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Equivalence Classes of Full-Dimensional $0/1$-Polytopes with Many Vertices

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Abstract

Let Q_n denote the n-dimensional hypercube with the vertex set $V_n = \{0, 1\}^n$. A 0/1-polytope of Q_n is a convex hull of a subset of V_n. This paper is concerned with the enumeration of equivalence classes of full-dimensional 0/1-polytopes under the symmetries of the hypercube. With the aid of a computer program, Aichholzer completed the enumeration of equivalence classes of full-dimensional 0/1-polytopes for Q_4, Q_5, and those of Q_6 up to 12 vertices. In this paper, we present a method to compute the number of equivalence classes of full-dimensional 0/1-polytopes of Q_n with more than 2^{n-3} vertices. As an application, we finish the counting of equivalence classes of full-dimensional 0/1-polytopes of Q_6 with more than 12 vertices.

Keywords: n-cube, full-dimensional 0/1-polytope, symmetry, hyperplane, Pólya theory.

AMS Classification: 05A15, 52A20, 52B12, 05C25

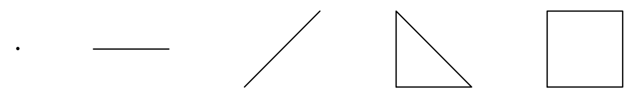
1 Introduction

Let Q_n denote the n-dimensional hypercube with vertex set $V_n = \{0, 1\}^n$. A 0/1-polytope of Q_n is defined to be the convex hull of a subset of V_n. The study of 0/1-polytopes has drawn much attention from different points of view, see, for example, [7, 8, 12, 13, 14, 16, 22], see also the survey of Ziegler [21].

In this paper, we are concerned with the problem of determining the number of equivalence classes of n-dimensional 0/1-polytopes of Q_n under the symmetries of Q_n, which has been considered as a difficult problem, see Ziegler [21]. It is also listed by Zong [22, Problem 5.1] as one of the fundamental problems concerning 0/1-polytopes.

An n-dimensional 0/1-polytope of Q_n is also called a full-dimensional 0/1-polytope of Q_n. Two 0/1-polytopes are said to be equivalent if one can be transformed to the other by a symmetry of Q_n. Such a equivalence relation is also called the 0/1-equivalence relation. Figure 1 gives representatives of 0/1-equivalence classes of Q_2, among which (d) and (e) are full-dimensional.

Sarangarajan and Ziegler [21, Proposition 8] found an lower bound on the number of equivalence classes of full-dimensional 0/1-polytopes of Q_n. As far as exact enumeration is



(a) (b) (c) (d) (e)

Figure 1: 0/1-Polytopes of the square

concerned, full-dimensional 0/1-equivalence classes of Q_4 were counted by Alexx Below, see Ziegler [21]. With the aid of a computer program, Aichholzer [1] completed the enumeration of full-dimensional 0/1-equivalence classes of of Q_5, and those of Q_6 up to 12 vertices, see Aichholzer [3] and Ziegler [21]. The 5-dimensional hypercube Q_5 has been considered as the last case that one can hope for a complete solution to the enumeration of full-dimensional 0/1-equivalence classes.

The objective of this paper is to present a method to compute the number of full-dimensional 0/1-equivalence classes of Q_n with more than 2^{n-3} vertices. As an application, we solve the enumeration problem for full-dimensional 0/1-equivalence classes of the 6-dimensional hypercube with more than 12 vertices.

To describe our approach, we introduce some notation. Denote by $\mathcal{A}_n(k)$ (resp., $\mathcal{F}_n(k)$) the set of (resp., full-dimensional) 0/1-equivalence classes of Q_n with k vertices. Let $\mathcal{H}_n(k)$ be the set of 0/1-equivalence classes of Q_n with k vertices that are not full-dimensional. The cardinalities of $\mathcal{A}_n(k)$, $\mathcal{F}_n(k)$ and $\mathcal{H}_n(k)$ are denoted respectively by $A_n(k)$, $F_n(k)$ and $H_n(k)$. It is clear that any full-dimensional 0/1-polytope of Q_n has at least $n+1$ vertices, i.e., $F_n(k) = 0$ for $1 \leq k \leq n$.

The starting point of this paper is the following obvious relation

$$F_n(k) = A_n(k) - H_n(k). \tag{1.1}$$

The number $A_n(k)$ can be computed based on the cycle index of the hyperoctahedral group. We can deduce that $H_n(k) = 0$ for $k > 2^{n-1}$ based on a result duo to Saks. For the purpose of computing $H_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$, we transform the computation of $H_n(k)$ to the determination of the number of equivalence classes of 0/1-polytopes with k vertices that are contained in the spanned hyperplanes of Q_n. To be more specific, we show that $\mathcal{H}_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$ can be decomposed into a disjoint union of equivalence classes of 0/1-polytopes that are contained in the spanned hyperplanes of Q_n. In particular, for $n = 6$ and $k > 16$, we obtain the number of full-dimensional 0/1-equivalence classes of Q_6 with k vertices.

Using a similar idea as in the case $2^{n-2} < k \leq 2^{n-1}$, we can compute $H_n(k)$ for $2^{n-3} < k \leq 2^{n-2}$. For $n = 6$ and $13 \leq k \leq 16$, we obtain the number of full-dimensional 0/1-equivalence classes of Q_6 with k vertices. Together with the computation of Aichholzer up to 12 vertices, we have completed the enumeration of full-dimensional 0/1-equivalence classes of the 6-dimensional hypercube.

2 The cycle index of the hyperoctahedral group

The group of symmetries of Q_n is known as the hyperoctahedral group B_n. In this section, we review the cycle index of B_n acting on the vertex set V_n. Since 0/1-equivalence classes of Q_n coincide with nonisomorphic vertex colorings of Q_n by using two colors, we may compute the number $A_n(k)$ from the cycle index of B_n.

Let G be a group acting on a finite set X. For any $g \in G$, g induces a permutation on X. The cycle type of a permutation is defined to be a multiset $\{1^{c_1}, 2^{c_2}, \ldots\}$, where c_i is the number of cycles of length i that appear in the cycle decomposition of the permutation. For $g \in G$, denote by $c(g) = \{1^{c_1}, 2^{c_2}, \ldots\}$ the cycle type of the permutation on X induced by g. Let $z = (z_1, z_2, \ldots)$ be a sequence of indeterminants, and let

$$z^{c(g)} = z_1^{c_1} z_2^{c_2} \cdots .$$

The cycle index of G is defined as follows

$$Z_G(z) = Z_G(z_1, z_2, \ldots) = \frac{1}{|G|} \sum_{g \in G} z^{c(g)}. \tag{2.1}$$

According to Pólya's theorem, the cycle index in (2.1) can be applied to count nonisomorphic colorings of X by using a given number of colors.

For a vertex coloring of Q_n with two colors, say, black and white, the black vertices can be considered as vertices of a 0/1-polytope of Q_n. This establishes a one-to-one correspondence between equivalence classes of vertex colorings and 0/1-equivalence classes of Q_n. Let $Z_n(z)$ denote the cycle index of B_n acting on the vertex set V_n. Then, by Pólya's theorem

$$A_n(k) = \left[u_1^k u_2^{2^n-k}\right] C_n(u_1, u_2), \tag{2.2}$$

where $C_n(u_1, u_2)$ is the polynomial obtained from $Z_n(z)$ by substituting z_i with $u_1^i + u_2^i$, and $[u_1^p u_2^q]C_n(u_1, u_2)$ denotes the coefficient of $u_1^p u_2^q$ in $C_n(u_1, u_2)$.

Clearly, the total number of 0/1-equivalence classes of Q_n is given by

$$\sum_{k=1}^{2^n} A_n(k) = C_n(1, 1). \tag{2.3}$$

It should be noted that $C_n(1, 1)$ also equals the number of types of Boolean functions, see Chen [10] and references therein. This number is also related to configurations of n-dimensional Orthogonal Pseudo-Polytopes, see, e.g., Aguila [5]. The computation of $Z_n(z)$ has been studied by Chen [10], Harrison and High [15], and Pólya [18], etc. Explicit expressions of $Z_n(z)$ for $n \leq 6$ can be found in [5], and we list them bellow.

$Z_1(z) = z_1,$

$$Z_2(z) = \frac{1}{8}\left(z_1^4 + 2z_1^2 z_2 + 3z_2^2 + 2z_4 \right),$$

$$Z_3(z) = \frac{1}{48}\left(z_1^8 + 6z_1^4 z_2^2 + 13z_2^4 + 8z_1^2 z_3^2 + 12z_4^2 + 8z_2 z_6 \right),$$

$$Z_4(z) = \frac{1}{384}\left(\begin{array}{l} z_1^{16} + 12z_1^8 z_2^4 + 12z_1^4 z_2^6 + 51z_2^8 + 48z_8^2 \\ +48z_1^2 z_2 z_4^3 + 84z_4^4 + 96z_2^2 z_6^2 + 32z_1^4 z_3^4 \end{array} \right),$$

$$Z_5(z) = \frac{1}{3840}\left(\begin{array}{l} z_1^{32} + 20z_1^{16} z_2^8 + 60z_1^8 z_2^{12} + 231z_2^{16} + 80z_1^8 z_3^8 + 240z_1^4 z_2^2 z_4^6 \\ +240z_2^4 z_4^6 + 520z_4^8 + 384z_1^2 z_5^6 + 160z_1^4 z_2^2 z_3^4 z_6^2 + 720z_2^4 z_6^4 \\ +480z_8^4 + 384z_2 z_{10}^3 + 320z_4^2 z_{12}^2 \end{array} \right),$$

$$Z_6(z) = \frac{1}{46080}\left(\begin{array}{l} z_1^{64} + 30z_1^{32} z_2^{16} + 180z_1^{16} z_2^{24} + 120z_1^8 z_2^{28} + 1053z_2^{32} + 160z_1^{16} z_3^{16} + \\ 640z_1^4 z_3^{20} + 720z_1^8 z_2^4 z_4^{12} + 1440z_1^4 z_2^6 z_4^{12} + 2160z_2^8 z_4^{12} + 4920z_4^{16} + \\ 2304z_1^4 z_5^{12} + 960z_1^8 z_2^4 z_3^8 z_6^4 + 5280z_2^8 z_6^8 + 3840z_1^2 z_2 z_3^2 z_6^9 + 5760z_8^8 \\ +1920z_2^2 z_6^{10} + 6912z_2^2 z_{10}^6 + 3840z_4^4 z_{12}^4 + 3840z_4 z_{12}^5 \end{array} \right).$$

The method of Chen for computing $Z_n(z)$ is based on the cycle structure of a power of a signed permutation. Let us recall the notation of a signed permutation. A signed permutation on $\{1, 2, \ldots, n\}$ is a permutation on $\{1, 2, \ldots, n\}$ with a $+$ or a $-$ sign attached to each element $1, 2, \ldots, n$. Following the notation in Chen [10] or Chen and Stanley [11], we may write a signed permutation in terms of the cycle decomposition and ignore the plus sign $+$. For example, $(\overline{2}4\overline{5})(3)(1\overline{6})$ represents a signed permutation, where $(245)(3)(16)$ is called its underlying permutation. The action of a signed permutation w on the vertices of Q_n is defined as follows. For a vertex $(x_1, x_2, \ldots, x_n)$ of Q_n, we define $w((x_1, x_2, \ldots, x_n))$ to be the vertex $(y_1, y_2, \ldots, y_n)$ as given by

$$y_i = \begin{cases} x_{\pi(i)}, & \text{if } i \text{ has the sign } +, \\ 1 - x_{\pi(i)}, & \text{if } i \text{ has the sign } -, \end{cases} \tag{2.4}$$

where π is the underlying permutation of w.

For the purpose of this paper, we define the cycle type of a signed permutation $w \in B_n$ as the cycle type of its underlying permutation. For example, $(\overline{2}4\overline{5})(3)(1\overline{6})(7)$ has cycle type $\{1^2, 2, 3\}$. We should note that the above definition of a cycle type of a signed permutation is different from the definition in terms of double partitions as in [10] because it will be shown in Section 5 that any signed permutation that fixes a spanned hyperplane of Q_n either have all positive cycles or all negative cycles.

We end this section with the following formula of Chen [10], which will be used in Section 6 to compute the cycle index of the group that fixes a spanned hyperplane of Q_n.

Theorem 2.1 *Let G be a group that acts on some finite set X. For any $g \in G$, the number of i-cycles of the permutation on X induced by g is given by*

$$\frac{1}{i}\sum_{j|i}\mu(i/j)\psi(g^j),$$

where μ is the classical number-theoretic Möbius function and $\psi(g^j)$ is the number of fixed points of g^j on X.

3 0/1-Polytopes with many vertices

In this section, we find an inequality concerning the dimension of a 0/1-polytope of Q_n and the number of its vertices. This inequality plays a key role in the computation of $F_n(k)$ for $k > 2^{n-3}$.

The main theorem of this section is given below.

Theorem 3.1 *Let P be a 0/1-polytope of Q_n with more than 2^{n-s} vertices, where $1 \leq s \leq n$. Then we have*

$$\dim(P) \geq n - s + 1.$$

The above theorem can be deduced from the following assertion.

Theorem 3.2 *For any $1 \leq s \leq n$, the intersection of s hyperplanes in $\mathbb{R}^n$ with linearly independent normal vectors contains at most 2^{n-s} vertices of Q_n.*

Indeed, it is not difficult to see that Theorem 3.2 implies Theorem 3.1. Let P be a 0/1-polytope of Q_n with more than 2^{n-s} vertices. Suppose to the contrary that $\dim(P) \leq n-s$. It is known that the affine space spanned by P can be expressed as the intersection of a collection of hyperplanes. Since $\dim(P) \leq n-s$, there exist s hyperplanes $H_1, H_2, \ldots, H_s$ whose normal vectors are linearly independent such that the intersection of $H_1, H_2, \ldots, H_s$ contains P. Let $V(P)$ denote the vertex set of P. By Theorem 3.2, we have

$$|V(P)| \leq \left| \left(\bigcap_{i=1}^{s} H_i \right) \bigcap V_n \right| \leq 2^{n-s},$$

which is a contradiction to the assumption that P contains more than 2^{n-s} vertices of Q_n. So we conclude that $\dim(P) \geq n - s + 1$.

Proof of Theorem 3.2. Assume that, for $1 \leq i \leq s$,

$$H_i\colon a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

are s hyperplanes in $\mathbb{R}^n$, whose normal vectors $a_i = (a_{i1}, \ldots, a_{in})$ are linearly independent. We aim to show that the intersection of $H_1, H_2, \ldots, H_s$ contains at most 2^{n-s} vertices of Q_n. We may express the intersection of $H_1, H_2, \ldots, H_s$ as the solution of a system of linear equations, that is,

$$\bigcap_{i=1}^{s} H_i = \{x^T \colon Ax = b\}, \tag{3.1}$$

where A denotes the matrix $(a_{ij})_{1\le i\le s, 1\le j\le n}$, $x = (x_1, x_2, \ldots, x_n)^T$, and $b = (b_1, \ldots, b_s)^T$, T denotes the transpose of a vector. Then Theorem 3.2 is equivalent to the following inequality

$$\left| V_n \bigcap \left\{x^T \colon Ax = b\right\} \right| \le 2^{n-s}. \tag{3.2}$$

We now proceed to prove (3.2) by induction on n and s. We first consider the case $s = 1$. Suppose that $H\colon c_1x_1 + c_2x_2 + \cdots + c_nx_n = c$ is a hyperplane in $\mathbb{R}^n$. Assume that among the coefficients $c_1, c_2, \ldots, c_n$ there are i of them that are nonzero. Without loss of generality, we may assume that $c_1, c_2, \ldots, c_i$ are nonzero, and $c_{i+1} = c_{i+2} = \cdots = c_n = 0$. Clearly, H reduces to a hyperplane in the i-dimensional Euclidean space $\mathbb{R}^i$. Such a hyperplane with nonzero coefficients is called a skew hyperplane. Now the vertices of Q_n contained in H are of the form $(d_1, \ldots, d_i, d_{i+1}, \ldots, d_n)$ where $(d_1, \ldots, d_i)$ are vertices of Q_i contained in the skew hyperplane $H'\colon c_1x_1 + c_2x_2 + \cdots c_ix_i = b$. Clearly, for each vertex $(d_1, d_2, \ldots, d_i)$ in H', there are 2^{n-i} choices for $(d_{i+1}, d_{i+2}, \ldots, d_n)$ such that $(d_1, d_2, \ldots, d_n)$ is contained in H. Using Sperner's lemma (see, for example, Lubell [17]), Saks [19, Theorem 3.64] has shown that the number of vertices of Q_i contained in a skew hyperplane does not exceed $\binom{i}{\lfloor \frac{i}{2} \rfloor}$. Let

$$f(n,i) = 2^{n-i}\binom{i}{\lfloor \frac{i}{2} \rfloor}.$$

Thus the number of vertices of Q_n contained in H is at most $f(n,i)$. It is easy to check that

$$\frac{f(n,i)}{f(n,i+1)} = \begin{cases} \frac{i+2}{i+1}, & \text{if } i \text{ is even,} \\ 1, & \text{if } i \text{ is odd.} \end{cases}$$

This yields $f(n,i) \ge f(n,i+1)$ for any $i = 1, 2, \ldots, n-1$. Hence H contains at most $f(n,1) = 2^{n-1}$ vertices of Q_n, which implies (3.2) for $s = 1$.

We now consider the case $s = n$. In this case, since the normal vectors $a_1, \ldots, a_n$ are linearly independent, the square matrix A is nonsingular. It follows that $Ax = b$ has exactly one solution. Therefore, inequality (3.2) holds when $s = n$.

So we are left with cases of n, s such that $1 < s < n$. We shall use induction to complete the proof. Suppose that (3.2) holds for n', s' such that $n' \le n, s' \le s$ and $(n', s') \ne (n, s)$.

Since the normal vector a_1 is nonzero, there exists some j_0 $(1 \le j_0 \le n)$ such that $a_{1j_0} \ne 0$. Without loss of generality, we may assume $a_{ij_0} = 0$ for $2 \le i \le s$ since one can apply elementary row transformations to the system of linear equations $Ax = b$ to ensure that the assumption is valid. For a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, let $v^j = (v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n) \in \mathbb{R}^{n-1}$ be the vector obtained from v by deleting the j-th coordinate. We now have two cases.

Case 1. The vectors $a_1^{j_0}, \ldots, a_s^{j_0}$ are linearly dependent. Since $a_2, \ldots, a_s$ are linearly independent and $a_{ij_0} = 0$ for $2 \le i \le s$, it is clear that $a_2^{j_0}, \ldots, a_s^{j_0}$ are linearly independent. So the vector $a_1^{j_0}$ can be expressed as a linear combination of $a_2^{j_0}, \ldots, a_s^{j_0}$. Assume that $a_1^{j_0} = \alpha_2 a_2^{j_0} + \ldots + \alpha_s a_s^{j_0}$, where $\alpha_k \in \mathbb{R}$ for $2 \le k \le s$. For $2 \le k \le s$, multiplying the k-th row by α_k and subtracting it from the first row, then the first equation $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$ becomes

$$a_{1j_0}x_{j_0} = b_1 - \sum_{k=2}^{s} \alpha_k b_k. \tag{3.3}$$

Let $A' = (a_2, \ldots, a_s)^T$ and $b' = (b_2, \ldots, b_s)^T$. Note that all entries in the j_0-th column of A' are zero since we have assumed $a_{ij_0} = 0$ for $2 \le i \le s$. Let A'_{j_0} be the matrix obtained from A' by removing this zero column. From equation (3.3), the value x_{j_0} in the j_0-th coordinate of the solutions of $Ax = b$ is fixed. Then solutions of $Ax = b$ can be obtained from the solutions of $A'_{j_0}x = b'$ by adding the value of x_{j_0} to the j_0-th coordinate. Concerning the number of vertices of Q_n contained in $\{x^T \colon Ax = b\}$, we consider the following two cases.

(1). The value x_{j_0} is not equal to 0 or 1. In this case, no vertex of Q_n is contained in $\{x^T \colon Ax = b\}$. Hence inequality (3.2) holds.

(2). The value x_{j_0} is equal to 0 or 1. Since every vertex of Q_n contained in $\{x^T \colon Ax = b\}$ is obtained from a vertex of Q_{n-1} contained in $\{x^T \colon A'_{j_0}x = b'\}$ by adding x_{j_0} in the j_0-th coordinate, it follows that

$$\left| V_n \bigcap \{x^T \colon Ax = b\} \right| = \left| V_{n-1} \bigcap \{x^T \colon A'_{j_0}x = b'\} \right|. \tag{3.4}$$

By the induction hypothesis, we find

$$\left| V_{n-1} \bigcap \{x^T \colon A'_{j_0}x = b'\} \right| \le 2^{(n-1)-(s-1)} = 2^{n-s}.$$

In view of (3.4), we obtain (3.2).

Case 2. Suppose $a_1^{j_0}, \ldots, a_s^{j_0}$ are linearly independent. Assume that the value of x_{j_0} in the solutions of $\{x^T \colon Ax = b\}$ can be taken 0 or 1. Then the vertices of Q_n contained in $\{x^T \colon Ax = b\}$ can be decomposed into a disjoint union of the following two sets

$$S_0 = V_n \bigcap \{x^T \colon x_{j_0} = 0,\ Ax = b\}$$

and

$$S_1 = V_n \bigcap \{x^T \colon x_{j_0} = 1,\ Ax = b\}.$$

We first consider the set S_0. Let $A'' = (a_1^{j_0}, \ldots, a_s^{j_0})^T$ be the matrix obtained from A by deleting the j_0-th column. Then vertices of Q_n contained in S_0 are obtained from the vertices of Q_{n-1} contained in $\{x^T\colon A''x = b\}$ by adding 0 to the j_0-th coordinate. So we have

$$|S_0| = \left|V_{n-1} \bigcap \{x^T\colon\ A''x = b\}\right| \leq 2^{n-1-s},$$

where the inequality follows from the induction hypothesis. Similarly, we get $|S_1| \leq 2^{n-1-s}$. Hence

$$\left|V_n \bigcap \{x^T\colon Ax = b\}\right| = |S_0| + |S_1| \leq 2^{n-s}.$$

Combining the above two cases, inequality (3.2) is true for $1 \leq s \leq n$. This completes the proof. ∎

Note that the upper bound 2^{n-s} is sharp. For example, it is easy to see the intersection of hyperplanes $x_i = 0$ $(1 \leq i \leq s)$ contains exactly 2^{n-s} vertices of Q_n.

By Theorem 3.2, we see that every 0/1-polytope of Q_n with more than 2^{n-1} vertices is full-dimensional. As a direct consequence, we obtain the following relation.

Corollary 3.3 *For $k > 2^{n-1}$, we have*

$$F_n(k) = A_n(k).$$

Form Corollary 3.3, the number $F_n(k)$ for $k > 2^{n-1}$ can be computed from the cycle index of the hyperoctahedral group, that is, for $k > 2^{n-1}$

$$F_n(k) = \left[u_1^k u_2^{2^n-k}\right] C_n(u_1, u_2).$$

For $n = 4, 5$ and 6, the values of $F_n(k)$ for $k > 2^{n-1}$ are given in Tables 1, 2 and 3.

k	9	10	11	12	13	14	15	16
$F_4(k)$	56	50	27	19	6	4	1	1

Table 1: $F_4(k)$ for $k > 8$.

k	17	18	19	20	21	22	23	24
$F_5(k)$	158658	133576	98804	65664	38073	19963	9013	3779
k	25	26	27	28	29	30	31	32
$F_5(k)$	1326	472	131	47	29	5	1	1

Table 2: $F_5(k)$ for $k > 16$.

k	$F_6(k)$	k	$F_6(k)$
33	38580161986426	49	3492397119
34	35176482187398	50	1052201890
35	30151914536933	51	290751447
36	24289841497881	52	73500514
37	18382330104696	53	16938566
38	13061946976545	54	3561696
39	8708686182967	55	681474
40	5443544478011	56	120843
41	3186944273554	57	19735
42	1745593733454	58	3253
43	893346071377	59	497
44	426539774378	60	103
45	189678764492	61	16
46	78409442414	62	6
47	30064448972	63	1
48	10666911842	64	1

Table 3: $F_6(k)$ for $k > 32$.

4 $H_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$

In this section, we shall aim to compute $H_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$. We shall show that in this case the number $H_n(k)$ is determined by the number of (partial) 0/1-equivalence classes of a spanned hyperplane of Q_n with k vertices. To this end, it is necessary to consider all possible spanned hyperplanes of Q_n. More precisely, we need representatives of equivalence classes of such spanned hyperplanes.

Recall that a spanned hyperplane of Q_n is a hyperplane in $\mathbb{R}^n$ spanned by n affinely independent vertices of Q_n, that is, the affine space spanned by the vertices of Q_n contained in this hyperplane is of dimension $n-1$. Let

$$H\colon a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

be a spanned hyperplane of Q_n, where $|a_1|, \ldots, |a_n|, |b|$ are positive integers with greatest common divisor 1. Let

$$\mathrm{coeff}(n) = \max\{|a_1|, \ldots, |a_n|\}.$$

It is clear that $\mathrm{coeff}(2) = \mathrm{coeff}(3) = 1$. The study of upper and lower bounds on the number $\mathrm{coeff}(n)$ has drawn much attention, see, for example, [4, 6, 9, 21]. The following are known bounds on $\mathrm{coeff}(n)$ and $|b|$, see, e.g., [21, Corollary 26] and [4, Theorem 5],

$$\frac{(n-1)^{(n-1)/2}}{2^{2n+o(n)}} \leq \mathrm{coeff}(n) \leq \frac{n^{n/2}}{2^{n-1}} \qquad \text{and} \qquad |b| \leq 2^{-n}(n+1)^{\frac{n+1}{2}}.$$

Using the above bounds, Aichholzer and Aurenhammer [4] obtained the exact values of coeff(n) for $n \leq 8$ by computing all possible spanned hyperplanes of Q_n up to dimension 8. For example, they showed that coeff$(4) = 2$, coeff$(5) = 3$, and coeff$(6) = 5$.

As will be seen, in order to compute $H_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$, we need to consider equivalence classes of spanned hyperplanes of Q_n under the symmetries of Q_n. Note that the symmetries of Q_n can be expressed by permuting the coordinates and changing x_i to $1 - x_i$ for some indices i. Therefore, for each equivalence class of spanned hyperplanes of Q_n, we can choose a representative of the following form

$$a_1x_1 + a_2x_2 + \cdots + a_tx_t = b, \tag{4.1}$$

where $t \leq n$ and $0 < a_1 \leq a_2 \leq \cdots \leq a_t \leq \text{coeff}(n)$.

A complete list of spanned hyperplanes of Q_n for $n \leq 6$ can been found in [2]. The following hyperplanes are representatives of equivalence classes of spanned hyperplanes of Q_4:

$$\begin{aligned}
&x_1 = 0,\\
&x_1 + x_2 = 1,\\
&x_1 + x_2 + x_3 = 1,\\
&x_1 + x_2 + x_3 + x_4 = 1 \text{ or } 2,\\
&x_1 + x_2 + x_3 + 2x_4 = 2.
\end{aligned}$$

In addition to the above hyperplanes of $\mathbb{R}^4$, which can also be viewed as spanned hyperplanes of Q_5, we have the following representatives of equivalence classes of spanned hyperplanes of Q_5:

$$\begin{aligned}
&x_1 + x_2 + x_3 + x_4 + x_5 = 1 \text{ or } 2,\\
&x_1 + x_2 + x_3 + x_4 + 2x_5 = 2 \text{ or } 3,\\
&x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 2 \text{ or } 3,\\
&x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 = 3 \text{ or } 4,\\
&x_1 + x_2 + x_3 + x_4 + 3x_5 = 3,\\
&x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 3,\\
&x_1 + x_2 + 2x_3 + 2x_4 + 3x_5 = 4.
\end{aligned}$$

When $n = 6$, for the purpose of computing $F_6(k)$ for $16 < k \leq 32$, we need the representatives of equivalence classes of spanned hyperplanes of Q_6 containing more than 16 vertices of Q_6. There are 6 such representatives as given below:

$$\begin{aligned}
&x_1 = 0,\\
&x_1 + x_2 = 1,
\end{aligned}$$

$$\begin{aligned}
&x_1 + x_2 + x_3 = 1,\\
&x_1 + x_2 + x_3 + x_4 = 2,\\
&x_1 + x_2 + x_3 + x_4 + x_5 = 2,\\
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 3.
\end{aligned}$$

Clearly, two spanned hyperplanes of Q_n in the same equivalence class contain the same number of vertices of Q_n. So we may say that an equivalence class of spanned hyperplanes of Q_n contains k vertices of Q_n if every hyperplane in this class contains k vertices of Q_n.

To state the main result of this section, we need to define the equivalence classes of 0/1-polytopes contained in a set of points in $\mathbb{R}^n$. Given a set $\mathcal{S} \subset \mathbb{R}^n$, consider the set of 0/1-polytopes of Q_n that are contained in $\mathcal{S}$, denoted by $\mathcal{S}(Q_n)$. Restricting the 0/1-equivalence relation to the set $\mathcal{S}(Q_n)$ indicates a equivalence relation on $\mathcal{S}(Q_n)$. More precisely, two 0/1-polytopes in $\mathcal{S}(Q_n)$ are equivalent if one can be transformed to the other by a symmetry of Q_n. We call equivalence classes of 0/1-polytopes in $\mathcal{S}(Q_n)$ partial 0/1-equivalence classes of $\mathcal{S}$ for the reason that any partial equivalence class of $\mathcal{S}$ is a subset of a (unique) 0/1-equivalence class of Q_n. Notice that for a 0/1-polytope P contained in $\mathcal{S}$ and a symmetry w, $w(P)$ is not in the partial 0/1-equivalence class of P when $w(P)$ is not in $\mathcal{S}(Q_n)$. Denote by $\mathcal{P}(\mathcal{S}, k)$ the set of partial 0/1-equivalence classes of $\mathcal{S}$ with k vertices. Let $N_{\mathcal{S}}(k)$ be the cardinality of $\mathcal{P}(\mathcal{S}, k)$.

Let $h(n, k)$ denote the number of equivalence classes of spanned hyperplanes of Q_n that contain at least k vertices of Q_n. Assume that $H_1, H_1, \ldots, H_{h(n,k)}$ are the representatives of equivalence classes of spanned hyperplanes of Q_n containing at least k vertices of Q_n. Recall that $\mathcal{H}_n(k)$ denotes the set of 0/1-equivalence classes of Q_n with k vertices that are not full-dimensional. We shall define a map, denoted by Φ, from the (disjoint) union of $\mathcal{P}(H_i, k)$ for $1 \le i \le h(n, k)$ to $\mathcal{H}_n(k)$. Given a partial 0/1-equivalence class $\mathcal{P}_i \in \mathcal{P}(H_i, k)$ ($1 \le i \le h(n, k)$), then we define $\Phi(\mathcal{P}_i)$ to be the (unique) 0/1-equivalence class in $\mathcal{H}_n(k)$ containing $\mathcal{P}_i$. Then we have the following theorem.

Theorem 4.1 *If $2^{n-2} < k \le 2^{n-1}$, then the map Φ is a bijection.*

Proof. We proceed to show that Φ is injective. To this end, we shall prove that for any two distinct partial 0/1-equivalence classes $\mathcal{P}_1$ and $\mathcal{P}_2$ with k vertices, their images, denoted by $\mathcal{C}_1$ and $\mathcal{C}_2$, are distinct 0/1-equivalence classes. Assume that $\mathcal{P}_1 \in \mathcal{P}(H_i, k)$ and $\mathcal{P}_2 \in \mathcal{P}(H_j, k)$, where $1 \le i, j \le h(n, k)$. Let P_1 (resp. P_2) be any 0/1-polytope in $\mathcal{P}_1$ (resp. $\mathcal{P}_2$). Evidently, P_1 (resp. P_2) is a 0/1-polytope in $\mathcal{C}_1$ (resp. $\mathcal{C}_2$). To prove that $\mathcal{C}_1 \ne \mathcal{C}_2$, it suffices to show that P_1 and P_2 are not in the same 0/1-equivalence class. We have two cases.

Case 1. $i = j$. In this case, it is clear that P_1 and P_2 are not equivalent.

Case 2. $i \ne j$. Suppose to the contrary that P_1 and P_2 are in the same 0/1-equivalence class. Then there exists a symmetry $w \in B_n$ such that $w(P_1) = P_2$. Since $2^{n-2} < k \le 2^{n-1}$, by Theorem 3.1 we see that P_1 and P_2 are of dimension $n-1$. Since P_1 is contained in H_i, H_i coincides with the affine space spanned by P_1. Similarly, H_j is the affine space spanned by P_2. This implies that $w(H_i) = H_j$, contradicting the assumption that H_i and H_j belong to distinct equivalence classes of spanned hyperplanes of Q_n. Consequently, P_1 and P_2 are not in the same 0/1-equivalence class.

It remains to show that Φ is surjective. For any $\mathcal{C} \in \mathcal{H}_n(k)$, we aim to find a partial 0/1-equivalence class such that its image is $\mathcal{C}$. Let P be any 0/1-polytope in $\mathcal{C}$. Since P is not full-dimensional, we can find a spanned hyperplane H of Q_n such that P is contained in H. It follows that H contains at leat k vertices of Q_n. Thus there exists a representative H_j $(1 \le j \le h(n,k))$ such that H is in the equivalence class of H_j. Assume that $w(H) = H_j$ for some $w \in B_n$. Then $w(P)$ is contained in H_j. It is easily seen that under the map Φ, $\mathcal{C}_i$ is the image of the partial 0/1-equivalence class of H_j containing $w(P)$. Thus we conclude that the above map is a bijection. This completes the proof. ∎

It should also be noted that in the above proof of Theorem 4.1, the condition $2^{n-2} < k \le 2^{n-1}$ is required only in Case 2. When $k < 2^{n-2}$, the map Φ may be no longer an injection. For the case $2^{n-3} < k \le 2^{n-2}$, we will consider the computation of $H_n(k)$ in Section 8.

As a direct consequence of Theorem 4.1, we obtain that for $2^{n-2} < k \le 2^{n-1}$,

$$H_n(k) = \sum_{i=1}^{h(n,k)} N_{H_i}(k). \tag{4.2}$$

Thus, for $2^{n-2} < k \le 2^{n-1}$ the computation of $H_n(k)$ is reduced to the determination of the number $N_H(k)$ of partial 0/1-equivalence classes of H with k vertices. In the rest of this section, we shall explain how to compute $N_H(k)$.

For $2^{n-2} < k \le 2^{n-1}$, let H be a spanned hyperplane of Q_n containing at least k vertices. Let P and P' be two distinct 0/1-polytoeps of Q_n with k vertices that are contained in H. Assume that P and P' belong to the same partial 0/1-equivalence class of H. Then there exists a symmetry $w \in B_n$ such that $w(P) = P'$. It is clear from Theorem 3.1 that both P and P' have dimension $n-1$. Then H is the affine space spanned by P or P'. So we deduce that $w(H) = H$. Let

$$F(H) = \{w \in B_n \colon w(H) = H\}$$

be the stabilizer subgroup of H, namely, the subgroup of B_n that fixes H. So we have shown that P and P' belong to the same partial 0/1-equivalence class of H if and only if one can be transformed to the other by a symmetry in $F(H)$.

The above fact allows us to use Pólya's theorem to compute the number $N_H(k)$ for $2^{n-2} < k \le 2^{n-1}$. Denote by $V_n(H)$ the set of vertices of Q_n that are contained in H. Let us consider the action of $F(H)$ on $V_n(H)$. Assume that each vertex in $V_n(H)$ is assigned one of the two colors, say, black and white. For such a 2-coloring of the vertices in $V_n(H)$, consider the black vertices as vertices of a 0/1-polytope contained in H. Clearly, for $2^{n-2} < k \leq 2^{n-1}$, this establishes a one-to-one correspondence between partial 0/1-equivalence classes of H with k vertices and equivalence classes of 2-colorings of the vertices in $V_n(H)$ with k black vertices.

Write $Z_H(z)$ for the cycle index of $F(H)$, and let $C_H(u_1, u_2)$ denote the polynomial obtained from $Z_H(z)$ by substituting z_i with $u_1^i + u_2^i$.

Theorem 4.2 *Assume that $2^{n-2} < k \leq 2^{n-1}$, and let H be a spanned hyperplane of Q_n containing at least k vertices of Q_n. Then we have*

$$N_H(k) = \left[u_1^k u_2^{|V_n(H)|-k}\right] C_H(u_1, u_2).$$

We will compute the cycle index $Z_H(z)$ in Section 5 and Section 6. Section 5 is devoted to the characterization of the stabilizer $F(H)$. In Section 6, we will give an explicit expression for $Z_H(z)$.

5 The structure of the stabilizer $F(H)$

In this section, we aim to characterize the stabilizer $F(H)$ for a given spanned hyperplane H of Q_n.

Let

$$H\colon a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

be a spanned hyperplane of Q_n. Given $w \in B_n$, let $s(w)$ be the set of entries of w that are assigned the minus sign. In view of (2.4), it is easy to see that $w(H)$ is of the following form

$$\sum_{i \notin s(w)} a_{\pi(i)}x_i + \sum_{j \in s(w)} a_{\pi(j)}(1 - x_j) = b, \tag{5.1}$$

where π is the underlying permutation of w. The hyperplane $w(H)$ in (5.1) can be rewritten as

$$s(w,1) \cdot a_{\pi(1)}x_1 + s(w,2) \cdot a_{\pi(2)}x_2 + \cdots + s(w,n) \cdot a_{\pi(n)}x_n = b - \sum_{j \in s(w)} a_{\pi(j)}, \tag{5.2}$$

where $s(w,j) = -1$ if $j \in s(w)$ and $s(w,j) = 1$ otherwise.

As an example, let

$$H\colon x_1 - x_2 - x_3 + 2x_4 = 1$$

be a spanned hyperplane of Q_4. Upon the action of the symmetry $w = (1)(\bar{2}\bar{3})(4) \in B_4$, H is transformed into the following hyperplane

$$x_1 + x_2 + x_3 + 2x_4 = 3.$$

As mentioned in Section 4, for every equivalence class of spanned hyperplanes of Q_n, we can choose a representative of the following form

$$H\colon\ a_1x_1 + a_2x_2 + \cdots + a_tx_t = b, \tag{5.3}$$

where $a_1 \le a_2 \le \cdots \le a_t$ ($t \le n$), and the coefficients a_i's and b are positive integers. Note that this observation also follows from (5.2). From now on, we shall restrict our attention only to spanned hyperplanes of Q_n of the form as in (5.3). The following definition is required for the determination of $F(H)$.

Definition 5.1 *Let H be a spanned hyperplane of the form as in (5.3). The type of H is defined to be a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, where α_i is the multiplicity of i occurring in the set $\{a_1, a_2, \ldots, a_t\}$.*

For example, let

$$H\colon\ x_1 + x_2 + 2x_3 + 2x_4 + 3x_5 = 4 \tag{5.4}$$

be a spanned hyperplane of Q_5. Then, the type of H is $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (2, 2, 1)$.

For positive integers i and j such that $i \le j$, let $[i, j]$ denote the interval $\{i, i+1, \ldots, j\}$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be the type of a spanned hyperplane. Under the assumption that $\alpha_0 = 0$, the following set

$$\{[\alpha_1 + \cdots + \alpha_{i-1} + 1, \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i] : 1 \le i \le \ell\} \tag{5.5}$$

is a partition of the set $\{1, 2, \ldots, t\}$. For example, let $\alpha = (2, 2, 1)$. Then the corresponding partition is $\{\{1, 2\}, \{3, 4\}, \{5\}\}$.

Since (5.5) is a partition of $\{1, 2, \ldots, t\}$, we can define the corresponding Young subgroup S_α of the permutation group on $\{1, 2, \ldots, t\}$, namely,

$$S_\alpha = S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_\ell}, \tag{5.6}$$

where $\times$ denotes the direct product of groups, and for $i = 1, 2, \ldots, \ell$, S_{α_i} is the permutation group on the interval

$$[\alpha_1 + \cdots + \alpha_{i-1} + 1, \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i]\,. \tag{5.7}$$

Let

$$\overline{S}_\alpha = \overline{S}_{\alpha_1} \times \overline{S}_{\alpha_2} \times \cdots \times \overline{S}_{\alpha_\ell}, \tag{5.8}$$

where $\overline{S}_{\alpha_i}$ is the set of signed permutations on the interval (5.7) with all elements assigned the minus sign. Define

$$P(H) = \begin{cases} S_\alpha, & \text{if } \sum_{i=1}^t a_i \ne 2b, \\ S_\alpha \bigcup \overline{S}_\alpha, & \text{if } \sum_{i=1}^t a_i = 2b. \end{cases} \tag{5.9}$$

The following theorem gives a characterization of the stabilizer of a spanned hyperplane.

Theorem 5.2 *Let $H\colon a_1x_1 + a_2x_2 + \cdots + a_tx_t = b$ be a spanned hyperplane of Q_n. Then*

$$F(H) = P(H) \times B_{n,t},$$

where $B_{n,t}$ is the group of all signed permutations on the interval $[t+1, n]$.

Proof. Assume that $w \in F(H)$ and π is the underlying permutation of w. We aim to show that $w \in P(H) \times B_{n,t}$. Consider the expression of $w(H)$ as in (5.2), that is,

$$s(w,1)\cdot a_{\pi(1)}x_1 + s(w,2)\cdot a_{\pi(2)}x_2 + \cdots + s(w,n)\cdot a_{\pi(n)}x_n = b - \sum_{j \in s(w)} a_{\pi(j)}. \qquad (5.10)$$

We claim that $s(w,j)$ are either all positive or all negative for $1 \le j \le t$. Suppose otherwise that there exist $1 \le i, j \le t$ $(i \ne j)$ such that $s(w,i) > 0$ and $s(w,j) < 0$. Since the a_i's are all positive, we see that the coefficients $s(w,i)a_{\pi(i)}$ and $s(w,j)a_{\pi(j)}$ for the hyperplane $w(H)$ have opposite signs. This implies that $w(H)$ and H are distinct, which contradicts the assumption that w fixes H. We now have the following two cases.

Case 1. The signs $s(w,j)$ are all positive for $1 \le j \le t$. In this case, since $w(H) = H$ it is clear that $w(H)$ is of the following form

$$a_{\pi(1)}x_1 + a_{\pi(2)}x_2 + \cdots + a_{\pi(t)}x_t = b,$$

where $a_{\pi(j)} = a_j$ for $1 \le j \le t$. Hence we deduce that, for any $1 \le j \le t$, $\pi(j)$ is in the interval $[\alpha_1 + \cdots + \alpha_{i-1} + 1, \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i]$ that contains the element j. Thus we obtain that $w \in S_\alpha \times B_{n,t}$.

Case 2. The signs $s(w,j)$ are all negative for $1 \le j \le t$. In this case, we see that $w(H)$ is of the following form

$$-a_{\pi(1)}x_1 - a_{\pi(2)}x_2 - \cdots - a_{\pi(t)}x_t = b - (a_1 + \cdots + a_t).$$

Since $w(H) = H$, we have $a_{\pi(j)} = a_j$ for $1 \le j \le t$ and $b - (a_1 + \cdots + a_t) = -b$. Thus we obtain $w \in \overline{S}_\alpha \times B_{n,t}$. Combining the above two cases, we conclude that $w \in P(H) \times B_{n,t}$.

On the other hand, from the expression (5.10) for $w(H)$, it is not difficult to check that every symmetry w in $P(H) \times B_{n,t}$ fixes H. This completes the proof. ∎

As will been seen in Section 6, for the purpose of computing the cycle index $Z_H(z)$ with respect to a spanned hyperplane H of Q_n, it is often necessary to consider the structure of the subgroup $P(H)$ of $F(H)$. We sometimes write a symmetry $\pi \in P(H)$ as a product form $\pi = \pi_1\pi_2\cdots\pi_\ell$, which means that for $i = 1, 2\ldots, \ell$, $\pi_i \in S_{\alpha_i}$ if $\pi \in S_\alpha$, and $\pi_i \in \overline{S}_{\alpha_i}$ if $\pi \in \overline{S}_\alpha$, where α is the type of H. We conclude this section with the following proposition, which will be required for the computation of $Z_H(z)$ in Section 6.

Proposition 5.3 *Let H be a spanned hyperplane of Q_n of type α. Let $\pi = \pi_1\pi_2\cdots\pi_\ell$ and $\pi' = \pi'_1\pi'_2\cdots\pi'_\ell$ be two symmetries in $P(H)$, and assume that both π and π' are either in S_α or in $\overline{S}_\alpha$. If π_i and π'_i have the same cycle type for $1 \le i \le \ell$, then π and π' are in the same conjugacy class of $P(H)$.*

Proof. To prove that π and π' are conjugate in $P(H)$, it suffices to show that there exists a symmetry $w \in P(H)$ such that $\pi = w\pi' w^{-1}$. First, we consider the case when both π and π' are in S_α. Since π_i and π'_i are of the same cycle type, they are in the same conjugacy class. So there is a permutation $w_i \in S_{\alpha_i}$ such that $\pi_i = w_i \pi'_i w_i^{-1}$. It follows that $\pi = (w_1\pi'_1 w_1^{-1}) \cdots (w_\ell \pi'_\ell w_\ell^{-1}) = w\pi' w^{-1}$, where $w = w_1 \cdots w_\ell \in S_\alpha$. This implies that π and π' are conjugate in $P(H)$.

It remains to consider the case when both π and π' are in $\overline{S}_\alpha$. Let π_0 (resp. π'_0) be the underlying permutation of π (resp. π'). Then there is a symmetry $w \in S_\alpha$ such that $\pi_0 = w\pi'_0 w^{-1}$. We claim that $\pi = w\pi' w^{-1}$. Indeed, it is enough to show that $\pi(x_1, x_2, \ldots, x_t) = w\pi' w^{-1}(x_1, x_2, \ldots, x_t)$ for any point $(x_1, x_2, \ldots, x_t)$ in $\mathbb{R}^t$. Assume that $\pi(x_1, x_2, \ldots, x_t) = (y_1, y_2, \ldots, y_t)$ and $w\pi' w^{-1}(x_1, x_2, \ldots, x_t) = (z_1, z_2, \ldots, z_t)$. Since all elements of π are assigned the minus sign, we obtain from (2.4) that $y_i = 1 - x_{\pi_0(i)}$ for $1 \leq i \leq t$. On the other hand, using (2.4), it is not hard to check that $z_i = 1 - x_{w^{-1}\pi'_0 w(i)}$ for $1 \leq i \leq t$. Since $\pi_0 = w\pi'_0 w^{-1}$, we deduce that $\pi_0(i) = w^{-1}\pi'_0 w(i)$. Therefore, we have $y_i = z_i$ for $1 \leq i \leq t$. So the claim is justified. This completes the proof. ∎

6 The computation of $Z_H(z)$

In this section, we shall derive a formula for the cycle index $Z_H(z)$ for a spanned hyperplane H of Q_n. It turns out that $Z_H(z)$ depends only on the cycle structures of the symmetries in the subgroup $P(H)$ of $F(H)$.

Let

$$H\colon\ a_1x_1 + a_2x_2 + \cdots + a_tx_t = b \tag{6.1}$$

be a spanned hyperplane of Q_n. Recall that $V_n(H)$ is the set of vertices of Q_n contained in H. To compute the cycle index $Z_H(z)$, we need to determine the cycle structures of permutations on $V_n(H)$ induced by the symmetries in $F(H)$. By Theorem 5.2, each symmetry in $F(H)$ can be written uniquely as a product πw, where $\pi \in P(H)$ and $w \in B_{n,t}$. We shall define two group actions for the subgroups $P(H)$ and $B_{n,t}$, and shall derive an expression for the cycle type of the permutation on $V_n(H)$ induced by πw in terms of the cycle types of the permutations induced by π and w.

Let H be a spanned hyperplane of Q_n as in (6.1). However, to define the action for $P(H)$, we shall consider H as a hyperplane in $\mathbb{R}^t$. Denoted by $V_t(H)$ the set of vertices of Q_t that are contained in H, namely,

$$V_t(H) = \{(x_1, x_2, \ldots, x_t) \in V_t\colon a_1x_1 + a_2x_2 + \cdots + a_tx_t = b\}.$$

Since the vertices of Q_n contained in H span a hyperplane in $\mathbb{R}^n$, it can be seen that the vertices in $V_t(H)$ span a hyperplane in $\mathbb{R}^t$. Since H is considered as a hyperplane in $\mathbb{R}^t$, we deduce that H is a spanned hyperplane of Q_t. Setting $n = t$ in Theorem 5.2, it follows that the stabilizer of H is $P(H)$. Therefore, $P(H)$ stabilizes the set $V_t(H)$. So any symmetry in $P(H)$ induces a permutation on $V_t(H)$.

We also need an action of the group $B_{n,t}$ on the set of vertices of Q_{n-t}. Assume that $w \in B_{n,t}$, namely, w is a signed permutation on the interval $[t+1, n]$. Subtracting each element of w by t, we get a signed permutation on $[1, n-t]$. In this way, each signed permutation in $B_{n,t}$ corresponds to a symmetry of Q_{n-t}. Hence $B_{n,t}$ is isomorphic to the group B_{n-t} of symmetries of Q_{n-t}. This leads to an action of the group $B_{n,t}$ on V_{n-t}.

Let $\pi \in P(H)$ and $w \in B_{n,t}$. Recall that, for an element g in a group G acting on a finite set X, $c(g)$ denotes the cycle type of the permutation on X induced by g, which is written as a multiset $\{1^{c_1}, 2^{c_2}, \ldots\}$. In this notation, $c(\pi)$ (resp. $c(w)$) represents the cycle type of the permutation on $V_t(H)$ (resp. V_{n-t}) induced by π (resp. w). The following lemma gives an expression for the cycle type $c(w\pi)$ of the induced permutation of πw on $V_n(H)$ in terms of the cycle types $c(\pi)$ and $c(w)$.

Lemma 6.1 *Let $H\colon\ a_1x_1 + a_2x_2 + \cdots + a_tx_t = b$ be a spanned hyperplane of Q_n, and πw be a symmetry in $F(H)$, where $\pi \in P(H)$ and $w \in B_{n,t}$. Assume that $c(\pi) = \{1^{m_1}, 2^{m_2}, \ldots\}$ and $c(w) = \{1^{c_1}, 2^{c_2}, \ldots\}$. Then we have*

$$c(\pi w) = \biguplus_{i\geq 1} \biguplus_{j\geq 1} \left\{ (\mathrm{lcm}(i,j))^{\frac{ijm_ic_j}{\mathrm{lcm}(i,j)}} \right\}, \tag{6.2}$$

where $\biguplus$ denotes the disjoint union of multisets, and $\mathrm{lcm}(i,j)$ *denotes the least common multiple of integers i and j.*

Proof. Clearly, each vertex in $V_n(H)$ can be expressed as a vector of the following form

$$(x_1, \ldots, x_t, y_1, \ldots, y_{n-t}),$$

where $(x_1, \ldots, x_t)$ is a vertex in $V_t(H)$ and $(y_1, \ldots, y_{n-t})$ is a vertex of Q_{n-t}. Assume that $|V_t(H)| = n_0$. Let $V_t(H) = \{u_1, u_2, \ldots, u_{n_0}\}$ and $Q_{n-t} = \{v_1, v_2, \ldots, v_{2^{n-t}}\}$. Then each vertex in $V_n(H)$ can be expressed as an ordered pair (u_i, v_j), where $1 \leq i \leq n_0$ and $1 \leq j \leq 2^{n-t}$.

Let $C_i = (s_1, \ldots, s_i)$ be an i-cycle of the permutation on $V_t(H)$ induced by π, that is, C_i maps the vertex u_{s_k} to the vertex $u_{s_{k+1}}$ if $1 \leq k \leq i-1$, and to the vertex u_{s_1} if $k = i$. Similarly, let $C_j = (t_1, \ldots, t_j)$ be a j-cycle of the permutation on V_{n-t} induced by w, that is, C_j maps the vertex v_{t_m} to the vertex $v_{t_{m+1}}$ if $1 \leq m \leq j-1$, and to the vertex v_{t_1} if $m = j$. Define the direct product of C_i and C_j, denoted $C_i \times C_j$, to be the permutation on the subset $\{(u_{s_k}, v_{t_m})\colon 1 \leq k \leq i,\ 1 \leq m \leq j\}$ of $V_n(H)$ such that

$$C_i \times C_j(u_{s_k}, v_{t_m}) = (C_i(u_{s_k}), C_j(v_{t_m})).$$

It is not hard to check that the cycle type of $C_i \times C_j$ is

$$\left\{ (\mathrm{lcm}(i,j))^{\frac{ij}{\mathrm{lcm}(i,j)}} \right\}.$$

Note that the induced permutation of πw on $V_n(H)$ is the product of $C_i \times C_j$, where C_i (resp. C_j) runs over the cycles of the permutation on $V_t(H)$ (resp. V_{n-t}) induced by π (resp. w). Thus the cycle type of the induced permutation of πw on $V_n(H)$ is given by (6.2). This completes the proof. ∎

Before presenting a formula for the cycle index $Z_H(z)$, we need to introduce some notation. Assume that π is a symmetry in $P(H)$ such that the cycle type of the induced permutation of π is

$$c(\pi) = \{1^{m_1}, 2^{m_2}, \ldots\}.$$

For $j \geq 1$, we define

$$f_\pi(z_j) = \prod_{i \geq 1} (z_{\mathrm{lcm}(i,j)})^{\frac{ijm_i}{\mathrm{lcm}(i,j)}}. \tag{6.3}$$

Let

$$f_\pi(z) = (f_\pi(z_1), f_\pi(z_2), \ldots). \tag{6.4}$$

We have the following proposition.

Proposition 6.2 *Let H be a spanned hyperplane of Q_n with type α. Let $\pi = \pi_1\pi_2\cdots\pi_\ell$ and $\pi' = \pi'_1\pi'_2\cdots\pi'_\ell$ be two symmetries in $P(H)$. Assume that both π and π' are either in S_α or in $\overline{S}_\alpha$. If π_i and π'_i have the same cycle type for $1 \leq i \leq \ell$, then $f_\pi(z) = f_{\pi'}(z)$.*

Proof. It follows from Proposition 5.3 that π and π' are conjugate in $P(H)$. Since $P(H)$ acts on $V_t(H)$, the permutations on $V_t(H)$ induced by π and π' are conjugate. So they have the same cycle type, i.e., $c(\pi) = c(\pi')$. Since $f_\pi(z)$ depends only on $c(\pi)$, we see that $f_\pi(z) = f_{\pi'}(z)$. This completes the proof. ∎

We now give an overview of some notation related to integer partitions. We shall write a partition λ of a positive integer n, denoted by $\lambda \vdash n$, in the multiset form, that is, write $\lambda = \{1^{m_1}, 2^{m_2}, \ldots\}$, where m_i is the number of parts of λ of size i. Denote by $\ell(\lambda)$ the number of parts of λ, that is, $\ell(\lambda) = m_1 + m_2 + \cdots$. For a partition $\lambda = \{1^{m_1}, 2^{m_2}, \ldots\}$, let

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots.$$

For two partitions λ and μ, define $\lambda \cup \mu$ to be the partition obtained by joining the parts of λ and μ together. For example, for $\lambda = \{1, 2\}$ and $\mu = \{1^2, 3\}$, then $\lambda \cup \mu = \{1^3, 2, 3\}$.

Let $H\colon a_1x_1 + a_2x_2 + \cdots + a_tx_t = b$ be a spanned hyperplane of Q_n, whose type is $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Assume that $\mu = \mu^1 \cup \cdots \cup \mu^\ell$ is a partition of t, where $\mu^i \vdash \alpha_i$ for $1 \leq i \leq \ell$. We can write $f_\mu(z)$ (resp. $\overline{f}_\mu(z)$) for $f_\pi(z)$, where $\pi = \pi_1\pi_2\cdots\pi_\ell$ is any symmetry in S_α (resp. $\overline{S}_\alpha$) such that π_i has cycle type μ^i for $1 \leq i \leq \ell$. By Proposition 6.2, the functions $f_\mu(z)$ and $\overline{f}_\mu(z)$ are well defined. We can now give a formula for the cycle index $Z_H(z)$.

Theorem 6.3 *Let $H\colon a_1x_1+a_2x_2+\cdots+a_tx_t=b$ be a spanned hyperplane of Q_n. Assume that H has type $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_\ell)$. Then we have*

$$Z_H(z)=\frac{1}{2^{\delta(H)}}\sum_{(\mu^1,\ldots,\mu^\ell)}\prod_{i=1}^{\ell} z_{\mu^i}^{-1}\left(Z_{n-t}(f_\mu(z))+\delta(H)Z_{n-t}(\overline{f}_\mu(z))\right), \tag{6.5}$$

where $\mu^i\vdash\alpha_i$, $\mu=\mu^1\cup\cdots\cup\mu^\ell$, $\delta(H)=1$ if $\sum_{i=1}^t a_i=2b$ and $\delta(H)=0$ otherwise.

Proof. Let $\pi\in P(H)$ and $w\in B_{n,t}$. Assume that $c(w)=\{1^{c_1},2^{c_2},\ldots\}$. By Lemma 6.1, we have

$$z^{c(\pi\cdot w)}=f_\pi(z_1)^{c_1}f_\pi(z_2)^{c_2}\cdots. \tag{6.6}$$

From (2.1) and (6.6), we deduce that

$$\begin{aligned}\sum_{\pi w} z^{c(\pi\cdot w)} &= \sum_{w} f_\pi(z_1)^{c_1}f_\pi(z_2)^{c_2}\cdots\\ &=(n-t)!2^{n-t}Z_{n-t}(f_\pi(z_1),f_\pi(z_2),\ldots)\\ &=(n-t)!2^{n-t}Z_{n-t}(f_\pi(z)),\end{aligned}$$

where w runs over the signed permutations in $B_{n,t}$. Thus

$$\begin{aligned}Z_H(z)&=\frac{1}{|F(H)|}\sum_{\pi w\in F(H)} z^{c(\pi w)}\\ &=\frac{1}{|F(H)|}\sum_{\pi\in P(H)}(n-t)!2^{n-t}Z_{n-t}(f_\pi(z))\\ &=\frac{(n-t)!2^{n-t}}{|F(H)|}\left(\sum_{\pi\in S_\alpha}Z_{n-t}(f_\pi(z))+\delta(H)\sum_{\pi'\in\overline{S}_\alpha}Z_{n-t}(f_{\pi'}(z))\right),\end{aligned} \tag{6.7}$$

where $\delta(H)=1$ if $\sum_{i=1}^t a_i=2b$ and $\delta(H)=0$ otherwise.

Recall that for any given partition $\nu\vdash m$, there are $\frac{m!}{z_\nu}$ permutations on $\{1,2,\ldots,m\}$ such that their cycle type is ν, see Stanley [20, Proposition 1.3.2]. So the number of symmetries $\pi=\pi_1\pi_2\cdots\pi_\ell$ in S_α (or, $\overline{S}_\alpha$) such that for $i=1,2\ldots,\ell$, π_i has cycle type μ^i is equal to

$$\prod_{i=1}^{\ell}\frac{\alpha_i!}{z_{\mu^i}}. \tag{6.8}$$

Combining (6.7), (6.8) and Proposition 6.2, we obtain that

$$Z_H(z)=\frac{(n-t)!2^{n-t}}{|F(H)|}\sum_{(\mu^1,\ldots,\mu^\ell)}\prod_{i=1}^{\ell}\frac{\alpha_i!}{z_{\mu^i}}\left(Z_{n-t}(f_\mu(z))+\delta(H)Z_{n-t}(\overline{f}_\mu(z))\right), \tag{6.9}$$

where $\mu^i \vdash \alpha_i$, and $\mu = \mu^1 \cup \cdots \cup \mu^\ell$.

Since

$$|F(H)| = (n-t)!2^{n-t+\delta(H)} \prod_{i=1}^{\ell} \alpha_i!, \tag{6.10}$$

by substituting (6.10) into (6.9), we are led to (6.5). This completes the proof. ∎

By Theorem 6.3, the cycle index $Z_H(z)$ depends only on $f_\pi(z)$ for $\pi \in P(H)$. In view of (6.3), we see that $f_\pi(z)$ depends only on $c(\pi)$. Assume that $c(\pi) = \{1^{m_1}, 2^{m_2}, \ldots\}$. By Theorem 2.1, we have

$$m_i = \frac{1}{i} \sum_{j|i} \mu(i/j)\psi(\pi^j), \tag{6.11}$$

where $\psi(\pi^j)$ is the number of vertices in $V_t(H)$ that are fixed by π^j. The following theorem gives a formula for $\psi(\pi)$, from which $\psi(\pi^j)$ is easily determined.

Theorem 6.4 *Let $H\colon a_1x_1 + a_2x_2 + \cdots + a_tx_t = b$ be a spanned hyperplane of Q_n. Assume that $\pi = \pi_1\pi_2\cdots\pi_\ell$ is a symmetry in $P(H)$ such that π_i has cycle type $\mu^i = \{1^{m_{i1}}, 2^{m_{i2}}, \ldots\}$ for $i = 1, 2, \ldots, \ell$. Then*

$$\psi(\pi) = \begin{cases} \left[x^b\right] \prod_{i=1}^{\ell} \prod_{j\geq 1} (1 + x^{ij})^{m_{ij}}, & \text{if } \pi \in S_\alpha, \\ \chi(\mu)2^{\ell(\mu)}, & \text{if } \pi \in \overline{S}_\alpha, \end{cases} \tag{6.12}$$

where $\mu = \mu^1 \cup \cdots \cup \mu^\ell$, $\chi(\mu) = 1$ if μ has no odd parts and $\chi(\mu) = 0$ otherwise.

Before we present the proof of the above theorem, we need to define 0/1-labelings of a symmetry $\pi \in P(H)$ for the purpose of characterizing the vertices of Q_t fixed by π. Let π be a symmetry in $P(H)$. A 0/1-labeling of π is a labeling of the cycles of π such that each cycle of π is assigned one of the two numbers 0 and 1.

Proof of Theorem 6.4. We first consider the case when π is in S_α. It is easy to observe that, a vertex $v = (v_1, v_2, \ldots, v_t)$ of Q_t is a fixed point of π, that is, $\pi(v) = v$ if and only if, for each i-cycle $(j_1, j_2, \ldots, j_i)$ of π and for any entry of v corresponding to $(j_1, j_2, \ldots, j_i)$, we have

$$v_{j_1} = v_{j_2} = \cdots = v_{j_i}$$

(or, more precisely, $v_{j_1} = v_{j_2} = \cdots = v_{j_i} = 0$ or $v_{j_1} = v_{j_2} = \cdots = v_{j_i} = 1$). The above characterization enables us to establish a one-to-one correspondence between 0/1-labelings of π and the vertices of Q_t fixed by π, that is, for any given 0/1-labeling of π, we can define a vertex $v = (v_1, v_2, \ldots, v_t)$ of Q_t fixed by π such that $v_i = 0$ $(1 \leq i \leq t)$ if and only if the cycle of π containing i is assigned 0. Moreover, if the vertex $v = (v_1, v_2, \ldots, v_t)$ corresponding to a 0/1-labeling of π is in $V_t(H)$, that is, $a_1v_1 + a_2v_2 + \cdots + a_tv_t = b$, then we have

$$b_1 + 2b_2 + \cdots + \ell b_\ell = b, \tag{6.13}$$

where b_i ($1 \le i \le \ell$) is the sum of the lengths of cycles of π_i which are labeled 1. It can be easily deduced that the number of 0/1-labelings of π satisfying (6.13) is

$$\psi(\pi) = [x^b] \prod_{i=1}^{\ell} \prod_{j \ge 1} \left(1 + x^{ij}\right)^{m_{ij}} .$$

We now consider the case when π is in $\overline{S}_\alpha$. As in the previous case, it can be seen that a vertex $v = (v_1, v_2, \ldots, v_t)$ of Q_t is fixed by π if and only if, for any (signed) i-cycle $(\overline{j_1}, \overline{j_2}, \ldots, \overline{j_i})$ of π, the following relation holds

$$(v_{j_1}, v_{j_2}, \ldots, v_{j_i}) = (1 - v_{j_2}, 1 - v_{j_3}, \ldots, 1 - v_{j_1}). \tag{6.14}$$

Consequently, if a vertex $v = (v_1, v_2, \ldots, v_t)$ of Q_t is fixed by π, then, for any (signed) i-cycle $(\overline{j_1}, \overline{j_2}, \ldots, \overline{j_i})$ of π, the vector $(v_{j_1}, v_{j_2}, \ldots, v_{j_i})$ is either $(0, 1, \ldots, 0, 1)$ or $(1, 0, \ldots, 1, 0)$. This implies that π does not have any fixed point if π has an odd cycle.

We now assume that π has only even (signed) cycles. In this case, we see that the number of vertices of Q_t fixed by π is equal to $2^{\ell(\mu)}$. To prove $\psi(\pi) = 2^{\ell(\mu)}$, we need to demonstrate that any vertex of Q_t fixed by π is in $V_t(H)$. Let $v = (v_1, v_2, \ldots, v_t)$ be any vertex of Q_t fixed by π. Using the fact that for each (signed) cycle $(\overline{j_1}, \overline{j_2}, \ldots, \overline{j_i})$ of π, the vector $(v_{j_1}, \ldots, v_{j_i})$ is $(1, 0, \ldots, 1, 0)$ or $(0, 1, \ldots, 0, 1)$, and applying the relation

$$a_1 + \cdots + a_t = 2b,$$

we deduce that $a_1 v_1 + a_2 v_2 + \cdots + a_t v_t = b$. Hence the vertex v is in $V_t(H)$. This completes the proof. ∎

By Theorem 6.4, we can compute $\psi(\pi^j)$. Let $\pi = \pi_1 \pi_2 \cdots \pi_\ell \in P(H)$, where π_i has cycle type $\mu^i = \{1^{m_{i1}}, 2^{m_{i2}}, \ldots\}$. Clearly, $\pi^j = \pi_1^j \pi_2^j \cdots \pi_\ell^j$. Let $\gcd(i, j)$ denote the greatest common divisor of i and j. As is easily checked, the cycle type of π_i^j ($1 \le i \le \ell$) is

$$\left\{ 1^{m_{i1}}, \gcd(2, j)^{\frac{2m_{i2}}{\gcd(2,j)}}, \gcd(3, j)^{\frac{3m_{i3}}{\gcd(3,j)}}, \ldots \right\} .$$

To apply Theorem 6.4, it is still necessary to determine whether the symmetry π^j belongs to S_α or $\overline{S}_\alpha$. It can be seen that if π is in S_α or π is in $\overline{S}_\alpha$ and j is even, then π^j belongs to S_α. Similarly, if π is in $\overline{S}_\alpha$ and j is odd, then π^j belongs to $\overline{S}_\alpha$.

7 $F_n(k)$ for $n = 4, 5, 6$ and $2^{n-2} < k \le 2^{n-1}$

This section is devoted to the computation of $F_n(k)$ for $n = 4, 5, 6$ and $2^{n-2} < k \le 2^{n-1}$. This requires the cycle indices $Z_H(z)$ for spanned hyperplanes of Q_n for $n = 4, 5, 6$ that contain more than 2^{n-2} vertices of Q_n.

Let $H_1, H_2, \ldots, H_{h(n,k)}$ be the representatives of equivalence classes of spanned hyperplanes of Q_n containing at least k vertices. When $2^{n-2} < k \leq 2^{n-1}$, combining relation (1.1), Theorem 4.1 and Theorem 4.2, we deduce that

$$\begin{aligned} F_n(k) &= A_n(k) - H_n(k) \\ &= A_n(k) - \sum_{i=1}^{h(n,k)} N_{H_i}(k) \\ &= A_n(k) - \sum_{i=1}^{h(n,k)} \left[u_1^k u_2^{|V_n(H_i)|-k}\right] C_{H_i}(z_1, z_2). \end{aligned} \tag{7.1}$$

We start with the computation of $F_4(k)$ for $k = 5, 6, 7, 8$. Observing that $F_4(k) = 0$ for $k < 5$, this gives the enumeration of full-dimensional 0/1-equivalence classes of Q_4. For brevity, we use H_n^t $(t \leq n)$ to denote the following hyperplane in $\mathbb{R}^n$

$$x_1 + x_2 + \cdots + x_t = \lfloor t/2 \rfloor .$$

In this notation, representatives of equivalence classes of spanned hyperplanes of Q_4 containing more than 4 vertices of Q_4 are as follows

$$\begin{aligned} &H_4^1\colon x_1 = 0, \\ &H_4^2\colon x_1 + x_2 = 1, \\ &H_4^3\colon x_1 + x_2 + x_3 = 1, \\ &H_4^4\colon x_1 + x_2 + x_3 + x_4 = 2. \end{aligned}$$

Employing the techniques in Section 6, we obtain the cycle indices $Z_{H_4^1}(z)$ and $Z_{H_4^2}(z)$ as given below:

$$\begin{aligned} Z_{H_4^1}(z) &= Z_3(z), \\ Z_{H_4^2}(z) &= \frac{1}{16}\left(9z_2^4 + 4z_4^2 + 2z_1^4 z_2^2 + z_1^8 \right). \end{aligned}$$

For the remaining two hyperplanes $H = H_4^3$ and H_4^4, it can be checked that $N_H(k) = 1$ for $k = 5, 6$. Thus, from (7.1) we can determine $F_4(k)$ for $k = 5, 6, 7, 8$. These values are given in Table 4, which agree with the results computed by Aichholzer [1].

We now compute $F_5(k)$ for $8 < k \leq 16$. Representatives of equivalence classes of spanned hyperplanes of Q_5 containing more than 8 vertices of Q_5 are $H_5^1, H_5^2, H_5^3, H_5^4, H_5^5$. By utilizing the the techniques in Section 6, we obtain that

$$\begin{aligned} Z_{H_5^1}(z) &= Z_4(z), \\ Z_{H_5^2}(z) &= \frac{1}{96}\left(z_1^{16} + 6z_1^8 z_2^4 + 33z_2^8 + 8z_1^4 z_3^4 + 24z_4^4 + 24z_2^2 z_6^2 \right), \end{aligned}$$

k	5	6	7	8
H_4^1	3	3	1	1
H_4^2	5	5	1	1
H_4^3	1	1		
H_4^4	1	1		
$F_4(k)$	17	40	54	72

Table 4: $F_4(k)$ for $k = 5, 6, 7, 8$.

$$Z_{H_5^3}(z) = \frac{1}{48}\left(12z_2^6 + 8z_4^3 + 2z_1^6 z_2^3 + z_1^{12} + 6z_1^2 z_2^5 + 3z_1^4 z_2^4 + 6z_6^2 + 4z_{12} + 4z_3^2 z_6 + 2z_3^4 \right),$$

$$Z_{H_5^4}(z) = \frac{1}{96}\left(z_1^{12} + 27z_2^6 + 9z_1^4 z_2^4 + 8z_3^4 + 24z_6^2 + 18z_2^2 z_4^2 + 6z_1^4 z_4^2 + 3z_1^8 z_2^2 \right),$$

$$Z_{H_5^5}(z) = \frac{1}{120}\left(24z_5^2 + 30z_2 z_4^2 + 20z_1 z_3 z_6 + 20z_1 z_3^3 + 15z_1^2 z_2^4 + 10z_1^4 z_2^3 + z_1^{10} \right).$$

Consequently, the values $F_5(k)$ for $8 < k \leq 16$ can be derived from (7.1), and they agree with the results of Aichholzer [1], see Table 5.

k	9	10	11	12	13	14	15	16
H_5^1	56	50	27	19	6	4	1	1
H_5^2	159	135	68	43	12	7	1	1
H_5^3	9	5	1	1				
H_5^4	7	5	1	1				
H_5^5	1	1						
$F_5(k)$	8781	19767	37976	65600	98786	133565	158656	159110

Table 5: $F_5(k)$ for $8 < k \leq 16$.

The main objective of this section is to compute $F_6(k)$ for $16 < k \leq 32$. As mentioned in Section 4, there are 6 representatives of equivalence classes of spanned hyperplanes of Q_6 containing more than 16 vertices of Q_6, i.e., $H_6^1, H_6^2, H_6^3, H_6^4, H_6^5, H_6^6$. Again, by applying the techniques in Section 6, we obtain that

$$Z_{H_6^1}(z) = Z_5(z),$$

$$Z_{H_6^2}(z) = \frac{1}{768}\left(\begin{array}{l} z_1^{32} + 12z_1^{16} z_2^8 + 12z_1^8 z_2^{12} + 127z_2^{16} + 32z_1^8 z_3^8 + \\ 48z_1^4 z_2^2 z_4^6 + 168z_4^8 + 224z_2^4 z_6^4 + 96z_8^4 + 48z_2^4 z_4^6 \end{array} \right),$$

$$Z_{H_6^3}(z) = \frac{1}{288}\left(\begin{array}{l} z_1^{24} + 6z_1^{12} z_2^6 + 52z_2^{12} + 18z_3^8 + 48z_4^6 + 32z_2^3 z_6^3 + 3z_1^8 z_2^8 + \\ 18z_1^4 z_2^{10} + 24z_1^2 z_3^2 z_2^2 z_6^2 + 8z_1^6 z_3^6 + 12z_3^4 z_6^2 + 42z_6^4 + 24z_{12}^2 \end{array} \right),$$

$$Z_{H_6^4}(z) = \frac{1}{384}\left(\begin{array}{l} z_1^{24} + 81z_2^{12} + +2z_1^{12}z_2^6 + 18z_1^4z_2^{10} + 15z_1^8z_2^8 + 72z_6^4 + 32z_{12}^2 \\ 64z_4^6 + 16z_3^4z_6^2 + 8z_3^8 + 54z_2^4z_4^4 + 12z_1^4z_2^2z_4^4 + 6z_1^8z_4^4 + 3z_1^{16}z_2^4 \end{array}\right),$$

$$Z_{H_6^5}(z) = \frac{1}{240}\left(\begin{array}{l} z_1^{20} + 24z_{10}^2 + 60z_2^2z_4^4 + 26z_2^{10} + 20z_1^2z_3^2z_6^2 + \\ 20z_1^2z_3^6 + 15z_1^4z_2^8 + 10z_1^8z_2^6 + 40z_2z_6^3 + 24z_5^4 \end{array}\right),$$

$$Z_{H_6^6}(z) = \frac{1}{1440}\left(\begin{array}{l} z_1^{20} + 144z_5^4 + 144z_{10}^2 + 320z_2z_6^3 + 270z_2^2z_4^4 + 76z_2^{10} \\ +90z_1^4z_4^4 + 30z_1^8z_2^6 + 45z_1^4z_2^8 + 240z_1^2z_3^2z_6^2 + 80z_1^2z_3^6 \end{array}\right).$$

Based on relation (7.1), we can compute $F_6(k)$ for $16 < k \leq 32$. These values are listed in Table 6.

	H_6^1	H_6^2	H_6^3	H_6^4	H_6^5	H_6^6	$F_6(k)$
17	158658	767103	1464	1334	12	5	30063520396
18	133576	642880	657	630	5	3	78408664654
19	98804	474635	220	216	1	1	189678190615
20	65664	312295	81	86	1	1	426539396250
21	38073	179829	19	20			893345853436
22	19963	92309	7	8			1745593621167
23	9013	40948	1	1			3186944223591
24	3779	16335	1	1			5443544457875
25	1326	5500					8708686176141
26	472	1753					13061946974320
27	131	441					18382330104124
28	47	129					24289841497705
29	10	23					30151914536900
30	5	9					35176482187384
31	1	1					38580161986424
32	1	1					39785643746724

Table 6: $F_6(k)$ for $16 < k \leq 32$.

8 $H_n(k)$ for $2^{n-3} < k \leq 2^{n-2}$

In this section, we shall present an approach for computing $H_n(k)$ for $2^{n-3} < k \leq 2^{n-2}$. This enables us to determine $F_6(k)$ for $k = 13, 14, 15, 16$. Together with the computation of Aichholzer up to 12 vertices for $n = 6$, we have completed the enumeration of full-dimensional 0/1-equivalence classes of the 6-dimensional hypercube.

Let us recall the map Φ defined in Section 4, which will be used in the computation of $H_n(k)$ for $2^{n-3} < k \leq 2^{n-2}$. Let $H_1, H_2, \ldots, H_{h(n,k)}$ be the representatives of equivalence classes of spanned hyperplanes of Q_n containing at least k vertices. As before, denote by $\mathcal{P}(H_i, k)$ $(1 \leq i \leq h(n,k))$ the set of partial 0/1-equivalence classes of H_i with k vertices. Let $\mathcal{P}_i$ be a partial 0/1-equivalence class in $\mathcal{P}(H_i, k)$ $(1 \leq i \leq h(n,k))$. So Φ maps $\mathcal{P}_i$ to the (unique) 0/1-equivalence class in $\mathcal{H}_n(k)$ containing $\mathcal{P}_i$. When $2^{n-2} < k \leq 2^{n-1}$, it has been shown in Theorem 4.1 that Φ is a bijection. However, as pointed out after the proof of Theorem 4.1, when $k \leq 2^{n-2}$, Φ is surjective but not necessarily injective.

For the purpose of computing $H_n(k)$ for $2^{n-3} < k \leq 2^{n-2}$, we shall first derive an expression for $H_n(k)$, which is valid for general k. Let $1 \leq i \leq h(n,k)$, and define

$$A_i = \Phi(\mathcal{P}(H_i, k)).$$

Since Φ is surjective, we see that

$$\mathcal{H}_n(k) = A_1 \cap A_2 \cup \cdots \cup A_{h(n,k)}.$$

It follows from the principle of inclusion-exclusion that

$$\begin{aligned} H_n(k) = & \sum_{1 \leq i \leq h(n,k)} |A_i| - \sum_{1 \leq i_1 < i_2 \leq h(n,k)} |A_{i_1} \cap A_{i_2}| \\ & + \sum_{1 \leq i_1 < i_2 < i_3 \leq h(n,k)} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \cdots . \end{aligned} \tag{8.1}$$

Hence the task of computing $H_n(k)$ reduces to evaluating $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}|$ for $1 \leq i_1 < i_2 < \cdots < i_m \leq h(n,k)$.

Assume that $2^{n-3} < k \leq 2^{n-2}$. In what follows, we shall focus on the computation of the cardinalities of A_i for $1 \leq i \leq h(n,k)$, and the cardinalities of $A_i \cap A_j$ for $1 \leq i < j \leq h(n,k)$. The computation for the cardinalities of $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}$ in the general case can be carried out in the same way. When $n = 6$ and $k = 13, 14, 15, 16$, the computations turn out to be quite simple.

We first compute $|A_i|$ $(1 \leq i \leq h(n,k))$ for $2^{n-3} < k \leq 2^{n-2}$. Since $A_i = \Phi(\mathcal{P}(H_i, k))$, we have $|A_i| = |\mathcal{P}(H_i, k)|$. Recall that $|\mathcal{P}(H_i, k)|$ is defined as $N_{H_i}(k)$ in Section 4 and has been computed for the case $2^{n-2} < k \leq 2^{n-1}$. To compute $N_{H_i}(k)$ for $2^{n-3} < k \leq 2^{n-2}$, we need some notation.

Let H be a spanned hyperplane of Q_n, and $\mathcal{S}$ be a subset of H. Recall that $\mathcal{S}(Q_n)$ is the set of 0/1-polytopes of Q_n contained in $\mathcal{S}$. In Section 4, we defined the partial 0/1-equivalence relation on $\mathcal{S}(Q_n)$. Here we need introduce another equivalence relation on $\mathcal{S}(Q_n)$, that is, two 0/1-polytopes in $\mathcal{S}(Q_n)$ are said to be equivalent if one can be transformed to the other by a symmetry in $F(H)$. The associated equivalence classes in $\mathcal{S}(Q_n)$ are called local 0/1-equivalence classes of $\mathcal{S}$. Since $F(H)$ is a subgroup of B_n, each local 0/1-equivalence class of $\mathcal{S}$ is contained in a (unique) partial 0/1-equivalence class of $\mathcal{S}$.

Denote by $\mathcal{L}(\mathcal{S}, k)$ the set of local 0/1-equivalence classes of $\mathcal{S}$ with k vertices. When $\mathcal{S} = H$, $\mathcal{L}(H, k)$ has appeared in Section 4, that is, $\mathcal{L}(H, k)$ is the set of equivalence classes of 0/1-polytopes contained in H with k vertices under the action of $F(H)$. So we have the following relation

$$|\mathcal{L}(H, k)| = \left[u_1^k u_2^{|V_n(H)-k|}\right] C_H(u_1, u_2). \tag{8.2}$$

In order to compute $N_H(k)$ for $2^{n-3} < k \leq 2^{n-2}$, we shall define a partition of $\mathcal{L}(H, k)$ into two subsets $\mathcal{L}_*(H, k)$ and $\mathcal{L}^*(H, k)$. This requires a property as given in Theorem 8.1.

Let H be a spanned hyperplane of Q_n containing at least k vertices of Q_n. Denote by $E(H, k)$ the set of intersections $H \cap w(H)$ such that

(1). The symmetry w of Q_n does not fix H, that is, $H \neq w(H)$;

(2). The intersection $H \cap w(H)$ contains at least k vertices of Q_n.

Denote by $h_1(H, k)$ the number of equivalence classes of $E(H, k)$ under the symmetries in $F(H)$. Let $E_1(H, k) = \{H \cap w_i(H) \colon 1 \leq i \leq h_1(H, k)\}$ be the set of representatives of these equivalence classes of $E(H, k)$.

Consider the (disjoint) union of $\mathcal{L}(H \cap w_i(H), k)$, where $1 \leq i \leq h_1(H, k)$. We shall define a map Φ_1 from this union to $\mathcal{L}(H, k)$. For $1 \leq i \leq h_1(H, k)$, let $\mathcal{L}_i$ be a local 0/1-equivalence class in $\mathcal{L}(H \cap w_i(H), k)$. Evidently, there is a (unique) local 0/1-equivalence class in $\mathcal{L}(H, k)$ containing $\mathcal{L}_i$, denoted $\mathcal{L}'_i$. Define $\Phi_1(\mathcal{L}_i) = \mathcal{L}'_i$. Then we have the following property.

Theorem 8.1 *If $2^{n-3} < k \leq 2^{n-2}$, then the map Φ_1 is an injection.*

Proof. Let $\mathcal{L}$ and $\mathcal{L}'$ be two distinct local 0/1-equivalence classes with k vertices. Assume that $\mathcal{L}$ (resp. $\mathcal{L}'$) is in $\mathcal{L}(H \cap w_i(H), k)$ (resp. $\mathcal{L}(H \cap w_j(H), k)$), where $1 \leq i, j \leq h_1(H, k)$. To prove that Φ_1 is an injection, we need to show that $\Phi_1(\mathcal{L}) \neq \Phi_1(\mathcal{L}')$. Clearly, if $i = j$ then we see that $\Phi_1(\mathcal{L}) \neq \Phi_1(\mathcal{L}')$. We now consider the case $i \neq j$.

Assume to the contrary that $\Phi_1(\mathcal{L}) = \Phi_1(\mathcal{L}')$. Let P (resp. P') be any 0/1-polytope in $\mathcal{L}$ (resp. $\mathcal{L}'$). Then there is a symmetry $w \in F(H)$ such that $P = w(P')$. Since both P and P' have more than 2^{n-3} vertices of Q_n, we see from Theorem 3.1 that $\dim(P) = \dim(P') \geq n - 2$. Since P (resp. P') is contained in $H \cap w_i(H)$ (resp. $H \cap w_j(H)$), both P and P' are of dimension $n - 2$. This implies that $H \cap w_i(H)$ (resp. $H \cap w_j(H)$) is the affine space spanned by P (resp. P'). Hence we deduce that $H \cap w_i(H) = w(H \cap w_j(H))$, which is contrary to the assumption that $H \cap w_i(H)$ and $H \cap w_j(H)$ are not equivalent under the symmetries in $F(H)$. This completes the proof. ∎

We are now ready to define $\mathcal{L}_*(H,k)$ to be the image of Φ_1. More precisely, $\mathcal{L}_*(H,k)$ is the (disjoint) union of $\Phi_1(\mathcal{L}(H\cap w_i(H),k))$, where $1\le i\le h_1(H,k)$. Let

$$\mathcal{L}^*(H,k)=\mathcal{L}(H,k)\backslash\mathcal{L}_*(H,k). \tag{8.3}$$

From the above definition (8.3), it can be seen that, for any local 0/1-equivalence class $\mathcal{L}\in\mathcal{L}^*(H,k)$ and any 0/1-polytope $P\in\mathcal{L}$, if $w\in B_n$ is a symmetry such that $w(P)$ is contained in H, then $w(H)=H$. This yields that $\mathcal{L}$ is also a partial 0/1-equivalence class of H. Consequently, $\mathcal{L}^*(H,k)$ is a subset of $\mathcal{P}(H,k)$. Let

$$\mathcal{P}_*(H,k)=\mathcal{P}(H,k)\backslash\mathcal{L}^*(H,k). \tag{8.4}$$

Combining (8.2), (8.3) and (8.4), we find that

$$\begin{aligned}
N_H(k)&=|\mathcal{P}(H,k)|\\
&=|\mathcal{L}^*(H,k)|+|\mathcal{P}_*(H,k)|\\
&=|\mathcal{L}(H,k)|-|\mathcal{L}_*(H,k)|+|\mathcal{P}_*(H,k)|\\
&=\left[u_1^k u_2^{|V_n(H)-k|}\right]C_H(u_1,u_2)-|\mathcal{L}_*(H,k)|+|\mathcal{P}_*(H,k)|.
\end{aligned} \tag{8.5}$$

Therefore, for $2^{n-3}<k\le 2^{n-2}$ $N_H(k)$ is determined by the cardinalities of $\mathcal{L}_*(H,k)$ and $\mathcal{P}_*(H,k)$. From Theorem 8.1, we see that for $2^{n-3}<k\le 2^{n-2}$, $|\mathcal{L}_*(H,k)|$ can be derived from the cardinalities of $\mathcal{L}(H\cap w(H),k)$, where $H\cap w(H)\in E_1(H,k)$. We shall demonstrate that the computation of $|\mathcal{P}_*(H,k)|$ for $2^{n-3}<k\le 2^{n-2}$ can be carried out in a similar fashion.

Denote by $h_2(H,k)$ the number of equivalence classes of $E(H,k)$ under the symmetries of Q_n. Let

$$E_2(H,k)=\{H\cap w_i(H)\colon 1\le i\le h_2(H,k)\}$$

be the set of representatives of these equivalence classes of $E(H,k)$. We define a map Φ_2 from the (disjoint) union of $\mathcal{P}(H\cap w_i(H),k)$, where $1\le i\le h_2(H,k)$, to $\mathcal{P}_*(H,k)$. Let $\mathcal{P}$ be a partial 0/1-equivalence class of $\mathcal{P}(H\cap w_i(H),k)$ $(1\le i\le h_2(H,k))$. Then the image $\Phi_2(\mathcal{P})$ is defined to be the (unique) partial 0/1-equivalence class of $\mathcal{P}_*(H,k)$ that contains $\mathcal{P}$. We reach the following assertion. The proof is similar to that of Theorem 8.1, hence it is omitted.

Theorem 8.2 *If $2^{n-3}<k\le 2^{n-2}$, then the map Φ_2 is a bijection.*

So far, we see that the number $N_H(k)$ for $2^{n-3}<k\le 2^{n-2}$ can be computed based on the cardinalities of $\mathcal{L}(H\cap w(H),k)$ and $\mathcal{P}(H\cap w(H),k)$, where $H\cap w(H)\in E(H,k)$. We shall illustrate how to compute $|\mathcal{L}(H\cap w(H),k)|$ and $|\mathcal{P}(H\cap w(H),k)|$ for $2^{n-3}<k\le 2^{n-2}$.

Assume that $H\cap w(H)\in E(H,k)$. Let P and P' be any two 0/1-polytopes belonging to the same local (resp. partial) 0/1-equivalence class of $H\cap w(H)$ with k vertices. Then there exists a symmetry in $F(H)$ (resp. B_n) such that $w(P) = P'$. It is clear from Theorem 3.1 that both P and P' have dimension $n-2$. Hence $H \cap w(H)$ is the affine space spanned by P, or, equivalently, by P'. So we deduce that $w(H \cap w(H)) = H \cap w(H)$. This implies that for $2^{n-3} < k \le 2^{n-2}$, we can use Pólya's theorem to compute the number of local (resp. partial) 0/1-equivalence classes of $H \cap w(H)$ with k vertices.

Let

$$F(H, w) = \{w' \in F(H)\colon\ w'(H \cap w(H)) = H \cap w(H)\}$$

and

$$F(H \cap w(H)) = \{w' \in B_n\colon\ w'(H \cap w(H)) = H \cap w(H)\}.$$

Denote by $V_n(H \cap w(H))$ the set of vertices of Q_n contained in $H \cap w(H)$, and denote by $Z_{(H,w)}(z)$ (resp. $Z_{H\cap w(H)}(z)$) the cycle index of $F(H,w)$ (resp. $F(H \cap w(H))$) acting on $V_n(H \cap w(H))$. Write $C_{(H,w)}(u_1, u_2)$ (resp. $C_{H\cap w(H)}(u_1, u_2)$) for the polynomial obtained from $Z_{(H,w)}(z)$ (resp. $Z_{H\cap w(H)}(z)$) by substituting z_i with $u_1^i + u_2^i$. Thus, for $2^{n-3} < k \le 2^{n-2}$, we obtain that

$$|\mathcal{L}(H \cap w(H), k)| = \left[u_1^k u_2^{|V_n(H\cap w(H))|-k}\right] C_{(H,w)}(u_1, u_2) \tag{8.6}$$

and

$$|\mathcal{P}(H \cap w(H), k)| = \left[u_1^k u_2^{|V_n(H\cap w(H))|-k}\right] C_{H\cap w(H)}(u_1, u_2). \tag{8.7}$$

Thus, applying and Theorems 8.1 and 8.2 and plugging the above formulas (8.6) and (8.7) into (8.5), we arrive at the following relation.

Theorem 8.3 *Let $2^{n-3} < k \le 2^{n-2}$, and H be a spanned hyperplane of Q_n containing at least k vertices of Q_n. Set $q(w) = |V_n(H \cap w(H))|$. Then we have*

$$\begin{aligned} N_H(k) = \left[u_1^k u_2^{|V_n(H)|-k}\right] C_H(u_1, u_2) - \sum_{H\cap w(H) \in E_1(H,k)} \left[u_1^k u_2^{q(w)-k}\right] C_{(H,w)}(u_1, u_2) \\ + \sum_{H\cap w(H) \in E_2(H,k)} \left[u_1^k u_2^{q(w)-k}\right] C_{H\cap w(H)}(u_1, u_2). \end{aligned} \tag{8.8}$$

Theorem 8.3 enables us to compute $N_H(k)$ for $k = 13, 14, 15, 16$, where H is a spanned hyperplane of Q_6 containing more than 12 vertices of Q_6. In addition to $H_6^1, H_6^2, H_6^3, H_6^4, H_6^5, H_6^6$, we have 8 representatives of equivalence classes of spanned hyperplanes of Q_6 containing more than 12 vertices of Q_6, namely,

$$H_1\colon x_1 + x_2 + x_3 + 2x_4 = 2,$$

$$H_2\colon x_1 + x_2 + x_3 + x_4 = 1,$$

$$H_3\colon x_1 + x_2 + x_3 + x_4 + 2x_5 = 3,$$

$$H_4\colon x_1+x_2+x_3+x_4+x_5+2x_6=3,$$
$$H_5\colon x_1+x_2+x_3+x_4+x_5+x_6=2,$$
$$H_6\colon x_1+x_2+x_3+x_4+2x_5=2,$$
$$H_7\colon x_1+x_2+x_3+2x_4+2x_5=3,$$
$$H_8\colon x_1+x_2+x_3+x_4+2x_5+2x_6=4.$$

It is easily checked that $E(H,k)=\emptyset$ for $k=13,14,15,16$, except for the two spanned hyperplanes H_6^1 and H_6^2. Therefore, we can deduce from Theorem 8.3 that

$$N_H(k)=\left[u_1^k u_2^{V_n(H)-k}\right] C_H(u_1,u_2), \tag{8.9}$$

where $H=H_6^3-H_6^6, H_1-H_8$. The cycle indices for $H=H_6^3-H_6^6$ have been given in Section 7. For $H=H_6$, H_7 and H_8, it is easily verified that $N_H(13)=2$ and $N_H(14)=1$. Using the techniques in Section 6, we can derive the cycle indices for H_1-H_5 as shown below:

$$Z_{H_1}(z)=\frac{1}{48}\begin{pmatrix} z_1^{16}+4z_{12}z_4+4z_3^2z_6z_1^2z_2+2z_3^4z_1^4+ \\ 12z_2^8+8z_4^4+6z_1^4z_2^6+5z_1^8z_2^4+6z_6^2z_2^2 \end{pmatrix},$$

$$Z_{H_2}(z)=\frac{1}{192}\begin{pmatrix} z_1^{16}+68z_4^4+24z_6^2z_2^2+16z_{12}z_4+8z_3^4z_1^4 \\ +39z_2^8+12z_1^4z_2^6+8z_1^8z_2^4+16z_3^2z_6z_1^2z_2 \end{pmatrix},$$

$$Z_{H_3}(z)=\frac{1}{96}\left(\ z_1^{16}+24z_6^2z_2^2+8z_3^4z_1^4+33z_2^8+6z_1^8z_2^4+24z_4^4\ \right),$$

$$Z_{H_4}(z)=\frac{1}{120}\left(\ z_1^{15}+24z_5^3+30z_2z_4^3z_1+20z_1z_3^2z_6z_2+20z_1^3z_3^4+15z_1^3z_2^6+10z_1^7z_2^4\ \right),$$

$$Z_{H_5}(z)=\frac{1}{720}\begin{pmatrix} z_1^{15}+120z_3z_6^2+144z_5^3+40z_3^5+180z_1z_2z_4^3 \\ +40z_1^3z_3^4+60z_1^3z_2^6+15z_1^7z_2^4+120z_1z_2z_3^2z_6 \end{pmatrix}.$$

It remains to compute $N_H(k)$ for $H=H_6^1$ and H_6^2 for $k=13,14,15,16$. For $H_6^1\colon x_1=0$ and $k=13,14,15,16$, it is routine to check that

$$E_1(H_6^1,k)=E_2(H_6^1,k)=\left\{H_6^1\cap w(H_6^1)\colon w=(1,2)(3)(4)(5)(6)\right\},$$

that is,

$$\{(x_1,\ldots,x_6)\colon x_1=0,x_2=0\}.$$

Thus, for $k=13,14,15,16$, it is clear that both the numbers of local and partial 0/1-equivalence classes of $H_6^1\cap w(H_6^1)$ with k vertices are given by

$$\left[u_1^k u_2^{16-k}\right] C_4(u_1,u_2).$$

Therefore, for $k = 13, 14, 15, 16$, by Theorem 8.3 we find that

$$N_{H_6^1}(k) = \left[u_1^k u_2^{32-k}\right] C_{H_6^1}(u_1, u_2). \tag{8.10}$$

Finally, we come to the computation of $N_{H_6^2}(k)$ for $k = 13, 14, 15, 16$. In this case, it is easy to check that

$$E_1(H_6^2, k) = E_2(H_6^2, k) = \{H_6^2 \cap w_1(H_6^2), H_6^2 \cap w_2(H_6^2)\},$$

where $w_1 = (1, 3, 2)(4)(5)(6)$ and $w_2 = (1, 3)(2, 4)(5)(6)$. Since

$$\begin{aligned} V_6(H_6^2 \cap w_1(H_6^2)) =& \{(1, 0, 1, x_4, x_5, x_6) \colon x_i = 0 \text{ or } 1 \text{ for } i = 4, 5, 6\} \cup \\ & \{(0, 1, 0, x_4, x_5, x_6) \colon x_i = 0 \text{ or } 1 \text{ for } i = 4, 5, 6\}, \end{aligned}$$

it can be easily checked that $\mathcal{L}(H_6^2 \cap w_1(H_6^2), k) = \mathcal{P}(H_6^2 \cap w_1(H_6^2), k)$ for $k = 13, 14, 15, 16$. By Theorem 8.3, we obtain that for $k = 13, 14, 15, 16$,

$$\begin{aligned} N_{H_6^2} =& \left[u_1^k u_2^{32-k}\right] C_{H_6^2}(u_1, u_2) - \left[u_1^k u_2^{16-k}\right] C_{(H_6^2, w_2)}(u_1, u_2) \\ &+ \left[u_1^k u_2^{16-k}\right] C_{H_6^2 \cap w_2(H_6^2)}(u_1, u_2). \end{aligned} \tag{8.11}$$

Next, we proceed to demonstrate how to compute $|A_i \cap A_j|$ for $1 \leq i < j \leq h(n, k)$. Let $E(H_i, H_j, k)$ be the set of intersections $H_i \cap w(H_j)$ ($w \in B_n$) that contain at least k vertices of Q_n. Denote by $h(H_i, H_j, k)$ the number of equivalence classes of $E(H_i, H_j, k)$ under the symmetries of Q_n. Let $m = h(H_i, H_j, k)$. Assume that $E_1(H_i, H_j) = \{H_i \cap w_1(H_j), \ldots, H_i \cap w_m(H_j)\}$ is the set of representatives of equivalence classes in $E(H_i, H_j, k)$. We define a map Φ_3 from the union of $\mathcal{P}(H_i \cap w_s(H_j), k)$, where $1 \leq s \leq h(H_i, H_j, k)$, to $A_i \cap A_j$. Let $\mathcal{P}_s$ be a partial 0/1-equivalence class in $\mathcal{P}(H_i \cap w_s(H_j), k)$. Clearly, there is a (unique) partial 0/1-equivalence class in $A_i \cap A_j$ containing $\mathcal{P}_s$, which will be denoted by $\mathcal{P}_s'$. Define $\Phi_3(\mathcal{P}_s) = \mathcal{P}_s'$. We have the following conclusion. We omit the proof since it is similar to that of Theorem 8.1.

Theorem 8.4 *If $2^{n-3} < k \leq 2^{n-2}$, then the map Φ_3 is a bijection.*

As a consequence of Theorem 8.4, for $2^{n-3} < k \leq 2^{n-2}$, we have

$$|A_i \cap A_j| = \sum_{s=1}^{h(H_i, H_j, k)} |\mathcal{P}(H_i \cap w_s(H_j), k)|.$$

The computation for $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}|$ ($m \geq 3$) in the general case can be done in a similar fashion. In fact, it will be shown that for $2^{n-3} < k \leq 2^{n-2}$, the computation can be reduced to the case $m = 2$.

Let $2^{n-3} < k \leq 2^{n-2}$, and $E(H_{i_1}, \ldots, H_{i_m}, k)$ be the set of intersections $H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m})$, where w_i for $2 \leq i \leq m$ are symmetries of Q_n, that contain at least k vertices of Q_n. Denote by $E_1(H_{i_1}, \ldots, H_{i_m}, k)$ the set of representatives of equivalence classes of $E(H_{i_1}, \ldots, H_{i_m}, k)$ under the symmetries of Q_n. We define a map Φ_m from the (disjoint) union of $\mathcal{P}(H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m}), k)$, where

$$H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m}) \in E_1(H_{i_1}, \ldots, H_{i_m}, k),$$

to the set $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}$. Let $\mathcal{P} \in \mathcal{P}(H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m}), k)$. The image $\Phi_m(\mathcal{P})$ is defined to be the unique partial 0/1-equivalence class in $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}$ containing $\mathcal{P}$. Similarly, we can prove that if $2^{n-3} < k \leq 2^{n-2}$, then Φ_m is a bijection. Thus we deduce that for $2^{n-3} < k \leq 2^{n-2}$,

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}| = \sum |\mathcal{P}(H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m}), k)|,$$

where the sum ranges over the representatives of $E_1(H_{i_1}, \ldots, H_{i_m}, k)$.

We further claim for $2^{n-3} < k \leq 2^{n-2}$, $E(H_{i_1}, \ldots, H_{i_m}, k)$ is a subset of $E(H_{i_1}, E_{i_2})$. This can be proved as follows. Assume that $2^{n-3} < k \leq 2^{n-2}$, and that $H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m})$ is in $E(H_{i_1}, \ldots, H_{i_m}, k)$. From Theorem 3.1 it can be seen that the dimension of $H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m})$ is at least $n-2$, since it contains more than 2^{n-3} vertices of Q_n. On the other hand, it is clear that $H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m})$ has dimension at most $n-2$. Hence, when $2^{n-3} < k \leq 2^{n-2}$, we conclude that $H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m})$ is of dimension $n-2$. Hence we obtain that $H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m}) = H_{i_1} \cap w_2(H_{i_2})$. Therefore, $E(H_{i_1}, \ldots, H_{i_m}, k)$ is a subset of $E(H_{i_1}, E_{i_2})$. This implies that the for $2^{n-3} < k \leq 2^{n-2}$, the computation for $E(H_{i_1}, \ldots, H_{i_m}, k)$ can be reduced to the case $m = 2$. More specifically, for $2^{n-3} < k \leq 2^{n-2}$, an intersection $H_{i_1} \cap w_2(H_{i_2}) \cap \cdots \cap w_m(H_{i_m})$ belongs to $E(H_{i_1}, \cdots, H_{i_m}, k)$ whenever (possibly after the action of some symmetry of Q_n) it belongs to $E(H_{i_{j_1}}, H_{i_{j_2}})$ for $1 \leq j_1 < j_2 \leq m$.

We now turn to the case when $n = 6$ and $k = 13, 14, 15, 16$. All possible pairs $\{H_i, H_j\}$ such that $E(H_i, H_j, k)$ is nonempty are listed below.

(1). $\{H_6^1, H_6^2\}$. In this case, it can be easily checked that

$$\begin{aligned} E_1(H_6^1, H_6^2, k) &= \{H_6^1 \cap H_6^2\} \cup \{H_6^1 \cap w(H_6^2) \colon w = (1,3,2)(4)(5)(6)\} \\ &= \{H_6^1 \cap H_6^2\} \cup \{H_6^1 \cap H_6^3\}. \end{aligned} \tag{8.12}$$

(2). $\{H_6^1, H_6^3\}$ and $\{H_6^2, H_6^3\}$. In these two cases, we have

$$E_1(H_6^1, H_6^3, k) = E_1(H_6^2 \cap H_6^3, k) = \{H_6^1 \cap H_6^3\}. \tag{8.13}$$

(3). $\{H_6^2, H_6^4\}$. In this case, it can be verified that

$$E_1(H_6^2, H_6^4, k) = \{H_6^2 \cap H_6^4\}. \tag{8.14}$$

From the above, we see that H_6^1, H_6^2 and H_6^3 are the only hyperplanes such that for $k = 13, 14, 15, 16$, $E(H_{i_1}, H_{i_2}, H_{i_3}, k)$ is nonempty. Moreover, for $k = 13, 14, 15, 16$ we have

$$E_1(H_6^1, H_6^2, H_6^3, k) = \{H_6^1 \cap H_6^3\}. \tag{8.15}$$

For $k = 13, 14, 15, 16$, it is easy to see that

$$\begin{aligned} |\mathcal{P}(H_6^1 \cap H_6^2, k)| &= \left[u_1^k u_2^{16-k}\right] C_4(u_1, u_2), \\ |\mathcal{P}(H_6^1 \cap H_6^3, k)| &= \left[u_1^k u_2^{16-k}\right] C_{H_5^2}(u_1, u_2), \\ |\mathcal{P}(H_6^2 \cap H_6^4, k)| &= \left[u_1^k u_2^{16-k}\right] C_{H_6^2 \cap w(H_6^2)}(u_1, u_2), \end{aligned} \tag{8.16}$$

where $w = (1,3)(2,4)(5)(6)$.

From (8.1) and the relations (8.9)–(8.16), we deduce that for $n = 6$ and $k = 13, 14, 15, 16$,

$$\begin{aligned} H_6(k) = &\sum_{i=1}^{6} \left[u_1^k u_2^{|V_6(H_6^i)|-k}\right] C_{H_6^i}(u_1, u_2) + \sum_{i=1}^{8} \left[u_1^k u_2^{|V_6(H_i)|-k}\right] C_{H_i}(u_1, u_2) \\ &- \left[u_1^k u_2^{16-k}\right] C_4(u_1, u_2) - 2\left[u_1^k u_2^{16-k}\right] C_{H_5^2}(u_1, u_2) - \left[u_1^k u_2^{16-k}\right] C_{(H_6^2, w)} \end{aligned}, \tag{8.17}$$

where $w = (1,3)(2,4)(5)(6)$. Using the argument in Section 6, for $w = (1,3)(2,4)(5)(6)$ we obtain that

$$Z_{(H_6^2, w)} = \frac{1}{32}\left(z_1^{16} + 21 z_2^8 + 8 z_4^4 + 2 z_1^8 z_2^4\right). \tag{8.18}$$

Hence, from (8.17) and (8.18) we obtain the values of $H_6(k)$ for $k = 13, 14, 15, 16$. Utilizing the relation $F_6(k) = A_6(k) - H_6(k)$, we deduce $F_6(k)$ for $k = 13, 14, 15, 16$ as given in Table 7.

k	13	14	15	16
$F_6(k)$	290159817	1051410747	3491461629	10665920350

Table 7: $F_6(k)$ for $k = 13, 14, 15, 16$.

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