On extremal graphs with at most two internally disjoint Steiner trees connecting any three vertices

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Abstract

The problem of determining the smallest number of edges, $h(n;\overline{\kappa} \geq r)$, which guarantees that any graph with n vertices and $h(n;\overline{\kappa} \geq r)$ edges will contain a pair of vertices joined by r internally disjoint paths was posed by Erdös and Gallai. Bollobás considered the problem of determining the largest number of edges $f(n;\overline{\kappa} \leq \ell)$ for graphs with n vertices and local connectivity at most ℓ. One can see that $f(n;\overline{\kappa} \leq \ell) = h(n;\overline{\kappa} \geq \ell+1) - 1$. These two problems had received a wide attention of many researchers in the last few decades. In the above problems, only pairs of vertices connected by internally disjoint paths are considered. In this paper, we study the number of internally disjoint Steiner trees connecting sets of vertices with cardinality at least 3.

Keywords: connectivity, Steiner tree, internally disjoint trees, generalized connectivity, generalized local connectivity.

AMS subject classification 2010: 05C40; 05C05; 05C35; 05C75.

1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to book [5] for graph theoretical notation and terminology not described here. We call the number of vertices in a graph as the *order* of the graph and the number of edges of it as its *size*. For two distinct vertices in a connected graph G, we can connect them by a path. Two paths are called *internally disjoint* if they have no common vertex except the end vertices. For any two distinct vertices x and y in G, the *local connectivity* $\kappa_G(x,y)$ is the maximum number of internally disjoint paths connecting x and y. Then $\min\{\kappa_G(x,y)|x,y \in V(G), x \neq y\}$ is usually the connectivity of G. In contrast to this parameter, $\overline{\kappa}(G) = \max\{\kappa_G(x,y)|x,y \in V(G), x \neq y\}$, introduced by Bollobás, is called the *maximum local connectivity* of G. The problem of determining the smallest number of edges, $h(n;\overline{\kappa} \geq r)$, which guarantees that any graph with n vertices and $h(n;\overline{\kappa} \geq r)$ edges will contain a pair of vertices joined by r internally disjoint paths was posed by Erdös and Gallai, see [1] for details.

Bollobás [2] considered the problem of determining the largest number of edges, $f(n;\overline{\kappa} \leq \ell)$, for graphs with n vertices and local connectivity at most ℓ, that is, $f(n;\overline{\kappa} \leq \ell) = \max\{e(G)||V(G)| = n \ and \ \overline{\kappa}(G) \leq \ell\}$. Motivated by determining the precise value of $f(n;\overline{\kappa} \leq \ell)$, this problem has obtained wide attention and many results have been worked out, see [2, 3, 4, 7, 8, 9, 16, 17, 18]. One can see that $h(n;\overline{\kappa} \geq \ell+1) = f(n;\overline{\kappa} \leq \ell) + 1$.

For $\overline{\kappa} \leq \ell$, it was showed that $f(n;\overline{\kappa} \leq \ell) \geq \lfloor \frac{\ell+1}{2}(n-1) \rfloor$. Since $f(n;\overline{\kappa} \leq \ell) = \lfloor \frac{\ell+1}{2}(n-1) \rfloor$ for $\ell = 2,3$, Bollobás and Erdös conjectured that the equality holds, but this was disproved by Leonard [7] for $\ell = 4$, and later Mader [16] constructed graphs disproving it for every $\ell \geq 4$.

For a graph $G(V,E)$ and a set $S \subseteq V$ of at least two vertices, *an S-Steiner tree* or *an Steiner tree connecting S* (or simply, *an S-tree*) is a such subgraph $T(V',E')$ of G that is a tree with $S \subseteq V'$. Two Steiner trees T and T' connecting S are said to be *internally disjoint* if $E(T) \cap E(T') = \varnothing$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa_G(S)$ is the maximum number of internally disjoint trees connecting S in G. The *generalized connectivity*, introduced by Chartrand et al. in 1984 [6], is defined as $\kappa_k(G) = min\{\kappa(S)|S \subseteq V(G), |S| = k\}$. There have been many results on the generalized connectivity, see [10, 11, 12, 13, 14]. Similar to the classical maximal local connectivity, we introduce another parameter $\overline{\kappa}_k(G) = max\{\kappa(S)|S \subseteq V(G), |S| = k\}$, which is called the *maximum generalized local connectivity* of G. It is easy to check that $0 \leq \overline{\kappa}_k(G) \leq \overline{\kappa}_k(K_n) \leq n - \lceil k/2 \rceil$ for a connected graph G.

In this paper, we mainly study the problem of determining the largest number of edges, $f(n;\overline{\kappa}_k \leq \ell)$, for graphs with n vertices and maximum generalized local connectivity at most ℓ, where $0 \leq \ell \leq n - \lceil k/2 \rceil$. That is, $f(n;\overline{\kappa}_k \leq \ell) = \max\{e(G)||V(G)| = n \ and \ \overline{\kappa}_k(G) \leq \ell\}$. We also study the smallest number of edges, $h(n;\overline{\kappa}_k \geq r)$, which guarantees that any graph with n vertices and $h(n;\overline{\kappa}_k \geq r)$ edges will contain a set S of k vertices such that there are r internally disjoint S-trees, where $0 \leq r \leq n - \lceil k/2 \rceil$. It is not difficult to see that $h(n;\overline{\kappa}_k \geq \ell+1) = f(n;\overline{\kappa}_k \leq \ell) + 1$ for $0 \leq \ell \leq n - \lceil k/2 \rceil$. For $k = 3$ and $\ell = 2$, we prove that $f(n;\overline{\kappa}_3 \leq 2) = 2n - 3$ for $n \geq 3$ and $n \neq 4$, and $f(n;\overline{\kappa}_3 \leq 2) = 2n - 2$ for $n = 4$. Furthermore, we characterize the graphs attaining these values. For $k = 3$ and a general ℓ, we construct some graphs to show that $f(n;\overline{\kappa}_3 \leq \ell) \geq \frac{\ell+2}{2}(n-2) + \frac{1}{2}$ for both n and ℓ odd, and $f(n;\overline{\kappa}_3 \leq \ell) \geq \frac{\ell+2}{2}(n-2) + 1$ otherwise.

2 Some basic results

As usual, the *union* of two graphs G and H is the graph, denoted by $G \cup H$, with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The disjoint union of k copies of the same graph G is denoted by kG. The *join* $G \vee H$ of two disjoint graphs G and H is obtained from $G \cup H$ by joining each vertex of G to every vertex of H.

In this section, we first introduce a graph operation and two graph classes.

Let H be a connected graph, and u a vertex of H. We define the *attaching operation at the vertex* u on H as follows: (1) identifying u and a vertex of a K_4; (2) u is attached with only one K_4. The vertex u is called *an attaching vertex*.

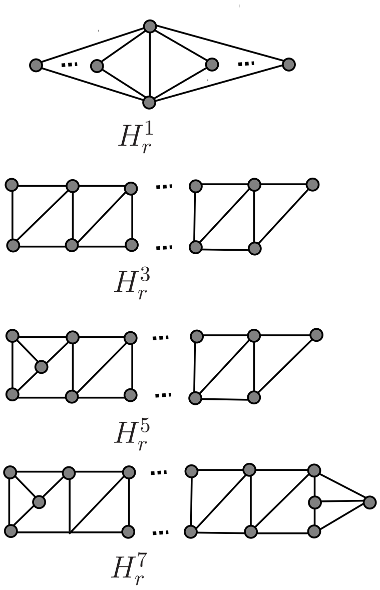

Figure 1. The graph class $\mathcal{G}_n$.

Now, we introduce two new graph classes. For $r \geq 3$, $\mathcal{G}_r = \{H_r^1, H_r^2, H_r^3, H_r^4, H_r^5, H_r^6, H_r^7\}$ is a class of graphs of order r (see Figure 1 for details). Let $\mathcal{H}_n^i$ $(1 \leq i \leq 7)$ be the class of graphs, each of them is obtained from a graph H_r^i by the attaching operation at some vertices of degree 2 on H_r^i, where $3 \leq r \leq n$ and $1 \leq i \leq 7$ (note that $H_n^i \in \mathcal{H}_n^i$). $\mathcal{G}_n^*$ is another class of graphs that contains $\mathcal{G}_n$, given as follows: $\mathcal{G}_3^* = \{K_3\}$, $\mathcal{G}_4^* = \{K_4, K_4 \setminus e\}$, $\mathcal{G}_5^* = \{G_1\} \cup (\bigcup_{i=1}^7 \mathcal{H}_5^i)$, $\mathcal{G}_6^* = \{G_3, G_4\} \cup (\bigcup_{i=1}^7 \mathcal{H}_6^i)$, $\mathcal{G}_7^* = \bigcup_{i=1}^7 \mathcal{H}_7^i$, $\mathcal{G}_8^* = \{G_2\} \cup (\bigcup_{i=1}^7 \mathcal{H}_8^i)$, $\mathcal{G}_n^* = \bigcup_{i=1}^7 \mathcal{H}_n^i$ for $n \geq 9$ (see Figure 2 for details).

It is easy to see that the following three observations hold.

Observation 1. *Let G and H be two connected graphs, and H' be a subdivision of H. If H' is a subgraph of G and $\overline{\kappa}_3(H) \geq 3$, then $\overline{\kappa}_3(G) \geq 3$.*

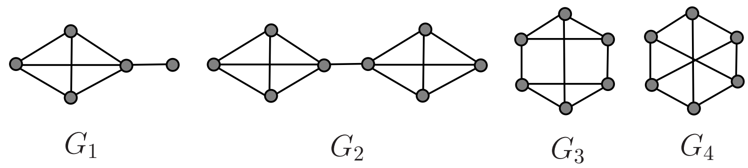

Figure 2. Some graphs in $\mathcal{G}_n^*$.

Observation 2. *Let H be a graph, u and v be two vertices in H, and G be a graph obtained from H by attaching a K_4 at u. If there are three internally disjoint paths between u and v in H, then $\overline{\kappa}_3(G) \geq 3$.*

Observation 3. *For each graph in Figure 3, $\overline{\kappa}_3 \geq 3$.*

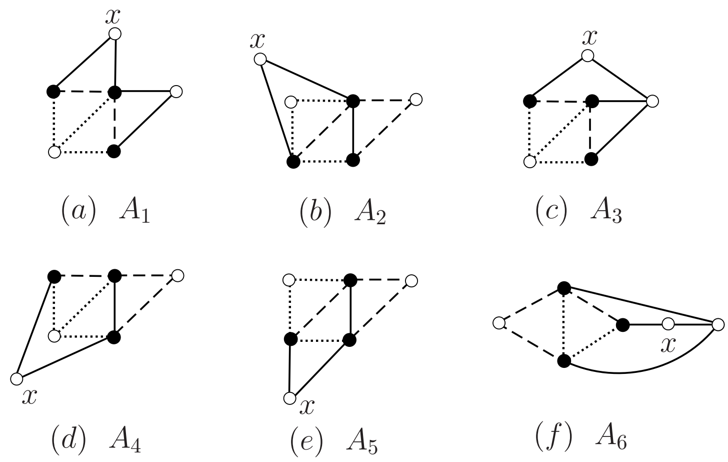

Figure 3. Graphs obtained from H_5^1 and H_5^3.

Lemma 1. *Let G be a graph containing a clique K_4. If there exists a path connecting two vertices of K_4 in $G \setminus E[K_4]$, then $\overline{\kappa}_3(G) \geq 3$.*

Proof. Let K_4 be a complete subgraph of G with vertex set $\{u_1, \cdots, u_4\}$, and P be a path connecting u_1 and u_2 in $G \setminus E[K_4]$. It suffices to show that there exists a set S such that $\kappa(S) \geq 3$. Choose $S = \{u_1, u_2, u_3\}$, clearly, $T_1 = u_1u_2 \cup u_1u_3$ and $T_2 = u_4u_1 \cup u_4u_2 \cup u_4u_3$ and $T_3 = P \cup u_2u_3$ form three internally disjoint S-trees. Thus, $\overline{\kappa}_3(G) \geq 3$. $\square$

Similarly, the following lemma holds.

Lemma 2. *Let G be a graph obtained from H_5^4 by adding a vertex x and two edges xy, xz, where $y, z \in V(H_5^4)$(see Figure 4). Then $\overline{\kappa}_3(G) \geq 3$ or $G = H_6^5$.*

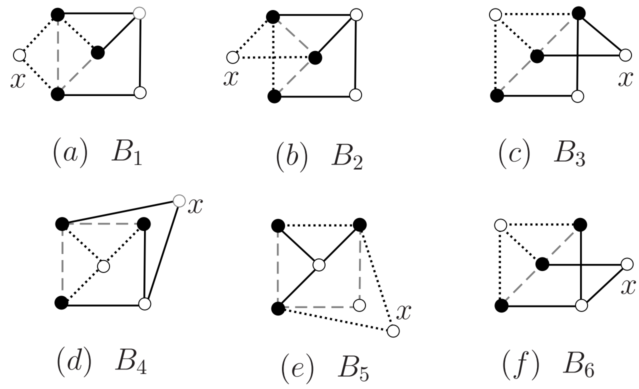

Figure 4. Graphs obtained from H_5^4 by adding a vertex of degree 2.

Lemma 3. *For any connected graph G with order 5 and size 8, $\overline{\kappa}_3(G) \geq 3$.*

Proof. We claim that $2 \leq \delta(G) \leq 3$. In fact, if $\delta(G) = 1$, without loss of generality, let $d(x) = 1$, then $|V(G-x)| = 4$ and $e(G-x) = 7$, a contradiction. If $\delta(G) \geq 4$, then $16 = 2e(G) \geq 5\delta \geq 20$, a contradiction.

If $\delta(G) = 2$, without loss of generality, let $d(x) = 2$, then $|V(G-x)| = 4$ and $e(G-x) = 6$, which implies that $G-x$ is a clique of order 4. From Lemma 1, $\overline{\kappa}_3(G) \geq 3$. So we suppose that $\delta(G) = 3$. Since $|V(G)| = 5$, $\Delta(G) \leq 4$. Since $\frac{2e(G)}{|V(G)|} = \frac{16}{5}$, there exists a vertex x in G such that $d(x) = 4$. Set $N_G(x) = \{u_1, u_2, u_3, u_4\}$. Since $\delta(G-x) \geq 2$ and $e(G-x) = 4$, $G-x$ is a cycle of order 4. Then G is a wheel of order 5 and the trees $T_1 = xu_2 \cup xu_4$ and $T_2 = u_3x \cup u_3u_2 \cup u_3u_4$ and $T_3 = u_1x \cup u_1u_4 \cup u_1u_2$ form 3 internally disjoint $\{x, u_2, u_4\}$-trees, namely, $\overline{\kappa}_3(G) \geq 3$. $\square$

Lemma 4. *For any connected graph G of order 5 and size 7, $\overline{\kappa}_3(G) \leq 2$ and $G \in \{G_1, H_5^1, H_5^3, H_5^4\}$.*

Proof. For each $S \subseteq V(G)$ with $|S| = 3$, a tree with two edges connecting S is called *Type I*, and the others with at least 3 edges are called *Type II*. One can see that three internally disjoint trees connecting S will use at least 8 edges since we only have one tree of Type I. So if G is a connected graph of order 5 and size 7, then $\overline{\kappa}_3(G) \leq 2$.

Suppose that $\delta(G) \geq 3$. Then $14 = 2e(G) \geq 5\delta \geq 15$, a contradiction. Thus, $\delta(G) \leq 2$. If $\delta(G) = 1$, without loss of generality, let $d(x) = 1$, then $|V(G-x)| = 4$ and $e(G-x) = 6$, which implies that $G-x$ is a clique of order 4. Then $G = G_1$ (see Figure 2).

If $\delta(G) = 2$, without loss of generality, let $d(x) = 2$, then $|V(G-x)| = 4$ and $e(G-x) = 5$, which implies that $G-x$ is a graph obtained from K_4 by deleting an edge. Thus, $G \in \{H_5^1, H_5^3, H_5^4\}$ (see Figure 1). $\square$

Lemma 5. *For any connected graph G with order 6 and size 10, $\overline{\kappa}_3(G) \geq 3$.*

Proof. If there exists a vertex $x \in V(G)$ such that $d(x) \leq 2$, then $|V(G-x)| = 5$ and $e(G-x) \geq 8$. From Lemma 3, $\overline{\kappa}_3(G-x) \geq 3$, which results in $\overline{\kappa}_3(G) \geq 3$.

Now we assume that $\delta(G) \geq 3$. If there exists a vertex $x \in V(G)$ such that $d(x) = 5$, then $|V(G-x)| = 5$ and $e(G-x) = 5$. Since $\delta(G-x) \geq 2$, $G-x$ is a cycle of order 5, which implies that G is a wheel of order 6. Clearly, $\overline{\kappa}_3(G) \geq 3$. So we can assume that $\Delta(G) \leq 4$. Let t be the number of vertices of degree 4 in G. Since $20 = 2e(G) = 4t + 3(6-t)$, $t = 2$, namely, there exist two vertices $x, y \in V(G)$ such that $d(x) = d(y) = 4$.

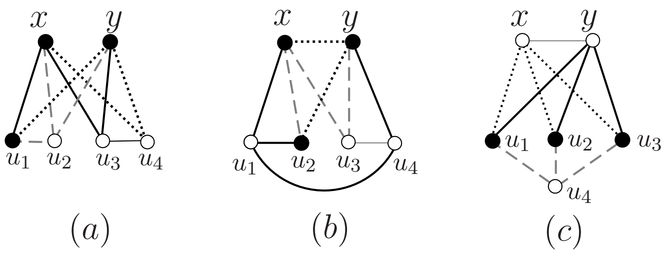

Figure 5. Graphs for Lemma 5.

If $xy \notin E(G)$, then G must be the graph shown in Figure 5 (a) since $\delta(G) \geq 3$. Then the trees $T_1 = u_2x \cup u_2y \cup u_2u_1$ and $T_2 = u_1x \cup xu_3 \cup u_3y$ and $T_3 = u_1y \cup yu_4 \cup u_4x$ form three $\{x, y, u_1\}$-trees, namely, $\overline{\kappa}_3(G) \geq 3$.

If $xy \in E(G)$ and $N_{G-xy}(x) \neq N_{G-xy}(y)$, then G must be the graph shown in Figure 5 (b) since $\delta(G) \geq 3$. Then the trees $T_1 = u_2x \cup xu_3 \cup u_3y$ and $T_2 = yx \cup yu_2$ and $T_3 = u_1x \cup u_1u_2 \cup u_1u_4 \cup u_4y$ form three $\{x, y, u_2\}$-trees, namely, $\overline{\kappa}_3(G) \geq 3$.

If $xy \in E(G)$ and $N_{G-xy}(x) = N_{G-xy}(y)$, then G must be the graph shown in Figure 5 (c) since $\delta(G) \geq 3$. Then the trees $T_1 = xu_1 \cup xu_2 \cup xu_3$ and $T_2 = yu_1 \cup yu_2 \cup yu_3$ and $T_3 = u_4u_1 \cup u_4u_2 \cup u_4u_3$ form three $\{u_1, u_2, u_3\}$-trees, namely, $\overline{\kappa}_3(G) \geq 3$. □

Lemma 6. *Let G be a connected graph of order 6 and size 9. If $\overline{\kappa}_3(G) \leq 2$, then $G \in \{G_3, G_4\}$ or $G \in \{H_6^1, H_6^2, H_6^5\}$ or $G \in \mathcal{H}_6^3$.*

Proof. We claim that $2 \leq \delta(G) \leq 3$. Suppose that $\delta(G) \geq 4$. Then $18 = 2e(G) \geq 6\delta \geq 24$, a contradiction. Suppose that $\delta(G) = 1$, without loss of generality, let $d(x) = 1$, then $|V(G-x)| = 5$ and $e(G-x) = 8$. From Lemma 3, $\overline{\kappa}_3(G-x) \geq 3$. Clearly, $\overline{\kappa}_3(G) \geq 3$ by Observation 1.

If $\delta(G) = 3$, then G is 3-regular. It is easy to check that $G = G_3$ or $G = G_4$. In the following, we assume that $\delta(G) = 2$. Without loss of generality, set $d(x) = 2$, then $|V(G-x)| = 5$ and $e(G-x) = 7$, which implies that $G-x = G_1$ or $G-x \in \{H_5^1, H_5^3, H_5^4\}$ by Lemma 4.

If $G-x = G_1$, then $G \in \mathcal{H}_6^3$. If $G-x = H_5^1$, then $G = H_6^1$ or $G = A_2$ or $G = A_6$ (see Figure

3), which results in $G = H_6^1$. If $G - x = H_5^3$, then $G = H_6^2$ or $G \in \{A_1, A_2, A_3, A_4, A_5\}$, which implies that $G = H_6^2$ by Observation 3. If $G - x = H_5^4$, then $G = H_6^5$ or $G \in \{B_1, B_2, B_3, B_4, B_5, B_6\}$, which implies that $G = H_6^5$ by Lemma 2. □

3 Main results

In this section, we give our main results. We first need some more lemmas. In Lemma 3 through Lemma 6, we have dealt with the cases $n \leq 6$. Now we assume that $n \geq 7$.

Lemma 7. *Let G' be a graph obtained from G by deleting a vertex of degree 2. If $G' \in \mathcal{G}_{n-1}^*(n \geq 7)$, then $G \in \mathcal{G}_n^*$ or $\overline{\kappa}_3(G) \geq 3$.*

Proof. Let x be the deleted vertex of degree 2 in G. Since $n \geq 7$, $G' \notin \{K_3, K_4, G_1\}$. From Observation 2 and Lemma 1, if $G' \in \{G_2, G_3, G_4\}$, then we can check that $G \in \mathcal{H}_9^3$ or $\overline{\kappa}_3(G) \geq 3$. From now on, we consider $G' \in \mathcal{G}_{n-1}^* \setminus \{G_2, G_3, G_4\}$.

Case 1. $G' \in \mathcal{H}_{n-1}^1$.

First we consider the case that there is no K_4 in G'. Thus, $G' = H_{n-1}^1$. Since $n \geq 7$, $G = H_n^1 \in \mathcal{H}_n^1$ or G must contain an A_2 or A_6 as its subgraph, which implies that $G \in \mathcal{G}_n^*$ or $\overline{\kappa}_3(G) \geq 3$ by Observation 1.

Next we consider the case that there exists at least one K_4 in G'. For each K_4, if $N_G(x) \cap (K_4 \setminus y) \neq \varnothing$, then we have $\overline{\kappa}_3(G) \geq 3$ by Lemma 1, where y is an attaching vertex in G'. Suppose that $N_G(x) \cap (K_4 \setminus y) = \varnothing$ for all $K_4 \subseteq G'$. Clearly, we can consider the graph $G' \in \mathcal{H}_{n-1}^1$ as the join of K_2 and r isolated vertices, and then doing the attaching operation at some vertices of degree 2 on $K_2 \vee rK_1$. So, we consider $N(x) \subseteq K_2 \vee rK_1$ ($r \geq 1$). For $r \geq 3$, it follows that $G \in \mathcal{H}_n^1$ or G contains the graph A_2 or A_6 as its subgraph, which implies that $G \in \mathcal{G}_n^*$ or $\overline{\kappa}_3(G) \geq 3$.

For $r = 2$, from Lemma 1, we only need to consider $N(x) \subseteq V(K_2 \vee 2K_1)$. By Observation 2, $G \in \mathcal{H}_{11}^1$ or $G \in \mathcal{H}_8^1$ or $G \in \mathcal{H}_8^3$ or $\overline{\kappa}_3(G) \geq 3$. For $r = 1$, $K_2 \vee K_1$ is a triangle and G' is a graph obtained from this triangle by the attaching operation at some vertices of this triangle since $n \geq 7$. Thus, from Observation 2 and Lemma 1, we can get $\overline{\kappa}_3(G) \geq 3$.

Case 2. $G' \in \mathcal{H}_{n-1}^2$ or $G' \in \mathcal{H}_{n-1}^3$.

We only prove the conclusion for $G' \in \mathcal{H}_{n-1}^2$, the same can be showed for $G' \in \mathcal{H}_{n-1}^3$ similarly. Without loss of generality, let $\mathcal{H}_{n-1}^2$ be the graph class obtained from H_r^2 by the attaching operation at some vertices of degree 2 on H_r^2, where $r = n-1, n-4, n-7$. One can see that u_1 and $v_{\frac{r}{2}}$ can be the attaching vertices. From Lemma 1, we only need to consider the case that $N_G(x) \subseteq H_r^2$. Set $N_G(x) = \{x_1, x_2\}$. Thus $x_1, x_2 \in V(H_r^2)$.

If $d_{H_r^2}(x_1) = d_{H_r^2}(x_2) = 2$, without loss of generality, let $x_1 = u_1$ and $x_2 = v_{\frac{r}{2}}$, then neither u_1 nor $v_{\frac{r}{2}}$ is an attaching vertex by Observation 2. We can choose a path $P := u_3u_4 \cdots u_{\frac{r}{2}}v_{\frac{r}{2}}xu_1$ connecting u_1 and u_3 in $G \setminus \{u_2, v_1, v_2\}$. Thus, G contains a subdivision of A_3 as its subgraph (see Figures 3 and 6 (a)), which results in $\overline{\kappa}_3(G) \geq 3$.

If $d_{H_r^2}(x_1) = 2$ and $d_{H_r^2}(x_2) = 3$, without loss of generality, let $x_1 = u_1$, then we can find a path connecting u_1 and u_3 and obtain $\overline{\kappa}_3(G) \geq 3$ for $x_2 \in H_r \setminus \{u_2, v_1, v_2\}$. For $x_2 = u_2$ and $x_2 = v_2$, G contains an A_1 and A_4 as its subgraph, which implies $\overline{\kappa}_3(G) \geq 3$. If $x_2 = v_1$, then $G \in \mathcal{H}_n^3$ and so $G \in \mathcal{G}_n^*$.

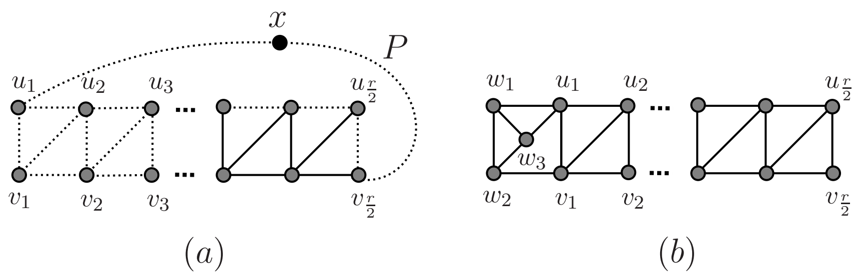

Figure 6. Graphs for Lemma 7.

For $3 \leq d_{H_r^2}(x_i) \leq 4$ $(i = 1, 2)$, one can check that G contains a subdivision of one of $\{A_1, A_2, \cdots, A_5\}$, which implies $\overline{\kappa}_3(G) \geq 3$.

Case 3. $G' \in \mathcal{H}_{n-1}^4$ or $G' \in \mathcal{H}_{n-1}^5$.

Note that only $v_{\frac{r}{2}}$ can be an attaching vertex in H_r^4 (see Figure 6 (b)), where $r = n-1, n-4$. From Lemma 1, we only need to consider $N(x) \subseteq H_r^4$. We can consider H_r^4 as a graph obtained from H_5^4 and H_{r-3}^2 by identifying one edge u_1v_1 in each of them.

Consider $N(x) \cap \{w_1, w_2, w_3\} \neq \varnothing$. If $N(x) \neq \{w_2, v_1\}$, then G contains a subdivision of one of $\{B_1, \cdots, B_6\}$ as its subgraph. So, $\overline{\kappa}_3(G) \geq 3$ by Lemma 2. If $N(x) = \{w_2, v_1\}$, then one can also get that $\overline{\kappa}_3(G) \geq 3$. Now we can assume that $N(x) \cap \{w_1, w_2, w_3\} = \varnothing$. For $|\{u_1, v_1\} \cap N(x)| = 2$, G contains an A_2 as its subgraph, which results in $\overline{\kappa}_3(G) \geq 3$. For $|\{u_{\frac{r}{2}}, v_{\frac{r}{2}}\} \cap N(x)| = 2$, if $v_{\frac{r}{2}}$ is not an attaching vertex in H_r^4, then $G \in \mathcal{H}_n^5$; if $v_{\frac{r}{2}}$ is an attaching vertex in H_r^4, then $\overline{\kappa}_3(G) \geq 3$ by Observation 2. For the other cases, we can also check that $\overline{\kappa}_3(G) \geq 3$.

Case 4. $G' \in \mathcal{H}_{n-1}^6$ or $G' \in \mathcal{H}_{n-1}^7$.

From the above Case 2 and Lemma 2, we can get $\overline{\kappa}_3(G) \geq 3$ in this case. □

Similarly, we have the following lemma.

Lemma 8. *Let G' be a graph obtained from G by deleting a vertex of degree 3. If $G' \in \mathcal{G}_{n-1}^*(n \geq 7)$, then $\overline{\kappa}_3(G) \geq 3$.*

Lemma 9. *Let G be a graph obtained from G' by deleting an edge $e = x_1x_2$ and adding a vertex x such that $N_G(x) = \{x_1, x_2, x_3\}$, where $x_3 \in V(G') \setminus \{x_1, x_2\}$. If $G' \in \mathcal{G}^*_{n-1}$ ($n \geq 7$), then $G \in \mathcal{G}^*_n$ or $\overline{\kappa}_3(G) \geq 3$.*

Proof. Since $n \geq 7$, $G' \notin \{K_3, K_4, G_1\}$. From Observation 2 and Lemma 1, if $G' \in \{G_2, G_3, G_4\}$, we can easily check that $\overline{\kappa}_3(G) \geq 3$ or $G \in \mathcal{G}^*_n$. Thus we consider $G' \in \mathcal{G}^*_{n-1} \setminus \{G_2, G_3, G_4\}$.

We claim that if there exists a K_4 in G' such that $e \in E(K_4)$, then $\overline{\kappa}_3(G) \geq 3$. Let $V(K_4) = \{u_1, u_2, u_3, u_4\}$. Without loss of generality, let $x_1 = u_2$ and $x_2 = u_4$.

If $x_3 \in V(K_4)$, then $x_3 = u_1$ or $x_3 = u_3$. It follows that $\overline{\kappa}_3(G) \geq 3$ (see Figure 7 (a)). So we assume that $x_3 \notin V(K_4)$. From Lemma 1, if x_3 belongs to another clique of order 4 such that x_3 is not an attaching vertex, then $\overline{\kappa}_3(G) \geq 3$ or $G \in \mathcal{G}^*_n$. So, we only need to consider $x_3 \in H^i_r (1 \leq i \leq 7)$. If neither u_2 nor u_4 is an attaching vertex, then u_1 or u_3 is an attaching vertex, say u_1. Then there must exist a path P connecting x_3 and u_1 such that $u_2, u_3, u_4 \notin V(P)$ since $H^i_r (1 \leq i \leq 7)$ is connected. Then the trees $T_1 = xu_2 \cup xu_4 \cup P$ and $T_2 = u_1u_2 \cup u_1u_4$ and $T_3 = u_3u_1 \cup u_3u_2 \cup u_3u_4$ form three $\{u_1, u_2, u_4\}$-trees, namely, $\overline{\kappa}_3(G) \geq 3$ (see Figure 7 (b)).

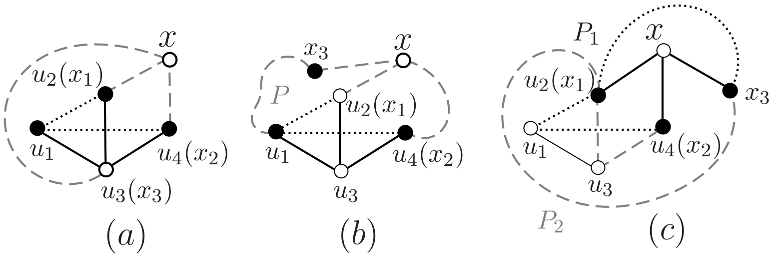

Figure 7. Graphs for the claim.

Suppose that one of $\{u_2, u_4\}$ is an attaching vertex, say u_2. Thus there must exist two paths P_1 and P_2 connecting x_3 and u_2 in H^i_r since H^i_r is 2-connected. Then the trees $T_1 = xu_2 \cup xu_4 \cup xx_3$ and $T_2 = u_4u_1 \cup u_1u_2 \cup P_1$ and $T_3 = u_4u_3 \cup u_3u_2 \cup P_2$ form three internally disjoint $\{u_2, u_4, x_3\}$-trees, namely, $\overline{\kappa}_3(G) \geq 3$ (see Figure 7 (c)).

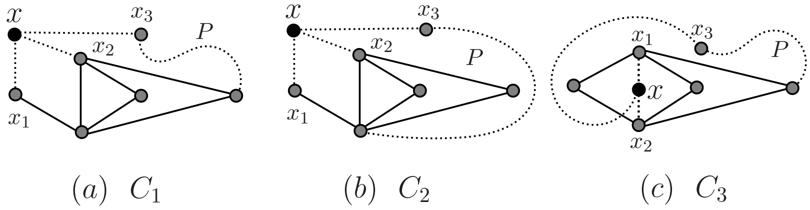

Figure 8. Graphs for Lemma 9.

Now we consider $e \notin E(K_4)$. Thus $e \in E(H_r^i)(1 \leq i \leq 7)$. We only consider $e \in E(H_r^1)$, and for $e \in E(H_r^i)(2 \leq i \leq 7)$ one can also check that $G \in \mathcal{G}_n^*$ or $\overline{\kappa}_3(G) \geq 3$. Since $H_r^1 = K_2 \vee (r-2)K_1$, we suppose that $e \in E(K_2 \vee (r-2)K_1)(r \geq 3)$. For $r \geq 5$, G must contain one of $\{C_1, C_2, C_3\}$ as its subgraph. One can check that $\overline{\kappa}_3(G) \geq 3$ by Observation 1 (see Figure 8). For $r = 4$, $G \in \mathcal{H}_8^3$ or $G \in \mathcal{H}_8^4$ or $G \in \mathcal{H}_{11}^3$ or $\overline{\kappa}_3(G) \geq 3$. For $r = 3$, we can obtain $G \in \mathcal{H}_7^2$ or $G \in \mathcal{H}_{10}^2$ or $\overline{\kappa}_3(G) \geq 3$ by Lemma 1 and Observation 2. $\square$

Theorem 1. *Let G be a connected graph of order n such that $\overline{\kappa}_3(G) \leq 2$. Then*

$$e(G) \leq \begin{cases} 2n-2 & if\ n = 4, \\ 2n-3 & if\ n \geq 3, n \neq 4. \end{cases}$$

with equality if and only if $G \in \mathcal{G}_n^*$.

Proof. We apply induction on $n(n \geq 7)$. For $n = 3, 4$, it is easy to see that $\mathcal{G}_n^* = \{K_n\}$. For $n = 5$ or $n = 6$, the assertion holds by Lemmas 4 and 6.

Suppose that the assertion holds for graphs of order less than $n \geq 7$. Now we show that the assertion holds for $n \geq 7$. We claim that $\delta(G) \leq 3$. Otherwise, $\delta(G) \geq 4$. Let G' be the graph obtained from G by deleting a vertex x such that $d(x) = \delta(G)$. Then, $2e(G') = 2e(G) - 2d(x) = 2e(G) - 2\delta(G) \geq (n-2)\delta(G) \geq 4(n-2)$. But, by the induction hypothesis, $2e(G') \leq 2[2(n-1)-3] = 4n-10$, a contradiction.

If $\delta(G) = 1$, then we let G' be the graph obtained from G by deleting a pendant vertex. Then by the induction hypothesis, $e(G) \leq e(G') + 1 = 2(n-1) - 3 + 1 = 2n-4 < 2n-3$.

If $\delta(G) = 2$, then we let G' be the graph obtained from G by deleting a vertex of degree 2. If $e(G') < 2(n-1) - 3$, then $e(G) = e(G') + 2 < 2(n-1) - 3 + 2 = 2n-3$. If $e(G') = 2(n-1) - 3$, then $e(G) = e(G') + 2 = 2(n-1) - 3 + 2 = 2n-3$. Since $G' \in \mathcal{G}_{n-1}^*$ and $\overline{\kappa}_3(G) \leq 2$, we can obtain $G \in \mathcal{G}_n^*$ by Lemma 7.

Suppose that $\delta(G) = 3$. Let G' be the graph obtained from G by deleting a vertex of degree 3, say x. If $e(G') = 2(n-1) - 3$, then $G' \in \mathcal{G}_{n-1}^*$. We can get a contradiction by Lemma 8. If $e(G') < 2(n-1) - 3$, then $e(G) = e(G') + 3 \leq 2(n-1) - 4 + 3 = 2n-3$.

Now we will show that $G \in \mathcal{G}_n^*$ for $e(G') = 2(n-1) - 4$. Suppose $N_G(x) = \{x_1, x_2, x_3\}$. We have the following two cases to consider.

Case 1. $G[N_G(x)]$ is not a triangle.

In this case, there exists an edge $x_ix_j \notin E(G)(1 \leq i, j \leq 3)$. Let $G'' = G' + x_ix_j$. Then we claim that $\overline{\kappa}_3(G'') \leq 2$. In fact, suppose that $\overline{\kappa}_3(G'') \geq 3$. Then there exists a 3-subset $S \subseteq V(G)$ such that G'' contains three internally disjoint S-trees, denoted by T_1, T_2, T_3. If $x_ix_j \notin \bigcup_{i=1}^3 E(T_i)$, then T_1, T_2, T_3 are 3 S-trees in G, which contradicts $\overline{\kappa}_3(G) \leq 2$.

Assume that x_ix_j belongs to some S-tree, without loss of generality, say $x_ix_j \in E(T_1)$, then $T_1' = (T_1 - x_ix_j) \cup x_ix \cup xx_j$ is an S-tree in G. Thus, T_1', T_2, T_3 are three internally disjoint S-trees in G, which implies that $\overline{\kappa}_3(G) \geq 3$, a contradiction.

Since $e(G'') = e(G') + 1 = 2(n-1) - 3$ and $\overline{\kappa}_3(G) \leq 2$, we have $G'' \in \mathcal{G}_{n-1}^*$. Furthermore, $G \in \mathcal{G}_n^*$ by Lemma 9.

Case 2. $G[N_G(x)]$ is a triangle.

Clearly, $G[N_G[x]]$ is a clique of order 4, where $N_G[x] = N_G(x) \cup \{x\}$. From Lemma 1, there is no path connecting any two vertices of $G[N_G[x]]$. So, $G \setminus E(G[N_G[x]])$ has three connected components except x. We denote them by G_1, G_2, G_3 (note that $G_i \neq K_4 (i = 1, 2, 3)$). By the induction hypothesis, $e(G) = \sum_{i=1}^{3} e(G_i) + 6 \leq 2\sum_{i=1}^{3} |G_i| - 3 = 2(n-1) - 3 < 2n - 3$.

□

Corollary 1.

$$f(n; \overline{\kappa}_3 \leq 2) = \begin{cases} 2n-2 & if\ n = 4, \\ 2n-3 & if\ n \geq 3, n \neq 4. \end{cases}$$

Since for $0 \leq \ell \leq n - k + \lfloor k/2 \rfloor - 1$, we have that $h(n; \overline{\kappa}_k \geq \ell + 1) = f(n; \overline{\kappa}_k \leq \ell) + 1$, the following corollary is immediate.

Corollary 2.

$$h(n; \overline{\kappa}_3 \geq 3) = \begin{cases} 2n-1 & if\ n = 4, \\ 2n-2 & if\ n \geq 3, n \neq 4. \end{cases}$$

Remark. Let n, ℓ be odd, and G' be a graph obtained from an $(\ell-3)$-regular graph of order $n-2$ by adding a maximum matching, and $G = G' \vee K_2$. Then $\delta(G) = \ell - 1$, $\overline{\kappa}_3(G) \leq \ell$ and $e(G) = \frac{\ell+2}{2}(n-2) + \frac{1}{2}$.

Otherwise, let G' be an $(\ell-2)$-regular graph of order $n-2$ and $G = G' \vee K_2$. Then $\delta(G) = \ell$, $\overline{\kappa}_3(G) \leq \ell$ and $e(G) = \frac{\ell+2}{2}(n-2) + 1$.

Therefore,

$$f(n; \overline{\kappa}_3 \leq \ell) \geq \begin{cases} \frac{\ell+2}{2}(n-2) + \frac{1}{2} & for\ n, \ell\ odd, \\ \frac{\ell+2}{2}(n-2) + 1 & otherwise. \end{cases}$$

One can see that for $\ell = 2$ this bound is the best possible ($f(n; \overline{\kappa}_3 \leq 2) = 2n - 3$). Actually, the graph constructed for this bound is $K_2 \vee (n-2)K_1$, which belongs to $\mathcal{G}_n^*$.

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