-q^{10})\sum_{n=0}^{\infty} b_5(n)q^n \pmod 2. \tag{3.8}$$

With the aid of the above results, we deduce many new congruences for $b_5(n)$.

Lemma 3.12. *For any prime $p \geq 5$, $\left(\frac{-10}{p}\right) = -1$, $\alpha \geq 0$, and $n \geq 0$,*

$$\sum_{n=0}^{\infty} b_5\left(4p^{2\alpha}n + \frac{7p^{2\alpha}-1}{6}\right)q^n \equiv f(-q^2)f(-q^5) \pmod 2.$$

Proof. From (3.7) and (3.8), it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} b_5(2(2n)+1)q^n &= \sum_{n=0}^{\infty} b_5(4n+1)q^n \\ &\equiv f(-q^5)\sum_{n=0}^{\infty} b_5(2n)q^n \pmod 2 \\ &\equiv f(-q^2)f(-q^5) \pmod 2. \end{aligned} \tag{3.9}$$

For a prime $p \geq 5$ and $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider

$$2\cdot\frac{3k^2+k}{2} + 5\cdot\frac{3m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod p,$$

namely,

$$(12k+2)^2+10(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-10}{p}\right)=-1$, we get $k=m=(\pm p-1)/6$. Therefore, using Theorem 2.2, we have

$$\sum_{n=0}^{\infty} b_5\left(4\left(p^2n+\frac{7p^2-7}{24}\right)+1\right)q^n \equiv f(-q^2)f(-q^5) \pmod{2}. \tag{3.10}$$

The combination of (3.9) and (3.10) gives

$$b_5(4n+1) \equiv b_5\left(4p^2n+\frac{7p^2-1}{6}\right) \pmod{2}.$$

According to the above relation, we obtain the lemma by induction on α. $\square$

Theorem 3.13. *For any prime $p \geq 5$, $\left(\frac{-10}{p}\right)=-1$, $\alpha \geq 1$, and $n \geq 0$,*

$$b_5\left(4p^{2\alpha}n+\frac{(24i+7p)p^{2\alpha-1}-1}{6}\right) \equiv 0 \pmod{2}, \quad i=1,2,\ldots,p-1.$$

Proof. Applying Theorem 2.2 and Lemma 3.12 yields

$$\begin{aligned}\sum_{n=0}^{\infty} b_5\left(4p^{2\alpha}\left(pn+\frac{7p^2-7}{24}\right)+\frac{7p^{2\alpha}-1}{6}\right)q^n &= \sum_{n=0}^{\infty} b_5\left(4p^{2\alpha+1}n+\frac{7p^{2\alpha+2}-1}{6}\right)q^n \\ &\equiv f(-q^{2p})f(-q^{5p}) \pmod{2}.\end{aligned}$$

Therefore, we deduce that for $i=1,2,\ldots,p-1$,

$$b_5\left(4p^{2\alpha+1}(pn+i)+\frac{7p^{2\alpha+2}-1}{6}\right) \equiv 0 \pmod{2}.$$

$\square$

Note that the congruence (1.5) can be derived from Theorem 3.13.

Lemma 3.14. *For any prime $p \geq 5$, $\left(\frac{-10}{p}\right)=-1$, and a given integer $0 \leq j \leq p-1$, there exist integers $0 \leq k, m \leq p-1$ such that*

$$2\cdot\frac{3k^2+k}{2}+5\cdot\frac{3m^2+m}{2} \equiv j \pmod{p}. \tag{3.11}$$

Proof. If (3.11) holds, then

$$2(6k+1)^2+5(6m+1)^2 \equiv 24j+7 \pmod{p}.$$

To prove Lemma 3.14, it suffices to show that there exist integers x and y such that

$$2x^2+5y^2-a \equiv 0 \pmod{p},$$

where $0 \leq x, y \leq p-1$ and $a=24j+7$. If $(2x^2+5y^2-a, p)=1$ for all x and y, then according to the fact

$$\ell^{p-1} \equiv 1 \pmod{p}, \text{ if } (\ell, p)=1,$$

it follows that

$$\sum_{x,y=0}^{p-1}\left(1-(2x^2+5y^2-a)^{p-1}\right)\equiv 0 \pmod p.$$

Therefore, we need to prove

$$\sum_{x,y=0}^{p-1}\left(1-(2x^2+5y^2-a)^{p-1}\right)\not\equiv 0 \pmod p.$$

Since

$$\begin{aligned}&\sum_{x,y=0}^{p-1}\left(1-(2x^2+5y^2-a)^{p-1}\right)\\ &\equiv -\sum_{x=0}^{p-1}\sum_{y=0}^{p-1}\sum_{m=0}^{p-1}\binom{p-1}{m}(2x^2+5y^2)^m(-a)^{p-1-m} \pmod p\\ &= -\sum_{m=0}^{p-1}\binom{p-1}{m}(-a)^{p-1-m}\sum_{i=0}^{m}\binom{m}{i}2^i5^{m-i}\sum_{x=0}^{p-1}x^{2i}\sum_{y=0}^{p-1}y^{2m-2i}\end{aligned}$$

and

$$\sum_{i=0}^{p-1} i^k \equiv \begin{cases} -1 & \pmod p, \text{ if } p-1 \mid k, \\ 0 & \pmod p, \text{ if } p-1 \nmid k, \end{cases} \qquad k>0,$$

we deduce that

$$\sum_{x,y=0}^{p-1}\left(1-(2x^2+5y^2-a)^{p-1}\right)\equiv -\binom{p-1}{\frac{p-1}{2}}2^{\frac{p-1}{2}}5^{\frac{p-1}{2}}\not\equiv 0 \pmod p,$$

as required. □

In view of Lemma 3.14, there does not exist any result for $b_5(n)$ which is similar to Theorem 3.8.

In the following, we use Ramanujan's identity (2.2) to get some further infinite families of congruences for $b_5(n)$. For convenience, set

$$a(q)=\frac{(q^{10},q^{15};q^{25})_\infty}{(q^5,q^{20};q^{25})_\infty} \qquad \text{and} \qquad b(q)=\frac{(q^5,q^{20};q^{25})_\infty}{(q^{10},q^{15};q^{25})_\infty}=\frac{1}{a(q)}.$$

Then we rewrite (2.2) as

$$f(-q)=f(-q^{25})\left(a(q)-q-q^2b(q)\right). \tag{3.12}$$

Lemma 3.15. *For $\alpha\ge 0$ and $n\ge 0$,*

$$\sum_{n=0}^{\infty} b_5\left(4\cdot 5^{2\alpha}n+\frac{7\cdot 5^{2\alpha}-1}{6}\right)q^n\equiv f(-q^2)f(-q^5) \pmod 2.$$

Proof. We prove the lemma by induction on α. Note that (3.9) is the case when $\alpha = 0$. Suppose that the congruence holds for α. Utilizing (3.12), we get

$$\begin{aligned}\sum_{n=0}^{\infty} b_5\left(4\cdot 5^{2\alpha}n+\frac{7\cdot 5^{2\alpha}-1}{6}\right)q^n &\equiv f(-q^2)f(-q^5) \pmod 2\\ &= f(-q^5)f(-q^{50})\left(a(q^2)-q^2-q^4b(q^2)\right). \end{aligned} \tag{3.13}$$

This implies that

$$\begin{aligned}&\sum_{n=0}^{\infty} b_5\left(4\cdot 5^{2\alpha}(5n+2)+\frac{7\cdot 5^{2\alpha}-1}{6}\right)q^n\\ =&\sum_{n=0}^{\infty} b_5\left(4\cdot 5^{2\alpha+1}n+\frac{11\cdot 5^{2\alpha+1}-1}{6}\right)q^n\\ \equiv& f(-q)f(-q^{10}) \pmod 2\\ =& f(-q^{10})f(-q^{25})\left(a(q)-q-q^2b(q)\right). \end{aligned} \tag{3.14}$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} b_5\left(4\cdot 5^{2\alpha+1}(5n+1)+\frac{11\cdot 5^{2\alpha+1}-1}{6}\right)q^n &= \sum_{n=0}^{\infty} b_5\left(4\cdot 5^{2\alpha+2}n+\frac{7\cdot 5^{2\alpha+2}-1}{6}\right)q^n\\ &\equiv f(-q^2)f(-q^5) \pmod 2.\end{aligned}$$

So the congruence holds for $\alpha+1$. □

Theorem 3.16. *For $\alpha \geq 0$ and $n \geq 0$,*

$$b_5\left(4\cdot 5^{2\alpha+1}n+\frac{31\cdot 5^{2\alpha}-1}{6}\right)\equiv 0 \pmod 2, \tag{3.15}$$

$$b_5\left(4\cdot 5^{2\alpha+1}n+\frac{79\cdot 5^{2\alpha}-1}{6}\right)\equiv 0 \pmod 2,$$

$$b_5\left(4\cdot 5^{2\alpha+2}n+\frac{83\cdot 5^{2\alpha+1}-1}{6}\right)\equiv 0 \pmod 2,$$

$$b_5\left(4\cdot 5^{2\alpha+2}n+\frac{107\cdot 5^{2\alpha+1}-1}{6}\right)\equiv 0 \pmod 2.$$

Proof. Since there are no terms with powers of q congruent to $1, 3$ modulo 5 on the right-hand side of (3.13), we obtain the first two congruences. Similarly, the other two can be derived from (3.14). □

Notice that the congruences (1.4) given by Calkin et al. [3] are contained in the above theorem.

Corollary 3.17. *For $\alpha \geq 0$ and $n \geq 0$,*

$$b_5\left(4\cdot 5^{2\alpha+2}n+\frac{31\cdot 5^{2\alpha}-1}{6}\right)\equiv b_5\left(4\cdot 5^{2\alpha+2}n+\frac{79\cdot 5^{2\alpha}-1}{6}\right)\equiv 0 \pmod{10},$$

$$b_5\left(4\cdot 5^{2\alpha+3}n+\frac{83\cdot 5^{2\alpha+1}-1}{6}\right)\equiv b_5\left(4\cdot 5^{2\alpha+3}n+\frac{107\cdot 5^{2\alpha+1}-1}{6}\right)\equiv 0 \pmod{10}.$$

Proof. Using the Ramanujan congruence $p(5n+4) \equiv 0 \pmod 5$, it is easy to show that $b_5(5n+4) \equiv 0 \pmod 5$, and so we have

$$b_5\left(5\left(4\cdot 5^{2\alpha+1}n+\frac{31\cdot 5^{2\alpha-1}-5}{6}\right)+4\right) \equiv 0 \pmod 5.$$

According to (3.15), we also have

$$b_5\left(4\cdot 5^{2\alpha+1}(5n)+\frac{31\cdot 5^{2\alpha}-1}{6}\right) \equiv 0 \pmod 2.$$

Combining these, we obtain the first congruence. The others are proved in a similar way. □

3.4. 8-regular partitions and 16-regular partitions.

Lemma 3.18. *For any prime $p \equiv -1 \pmod 6$, $\alpha \geq 0$, and $n \geq 0$,*

$$\sum_{n=0}^{\infty} b_8\left(p^{2\alpha}n+\frac{7p^{2\alpha}-7}{24}\right)q^n \equiv \psi(q)f(-q^4) \pmod 2.$$

Proof. It can be shown that

$$\sum_{n=0}^{\infty} b_8(n)q^n \equiv \frac{(q;q)_\infty^8}{(q;q)_\infty} \equiv (q;q)_\infty^3(q^4;q^4)_\infty \equiv \psi(q)f(-q^4) \pmod 2. \tag{3.16}$$

Employing Theorem 2.1 and Theorem 2.2, we consider that for $0 \leq k \leq (p-1)/2$ and $-(p-1)/2 \leq m \leq (p-1)/2$,

$$\frac{k^2+k}{2}+4\cdot\frac{3m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod p,$$

namely,

$$3(2k+1)^2+(12m+2)^2 \equiv 0 \pmod p.$$

Since $p \equiv -1 \pmod 6$, it implies that

$$2k+1 \equiv 12m+2 \equiv 0 \pmod p,$$

that is, $k=(p-1)/2$ and $m=(-p-1)/6$. Therefore, we arrive at

$$\sum_{n=0}^{\infty} b_8\left(pn+\frac{7p^2-7}{24}\right)q^n \equiv \psi(q^p)f(-q^{4p}) \pmod 2. \tag{3.17}$$

Then

$$\sum_{n=0}^{\infty} b_8\left(p^2n+\frac{7p^2-7}{24}\right)q^n \equiv \psi(q)f(-q^4) \pmod 2.$$

From (3.16) together with the above relation, it follows that

$$b_8(n) \equiv b_8\left(p^2n+\frac{7p^2-7}{24}\right) \pmod 2.$$

By induction on α, we prove the lemma based on the above relation. □

Theorem 3.19. *For any prime $p \equiv -1 \pmod 6$, $\alpha \geq 1$, and $n \geq 0$,*

$$b_8\left(p^{2\alpha}n + \frac{(24i+7p)p^{2\alpha-1}-7}{24}\right) \equiv 0 \pmod 2, \quad i = 1, 2, \ldots, p-1.$$

Proof. Combining (3.17) with Lemma 3.18, we derive that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} b_8\left(p^{2\alpha+1}n + \frac{7p^{2\alpha+2}-7}{24}\right) q^n \equiv \psi(q^p) f(-q^{4p}) \pmod 2.$$

Therefore, it follows that

$$b_8\left(p^{2\alpha+1}(pn+i) + \frac{7p^{2\alpha+2}-7}{24}\right) \equiv 0 \pmod 2,$$

where $i = 1, 2, \ldots, p-1$. □

Notice that for any prime $p \equiv -1 \pmod 6$ and a given integer $0 \leq j \leq p-1$, there exist integers $0 \leq k, m \leq p-1$ such that

$$\frac{k^2+k}{2} + 4 \cdot \frac{3m^2+m}{2} \equiv j \pmod p.$$

This implies that there does not exist any result for $b_8(n)$ which is analogous to Theorem 3.8.

In the following, we deal with $b_{16}(n)$ in the same manner.

Lemma 3.20. *For any prime $p \equiv -1 \pmod 4$, $\alpha \geq 0$, and $n \geq 0$,*

$$\sum_{n=0}^{\infty} b_{16}\left(p^{2\alpha}n + \frac{5p^{2\alpha}-5}{8}\right) q^n \equiv \psi(q)\psi(q^4) \pmod 2.$$

Proof. We show that

$$\sum_{n=0}^{\infty} b_{16}(n) q^n \equiv \frac{(q;q)_\infty^{16}}{(q;q)_\infty} \equiv (q;q)_\infty^3 (q^4;q^4)_\infty^3 \equiv \psi(q)\psi(q^4) \pmod 2.$$

For $0 \leq k, m \leq (p-1)/2$, the congruence relation

$$\frac{k^2+k}{2} + 4 \cdot \frac{m^2+m}{2} \equiv \frac{5p^2-5}{8} \pmod p$$

holds if and only if $k = m = (p-1)/2$. This implies that

$$b_{16}(n) \equiv b_{16}\left(p^2 n + \frac{5p^2-5}{8}\right) \pmod 2.$$

Using the above relation, we arrive at the lemma by induction on α. □

Theorem 3.21. *For any prime $p \equiv -1 \pmod 4$, $\alpha \geq 1$, and $n \geq 0$,*

$$b_{16}\left(p^{2\alpha}n + \frac{(8i+5p)p^{2\alpha-1}-5}{8}\right) \equiv 0 \pmod 2, \quad i = 1, 2, \ldots, p-1.$$

4. Concluding remarks

Let $b'_p(n)$ denote the number of p-regular partitions with distinct parts of n. Sellers [13, Theorem 2.1] found the following congruence

$$b'(pn+r) \equiv 0 \pmod 2, \tag{4.1}$$

where $p \geq 3$ is a prime and r is an integer with $1 \leq r \leq p-1$ such that $\left(\frac{24r+1}{p}\right) = -1$. From Theorem 2.2 together with the following relation

$$\sum_{n=0}^{\infty} b'_p(n)q^n = \frac{(-q;q)_\infty}{(-q^p;q^p)_\infty} \equiv \frac{(q;q)_\infty}{(q^p;q^p)_\infty} \pmod 2,$$

it is easy to obtain (4.1). Moreover, we see that for a given prime $p \geq 5$,

$$\sum_{n=0}^{\infty} b'_p\left(pn + \frac{p^2-1}{24}\right) q^n \equiv \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \pmod 2,$$

which implies that

$$b'_p\left(pn + \frac{p^2-1}{24}\right) \equiv b_p(n) \pmod 2.$$

Therefore, one can study congruences modulo 2 for $b'_p(n)$ from those for $b_p(n)$.

In conclusion, we point out that the techniques used in this paper can be also applied to other kinds of partition functions, for example, broken k-diamond partition functions, k dots bracelet partition functions, and t-core partition functions.

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