4-Regular oriented graphs with optimum skew energy*

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Abstract

Let G be a simple undirected graph, and G^σ be an oriented graph of G with the orientation σ and skew-adjacency matrix $S(G^\sigma)$. The skew energy of the oriented graph G^σ, denoted by $\mathcal{E}_S(G^\sigma)$, is defined as the sum of the absolute values of all the eigenvalues of $S(G^\sigma)$. In this paper, we characterize the underlying graphs of all 4-regular oriented graphs with optimum skew energy and give orientations of these underlying graphs such that the skew energy of the resultant oriented graphs indeed attain optimum. It should be pointed out that there are infinitely many 4-regular connected optimum skew energy oriented graphs, while the 3-regular case only has two graphs: K_4 the complete graph on 4 vertices and Q_3 the hypercube.
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1 Introduction

Let G be a simple undirected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let G^σ be an oriented graph of G with the orientation σ, which assigns to each edge of G a direction so that the induced graph G^σ becomes an oriented graph or a directed graph. Then G is called the underlying graph of G^σ. The skew-adjacency matrix of G^σ is the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$, where $s_{ij} = 1$ and $s_{ji} = -1$ if $\langle v_i, v_j\rangle$ is an arc of G^σ, otherwise $s_{ij} = s_{ji} = 0$. The skew energy [1] of G^σ, denoted by $\mathcal{E}_S(G^\sigma)$, is defined as the sum of the absolute values of all the eigenvalues of $S(G^\sigma)$. Obviously, $S(G^\sigma)$ is a skew-symmetric matrix, and thus all the eigenvalues are purely imaginary numbers.

In theoretical chemistry, the energy of a given molecular graph is related to the total π-electron energy of the molecule represented by that graph. Consequently, the graph energy

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has some specific chemistry interests and has been extensively studied, since the concept of the energy of simple undirected graphs was introduced by Gutman in [4]. We refer the survey [5] and the book [8] to the reader for details. Up to now, there are various generalizations of the graph energy, such as the Laplacian energy, signless Laplacian energy, incidence energy, distance energy, and the Laplacian-energy like invariant for undirected graphs, and the skew energy and skew Laplacian energy for oriented graphs.

Adiga et al. [1] first defined the skew energy of an oriented graph, and investigated some properties of the skew energy. Then, Shader et al. [9] studied the relationship between the spectra of a graph G and the skew-spectra of an oriented graph G^σ of G, which would be helpful to the study of the relationship between the energy of G and the skew energy of G^σ. Hou and Lei [6] characterized the coefficients of the characteristic polynomial of the skew-adjacency matrix of an oriented graph. Moreover, other bounds and extremal graphs of some classes of oriented graphs have been established. In [7] and [10], Hou et al. determined the oriented unicyclic graphs with minimal and maximal skew energy and the oriented bicyclic graphs with minimal and maximal shew energy, respectively. The skew energy of orientations of hypercubes were discussed by Tian [11]. Later, Gong and Xu [3] characterized the 3-regular oriented graphs with optimum skew energy. Recently, we [2] studied the skew energy of random oriented graphs.

Back to the paper Adiga et al. [1], where they derived a sharp upper bound for the skew energy of an oriented graph G^σ in terms of the order n and the maximum degree Δ of G^σ, that is,

$$\mathcal{E}_S(G) \leq n\sqrt{\Delta}\,.$$

They showed that the equality holds if and only if $S(G^\sigma)^T S(G^\sigma) = \Delta I_n$, which implies that G^σ is Δ-regular. In the following, we will call an oriented graph G^σ on n vertices with maximum degree Δ an *optimum skew energy oriented graph* if $\mathcal{E}_S(G^\sigma) = n\sqrt{\Delta}$. A natural question is proposed in [1]:

Which k-regular graphs on n vertices have orientations G^σ with $\mathcal{E}_S(G^\sigma) = n\sqrt{\Delta}$, or equivalently, $S(G^\sigma)^T S(G^\sigma) = \Delta I_n$?

In the same paper, they answer the question for $k = 1$ and $k = 2$. They showed that a 1-regular graph on n vertices has an orientation with $S(G^\sigma)^T S(G^\sigma) = I_n$ if and only if n is even and it is $\frac{n}{2}$ copies of K_2; while a 2-regular graph on n vertices has an orientation with $S(G^\sigma)^T S(G^\sigma) = 2I_n$ if and only if n is a multiple of 4 and it is a union of $\frac{n}{4}$ copies of C_4. Later, Gong and Xu [3] characterized all 3-regular connected oriented graphs on n vertices with $S(G^\sigma)^T S(G^\sigma) = 3I_n$, which in fact are only two special graphs, the complete graph K_4 and the hypercube Q_3.

In this paper, we further consider the above question. We characterize all 4-regular connected graphs G that have oriented graphs G^σ with $S(G^\sigma)^T S(G^\sigma) = 4I_n$. It should be noted that the 4-regular case is more complicated than the 3-regular case, and in fact, there are infinitely many 4-regular connected optimum skew energy oriented graphs.

2 Preliminaries

In this section, we do some preparations with some notations and a few known results. Besides, we also get some intuitive conclusions that will be frequently used in the sequel of the paper.

Let $G = G(V, E)$ be a graph with vertex set V and edge set E. For any $v \in V$, denote by $d_G(v)$ and $N_G(v)$ the degree and neighborhood of v in G, respectively. For any subset $S \subseteq V$, $G[S]$ denotes the subgraph of G induced by S. For a given orientation σ of G, the resultant oriented graph is denoted by $G^\sigma = (V(G^\sigma), \Gamma(G^\sigma))$ and the skew-adjacency matrix of G^σ by $S(G^\sigma)$.

The following result is due to Adiga et al. [1].

Lemma 2.1 *[1] Let $S(G^\sigma)$ be the skew-adjacency matrix of an oriented graph G^σ. If $S(G^\sigma)^T S(G^\sigma) = kI$, then $|N(u) \cap N(v)|$ is even for any two distinct vertices u and v of G^σ.*

Since our paper focuses on the investigation of 4-regular graphs, the following result is more directly applied, which is in fact implied in Lemma 2.1.

Proposition 2.2 *Let G^σ be a 4-regular oriented graph with skew-adjacency matrix $S(G^\sigma)$. If $S(G^\sigma)^T S(G^\sigma) = 4I$, then the underlying graph G satisfies that $|N(u) \cap N(v)| \in \{0, 2\}$ for any two adjacent vertices u and v and $|N(u) \cap N(v)| \in \{0, 2, 4\}$ for any two non-adjacent vertices u and v.*

Let $W = u_1 u_2 \cdots u_k$ (perhaps $u_i = u_j$ for $i \neq j$) be a walk from u_1 to u_k and $\widehat{W}$ be the inverse walk of W obtained from W by replacing the ordering of vertices by its inverses, i.e., $\widehat{W} = u_k u_{k-1} \cdots u_1$. The sign of W is defined as

$$\mathrm{sgn}(W) = \prod_{i=1}^{k-1} s_{u_i u_{i+1}}.$$

It is easy to check that

$$\mathrm{sgn}(\widehat{W}) = \begin{cases} \mathrm{sgn}(W) & \text{if } l(W) \text{ is even,} \\ -\mathrm{sgn}(W) & \text{if } l(W) \text{ is odd,} \end{cases}$$

where $l(W)$ denotes the length of the walk W. Moreover, let $w^+_{uv}(k)$ and $w^-_{uv}(k)$ denote the number of all positive walks and negative walks starting from u and ending at v with length k, respectively.

Gong and Xu [3] obtained the following result on the relationship between the entries of S^k and the number of walks between any pair of ordered vertices.

Lemma 2.3 *[3] Let S be the skew-adjacency matrix of an oriented graph G^σ and (u, v) be an* *arbitrary pair of ordered vertices of* G^σ. *Then*

$$(S^k)_{uv} = w^+_{uv}(k) - w^-_{uv}(k)$$

holds for any positive integer k.

For regular graphs, the following proposition is immediate.

Proposition 2.4 *Let* G^σ *be a* k*-regular oriented graph with skew-adjacency matrix* S. *Then* $S^T S = kI$ *if and only if for any two distinct vertices* u *and* v *of* G^σ,

$$w^+_{uv}(2) = w^-_{uv}(2).$$

Throughout this paper, we just need to consider connected graphs and connected oriented graphs due to the following basic lemma. Recall that the union $G_1^\sigma \cup G_2^\sigma$ of two disjoint oriented graphs $G_1^\sigma = (V_1, \Gamma_1)$ and $G_2^\sigma = (V_2, \Gamma_2)$ is the oriented graph $G^\sigma = (V, \Gamma)$ where $V = V_1 \cup V_2$ and $\Gamma = \Gamma_1 \cup \Gamma_2$.

Lemma 2.5 *[11] Let* G_1^σ, G_2^σ *be two disjoint oriented graphs of order* n_1, n_2 *with skew-adjacency matrices* $S(G_1^\sigma)$, $S(G_2^\sigma)$, *respectively. Then for some positive integer* k, $S(G_1^\sigma)^T S(G_1^\sigma) = kI_{n_1}$ *and* $S(G_2^\sigma)^T S(G_2^\sigma) = kI_{n_2}$ *if and only if the skew-adjacency matrix* $S(G_1^\sigma \cup G_2^\sigma)$ *of the union* $G_1^\sigma \cup G_2^\sigma$ *satisfies* $S(G_1^\sigma \cup G_2^\sigma)^T S(G_1^\sigma \cup G_2^\sigma) = kI_{n_1+n_2}$.

We end this section by recursively defining two graph classes $\mathcal{G}_i$ and $\mathcal{H}_j$ for all positive integers i and j, depicted in Figure 2.1 and Figure 2.2, respectively.

For the graph class $\mathcal{G}_i$, we define the initial graph $\mathcal{G}_1 = (V(\mathcal{G}_1), E(\mathcal{G}_1))$, where

$$\begin{aligned}
V(\mathcal{G}_1) =& \{u, v\} \cup \{u_1, u_2, v_1, v_2\} \cup \{u_3, u_4, v_3, v_4\},\\
E(\mathcal{G}_1) =& \{(u, u_1), (u, u_2), (u, v_1), (u, v_2), (v, u_1), (v, u_2), (v, v_1), (v, v_2)\}\\
& \cup \{(u_1, u_3), (u_1, u_4), (u_2, u_3), (u_2, u_4), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4)\}\\
& \cup \{(u_3, v_4), (v_4, u_4), (u_4, v_3), (v_3, u_3)\}.
\end{aligned}$$

Suppose that $\mathcal{G}_{i-1}$ is well defined. Below we will give the definition of $\mathcal{G}_i = (V(\mathcal{G}_i), E(\mathcal{G}_i))$.

$$\begin{aligned}
V(\mathcal{G}_i) =& V(\mathcal{G}_{i-1}) \cup \{u_{2i+1}, u_{2i+2}, v_{2i+1}, v_{2i+2}\},\\
E(\mathcal{G}_i) =& E(\mathcal{G}_{i-1}) \setminus \{(u_{2i-1}, v_{2i}), (v_{2i}, u_{2i}), (u_{2i}, v_{2i-1}), (v_{2i-1}, u_{2i-1})\}\\
& \cup \{(u_{2i-1}, u_{2i+1}), (u_{2i-1}, u_{2i+2}), (u_{2i}, u_{2i+1}), (u_{2i}, u_{2i+2})\}\\
& \cup \{(v_{2i-1}, v_{2i+1}), (v_{2i-1}, v_{2i+2}), (v_{2i}, v_{2i+1}), (v_{2i}, v_{2i+2})\}\\
& \cup \{(u_{2i+1}, v_{2i+2}), (v_{2i+2}, u_{2i+2}), (u_{2i+2}, v_{2i+1}), (v_{2i+1}, u_{2i+1})\}.
\end{aligned}$$

Observe that $|V(\mathcal{G}_i)| = 4i + 6$.

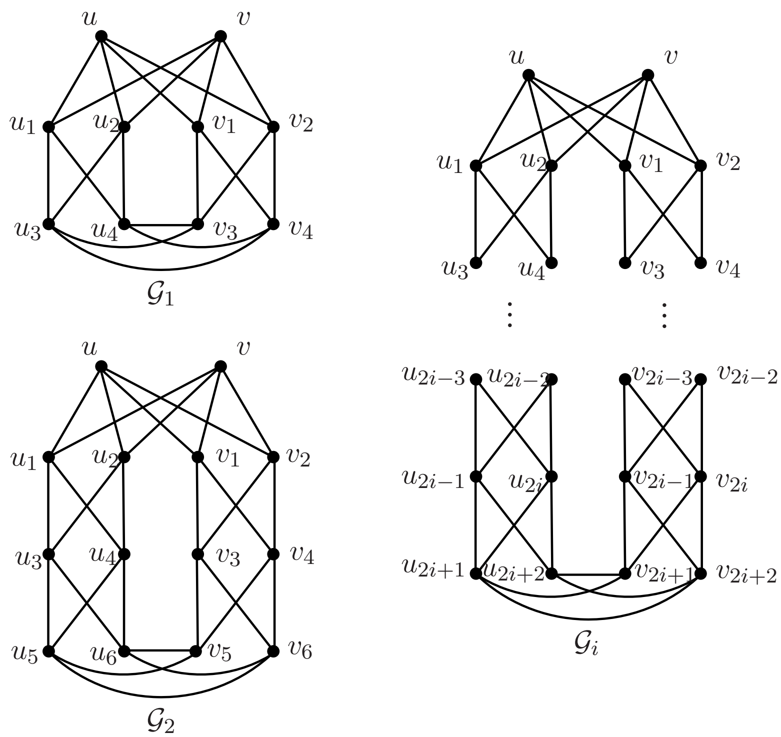


Figure 2.1: The graph class $\mathcal{G}_i$ for any positive integer i

For the other graph class $\mathcal{H}_j$, the initial graph $\mathcal{H}_1$ is defined as $\mathcal{H}_1 = (V(\mathcal{H}_1), E(\mathcal{H}_1))$, where

$$\begin{aligned} V(\mathcal{H}_1) =& \{u, v\} \cup \{u_1, u_2, v_1, v_2\} \cup \{u_3, u_4\}, \\ E(\mathcal{H}_1) =& \{(u, u_1), (u, u_2), (u, v_1), (u, v_2), (v, u_1), (v, u_2), (v, v_1), (v, v_2)\} \\ & \cup \{(u_1, u_3), (u_1, u_4), (u_2, u_3), (u_2, u_4), (v_1, u_3), (v_1, u_4), (v_2, u_3), (v_2, u_4)\}. \end{aligned}$$

Suppose now that we have given the definition of $\mathcal{H}_{j-1}$. Then $\mathcal{H}_j = (V(\mathcal{H}_j), E(\mathcal{H}_j))$ is defined as follows.

$$\begin{aligned} V(\mathcal{H}_j) =& V(\mathcal{H}_{j-1}) \cup \{v_{2j-1}, v_{2j}, u_{2j+1}, u_{2j+2}\}, \\ E(\mathcal{H}_j) =& E(\mathcal{H}_{j-1}) \setminus \{(v_{2j-3}, u_{2j-1}), (v_{2j-3}, u_{2j}), (v_{2j-2}, u_{2j-1}), (v_{2j-2}, u_{2j})\} \\ & \cup \{(v_{2j-3}, v_{2j-1}), (v_{2j-3}, v_{2j}), (v_{2j-2}, v_{2j-1}), (v_{2j-2}, v_{2j})\} \\ & \cup \{(u_{2j-1}, u_{2j+1}), (u_{2j-1}, u_{2j+2}), (u_{2j}, u_{2j+1}), (u_{2j}, u_{2j+2})\} \\ & \cup \{(v_{2j-1}, u_{2j+1}), (v_{2j-1}, u_{2j+2}), (v_{2j}, u_{2j+1}), (v_{2j}, u_{2j+2})\}. \end{aligned}$$

Obviously, $|V(\mathcal{H}_j)| = 4j + 4$.

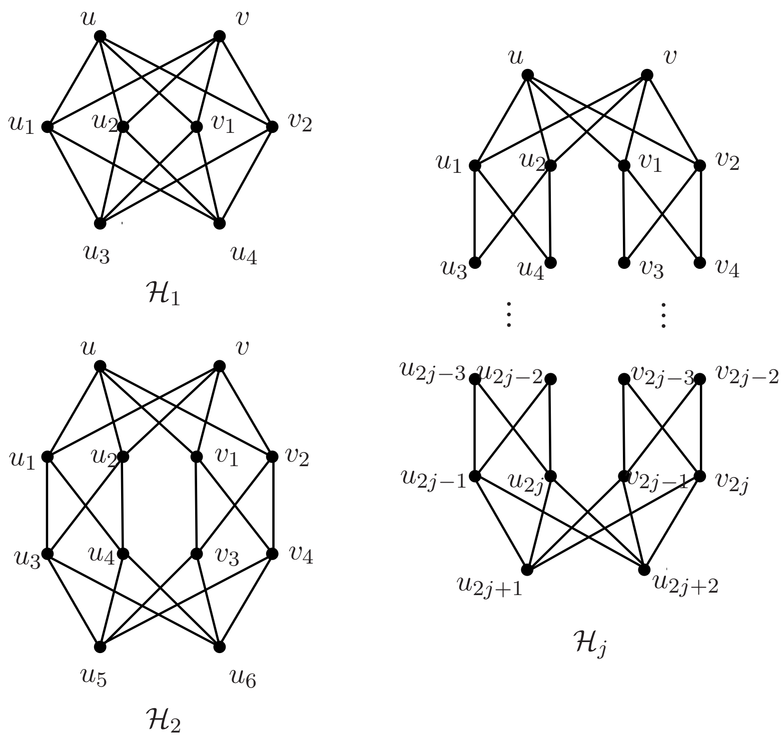


Figure 2.2: The graph class $\mathcal{H}_j$ for any positive integer j

3 Main results

In this section, we first characterize the underlying graphs of all 4-regular oriented graphs with optimum skew energy. Then we give orientations of these underlying graphs such that the resultant oriented graphs have optimum shew energy.

Theorem 3.1 *Let G^σ be a 4-regular oriented graph with optimum skew energy. If the underlying graph G contains triangles, then G is either G_1 or G_2 depicted in Figure 3.3.*

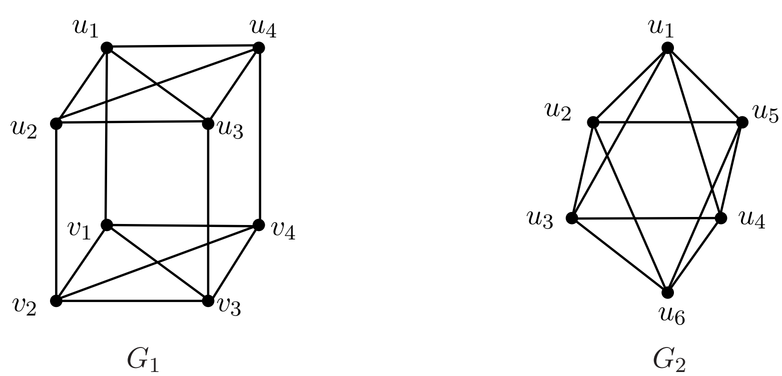


Figure 3.3: The underlying graphs containing triangles

Proof. Let $u_1u_2u_3u_1$ be a triangle in G. Since $u_2 \in N(u_1) \cap N(u_3)$, there is another common neighbor between u_1 and u_3 from Proposition 2.2, denoted by u_4. Observe that $u_3 \in N(u_1) \cap N(u_2)$. Then by Proposition 2.2 again, there is another vertex in $N(u_1) \cap N(u_2)$, which is either u_4 or a new vertex, say u_5.

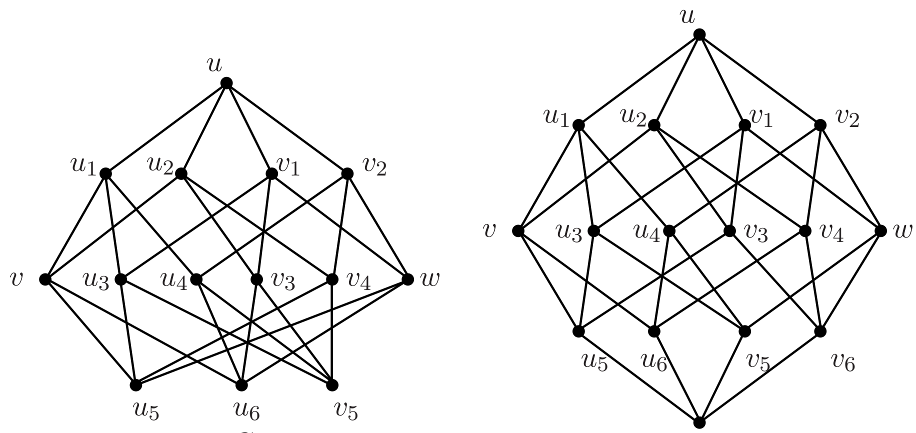
Firstly, assume that $u_4 \in N(u_1) \cap N(u_2)$, that is, $(u_2, u_4) \in G$. As G is 4-regular, u_1 has the fourth neighbor, denoted by v_1. We claim that $(v_1, u_2) \notin G$; otherwise $N(u_1) \cap N(u_2) = \{u_3, u_4, v_1\}$ which contradicts Proposition 2.2. Similarly, we have $(v_1, u_3) \notin G$ and $(v_1, u_4) \notin G$. We can further obtain that the new vertices v_2, v_3 and v_4 are the fourth neighbors of u_2, u_3 and u_4, respectively, and $(v_i, u_j) \notin G$ for $1 \leq i \neq j \leq 4$. Then we consider $N(v_1) \cap N(u_2)$. Note that $u_1 \in N(u_2) \cap N(v_1)$, $(v_1, u_3) \notin G$ and $(v_1, u_4) \notin G$ by the discussion above, which forces that v_2 becomes another common neighbor between v_1 and u_2, i.e., $(v_1, v_2) \in G$. By similar discussions on $N(v_1) \cap N(u_4)$, $N(v_3) \cap N(u_2)$ and $N(v_3) \cap N(u_4)$, respectively, we can deduce that $(v_1, v_4) \in G$, $(v_2, v_3) \in G$ and $(v_3, v_4) \in G$. Noticing that $u_1 \in N(v_1) \cap N(u_3)$, another common vertex must be v_3, since $d(u_3) = 4$ and the degrees of other neighbors of u_3 other than v_3 are equal to 4. By considering $N(u_2) \cap N(v_4)$ similarly, we have $(v_2, v_4) \in G$. Up to now, the degrees of all vertices of G attain 4. Hence the underlying graph G is the graph G_1 given in Figure 3.3.

Now we suppose that $N(u_1) \cap N(u_2)$ contains a new vertex u_5. We claim that $(u_2, u_4) \notin G$ and $(u_3, u_5) \notin G$; otherwise, $N(u_1) \cap N(u_2) = \{u_3, u_4, u_5\}$ or $N(u_1) \cap N(u_3) = \{u_2, u_4, u_5\}$, a contradiction to Proposition 2.2. Notice that $u_3 \in N(u_1) \cap N(u_4)$, $d(u_1) = 4$ and $(u_2, u_4) \notin G$, which implies $(u_4, u_5) \in G$. Since $d(u_5) = 3$ and $(u_3, u_5) \notin G$, u_5 has the fourth neighbor u_6. Now we consider $N(u_2) \cap N(u_5)$. Combining the observation that $u_1 \in N(u_2) \cap N(u_5)$ with the fact $(u_2, u_4) \notin G$, we deduce that $u_6 \in N(u_2) \cap N(u_5)$. Then by a similar way, we successively discuss $N(u_2) \cap N(u_3)$ and $N(u_3) \cap N(u_4)$ and obtain $(u_3, u_6) \in G$ and $(u_4, u_6) \in G$. It is easy to check that the graph has already been 4-regular and is just the graph G_2 depicted in Figure 3.3. ∎

Theorem 3.2 *Let G^σ be a 4-regular oriented graph with optimum skew energy. If the underlying graph G is triangle-free, then G is one of the following graphs: the hypercube Q_4 of dimension 4, the graph G_3, a graph in $\mathcal{G}_i$, or a graph in $\mathcal{H}_j$; see Figures 2.1, 2.2 and 3.4.*

Proof. Let u_1, u_2, v_1 and v_2 be all neighbors of a vertex u in G. Then the induced subgraph $G[\{u_1, u_2, v_1, v_2\}]$ contains no edge, since the graph G is triangle-free. Denote by v, u_3, u_4 be another three neighbors of u_1 other than u. Note that $u_1 \in N(u) \cap N(v)$. By Proposition 2.2, there is another one or three common neighbors in $\{u_2, v_1, v_2\}$ between u and v. We can obtain the same results by considering $N(u) \cap N(u_3)$ and $N(u) \cap N(u_4)$. Assume that a_1, a_2 and a_3 are the numbers of the common neighbors in $\{u_2, v_1, v_2\}$ between u and v, u and u_3, u and u_4, respectively. Obviously, $a_1, a_2, a_3 \in \{1, 3\}$. Without loss of generality, suppose $a_1 \geq a_2 \geq a_3$. We discuss the following four cases according to the values of a_1, a_2 and a_3.

Case 1. $(a_1, a_2, a_3) = (1, 1, 1)$.



G_3 Q_4

Figure 3.4: The underlying graphs containing no triangles

Without loss of generality, let $(u_2, v) \in G$. Then $(v_1, v) \notin G$ and $(v_2, v) \notin G$ as $a_1 = 1$. Observe that $u \in N(u_1) \cap N(v_1)$, which implies that there is another common neighbor in $\{u_3, u_4\}$ between u_1 and v_1. Let $u_3 \in N(u_1) \cap N(v_1)$. Then $(u_2, u_3) \notin G$ and $(v_2, u_3) \notin G$ as $a_2 = 1$. By considering $N(u_1) \cap N(v_2)$, we deduce that $(v_2, u_4) \in G$, since $(v_2, v) \notin G$ and $(v_2, u_3) \notin G$ by the discussion above. Obviously, $(u_2, u_4) \notin G$ and $(v_2, u_4) \notin G$ as $a_3 = 1$. Then it is known that u_2 contains another two neighbors, say v_3 and v_4. Since $u \in N(u_2) \cap N(v_1)$ and $(v_1, v) \notin G$, it follows that there is another common neighbor in $\{v_3, v_4\}$ between u_2 and v_1. Without loss of generality, $v_3 \in N(u_2) \cap N(v_1)$. Then $(v_1, v_4) \notin G$; otherwise, $|N(u_2) \cap N(v_1)| = 3$ and no other vertex can be chosen as the fourth common neighbor, which is a contradiction. In view of the observation that $u \in N(u_2) \cap N(v_2)$ and $(v_2, v) \notin G$, we have that another common neighbor between u_2 and v_2 belongs to $\{v_3, v_4\}$. We claim that $v_3 \notin N(u_2) \cap N(v_2)$; otherwise, $N(u) \cap N(v_3) = \{u_2, v_1, v_2\}$ and there is no other vertex in $N(u) \cap N(v_3)$, which contradicts Proposition 2.2. Therefore, $v_4 \in N(u_2) \cap N(v_2)$. We proceed to have w as the fourth neighbor of v_1. By considering $N(v_1) \cap N(v_2)$, we obtain $(v_2, w) \in G$.

Up to now, we have $d(u) = d(u_1) = d(u_2) = d(v_1) = d(v_2) = 4$ and $d(v) = d(u_3) = d(u_4) = d(v_3) = d(v_4) = d(w) = 2$. We claim that the deduced subgraph $G[\{v, u_3, u_4, v_3, v_4, w\}]$ is empty. Otherwise, the possible edges are $(v, w), (u_3, v_4)$ and (u_4, v_3) since G is triangle-free. If $(v, w) \in G$, then $|N(u_1) \cap N(w)| = 1$, which is a contradiction. We thus have $(v, w) \notin G$. Similarly, $(u_3, v_4) \notin G$ and $(u_4, v_3) \notin G$.

Suppose now that u_5 and u_6 are the other two neighbors of v. Note that $u_1 \in N(v) \cap N(u_3)$. Then we have either $(u_3, u_5) \in G$ or $(u_3, u_6) \in G$. Without loss of generality, $(u_3, u_5) \in G$, and hence $(u_3, u_6) \notin G$. Moreover, $(u_4, u_5) \notin G$, otherwise, $N(u_1) \cap N(u_5) = \{v, u_3, u_4\}$, a contradiction. By considering $N(u_1) \cap N(u_6)$, we get $(u_4, u_6) \in G$. Assume that v_5 is the fourth neighbor of u_3. It is obvious that $u_3 \in N(u_1) \cap N(v_5)$, which forces that u_4 becomes another common vertex between u_1 and v_5. We see that $v \in N(u_2) \cap N(u_5)$, which indicates that there is another common neighbor between u_2 and u_5. It means either $(v_3, u_5) \in G$ or $(v_4, u_5) \in G$. We discuss the two cases separately.

On the one hand, if $(v_3, u_5) \in G$, then $(v_4, u_5) \notin G$. It follows that $(v_4, u_6) \in G$ by considering $N(v) \cap N(v_4)$. We claim that $(v_3, u_6) \notin G$ and $(v_3, v_5) \notin G$; otherwise, $N(v) \cap N(v_3) = \{u_2, u_5, u_6\}$ or $N(u_3) \cap N(v_3) = \{v_1, u_5, v_5\}$, which is a contradiction. Therefore, v_3 contains a new neighbor, denoted by v_6. Since $v_3 \in N(v_1) \cap N(v_6)$, we have $(w, v_6) \in G$, since w is the unique neighbor of v_1 with degree less than 4. Similarly, we get $(v_4, v_6) \in G$ by considering $N(v_2) \cap N(v_6)$. We further obtain that $(w, v_5) \in G$ by considering $N(v_2) \cap N(v_5)$.

Up to now, $d(u_5) = d(u_6) = d(v_5) = d(v_6) = 3$ and other vertices above have degree 4. It is known that $G[\{u_5, u_6, v_5, v_6\}]$ contains no edges because of the triangle-free property of G. Suppose now that s is the fourth neighbor of u_5. Considering $N(v) \cap N(s)$, $N(u_3) \cap N(s)$ and $N(v_3) \cap N(s)$, respectively, we derive that $(u_5, s) \in G, (u_6, s) \in G, (v_5, s) \in G$ and $(v_6, s) \in G$. Now all vertices have degree 4. It can be verified that G is the hypercube Q_4.

On the other hand, $(v_4, u_5) \in G$. It follows that $v_4 \in N(v_2) \cap N(u_5)$. Then we have $(w, u_5) \in G$, since w is the unique neighbor of v_2 with degree less than 4 other than v_4. Note that $u_2 \in N(v) \cap N(v_3)$, which forces that u_6 becomes another common neighbor between v and v_3, since u_6 is the unique neighbor of v, whose degree is less than 4. By a similar discussion on $N(u_3) \cap N(v_3)$, we can deduce that $(v_3, v_5) \in G$. Since $u_5 \in N(u_3) \cap N(v_4)$, we get $(v_4, v_5) \in G$. We further consider $N(u_4) \cap N(w)$ and obtain $(w, u_6) \in G$. Now all vertices have degree 4. It can be easily verified that G is the graph G_3 depicted in Figure 3.4.

Case 2. $(a_1, a_2, a_3) = (3, 1, 1)$.

In this case, v is adjacent to all vertices of $\{u_2, v_1, v_2\}$, while u_3 and u_4 are adjacent to one of them, respectively. Without loss of generality, $(u_2, u_3) \in G$. Then $(v_1, u_3) \notin G$ and $(v_2, u_3) \notin G$ since $a_2 = 1$. It follows that $(u_2, u_4) \in G$. If not, then either $(v_1, u_4) \in G$ or $(v_2, u_4) \in G$, where the former possibility implies that $N((u_1) \cap N(v_1) = \{u, v, u_4\})$ and the latter implies that $N(u_1) \cap N(v_2) = \{u, v, u_4\}$, both of which contradict Proposition 2.2. Hence $(u_2, u_4) \in G$, $(v_1, u_4) \notin G$ and $(v_2, u_4) \notin G$. Now let v_3 and v_4 be another two neighbors of v_1. Observe that $v_1 \in N(v) \cap N(v_3)$, which forces v_2 to be another common vertex between v and v_3. By a similar discussion on $N(v) \cap N(v_4)$, we can derive $(v_2, v_4) \in G$.

Now, $d(u) = d(v) = d(u_1) = d(u_2) = d(v_1) = d(v_2) = 4$ and $d(u_3) = d(u_4) = d(v_3) = d(v_4) = 2$. We can divide our subsequent discussion into the following steps.

Step 1. If the induced subgraph $G[\{u_3, u_4, v_3, v_4\}]$ contains edges, then the edges can only be some of (u_3, v_3), (u_4, v_4), (u_3, v_4) and (u_4, v_3), since G is triangle-free. Without loss generality, assume $(u_3, v_4) \in G$. Then $u_3 \in N(u_1) \cap N(v_4)$ and $v_4 \in N(v_2) \cap N(u_3)$, which forces $(u_4, v_4) \in G$ and $(u_3, v_3) \in G$. We further consider $N(u_2) \cap N(v_3)$, and obtain $(u_4, v_3) \in G$. Consequently, each vertex above has degree 4. It is easy to verify that G is the graph $\mathcal{G}_1$ depicted in Figure 2.1. Now the discussion stop;

Step 2. If the induced subgraph $G[\{u_3, u_4, v_3, v_4\}]$ contains no edges, then there are another two neighbors of u_3, say u_5 and u_6. Considering $N(u_1) \cap N(u_5)$ and $N(u_1) \cap N(u_6)$, respectively, we have $(u_4, u_5) \in G$ and $(u_4, u_6) \in G$, since u_4 is the unique neighbor of u_1 whose degree is less than 4.

On the one hand, if u_5 or u_6 is adjacent to v_3 or v_4, then without loss generality we can suppose $(u_5, v_3) \in G$. Then $v_3 \in N(v_1) \cap N(u_5)$, which implies $(u_5, v_4) \in G$. Notice that $u_5 \in N(u_3) \cap N(v_3)$ and $u_5 \in N(u_3) \cap N(v_4)$. Then we deduce that $(v_3, u_6) \in G$ and $(v_4, u_6) \in G$, since u_6 is the unique neighbor of u_3 whose degree is less than 4. It can be verified that G is the graph $\mathcal{H}_2$ depicted in Figure 2.2.

On the other hand, both u_5 and u_6 are not adjacent to v_3 or v_4. Then v_3 has another two neighbors, denoted by v_5 and v_6. By a similar discussion on $N(v_2) \cap N(v_5)$ and $N(v_2) \cap N(v_6)$, respectively, we can obtain $(v_4, v_5) \in G$ and $(v_4, v_6) \in G$. Then continue the following step;

Step 3. If the induced subgraph $G[\{u_5, u_6, v_5, v_6\}]$ contains edges, we can discuss this case similar to Step 1. Consequently, we can obtain that G is the graph $\mathcal{G}_2$ depicted in Figure 2.1. The discussion stops; If the induced subgraph $G[\{u_5, u_6, v_5, v_6\}]$ contains no edges, we can also continue the discussion according to Step 2, until we get that G is the graph $\mathcal{H}_3$ depicted in Figure 2.2, or executing Step 3 again. The discussion continues.

It should be pointed out that the discussion will terminate by illustrating that G is either a graph in $\mathcal{G}_i$ or a graph in $\mathcal{H}_j$, which are shown in Figure 2.1 and Figure 2.2, respectively.

Case 3. $(a_1, a_2, a_3) = (3, 3, 1)$.

This case means that v and u_3 are adjacent to all vertices of $\{u_2, v_1, v_2\}$, while u_4 is precisely adjacent to one of them. Without loss generality, suppose $(u_2, u_4) \in G$. Then $(v_1, u_4) \notin G$ and $(v_2, u_4) \notin G$. Consequently, $|N(u_1) \cap N(v_1)| = |\{u, v, u_3\}| = 3$, which contradicts Proposition 2.2. Therefore, this case could not happen.

Case 4. $(a_1, a_2, a_3) = (3, 3, 3)$.

Obviously, v, u_3 and u_4 are adjacent to all vertices of $\{u_2, v_1, v_2\}$. It can be checked that all vertices have degree 4, and hence G is the complete bipartite graph $K_{4,4}$, which is also the graph $\mathcal{H}_1$ depicted in Figure 2.2.

To sum up the discussion above, G is the hypercube Q_4 or the graph G_3 or a graph in $\mathcal{G}_i$ or a graph in $\mathcal{H}_j$. The proof is now complete. ∎

For convenience, we denote the set of all graphs presented above by $\mathcal{F}$, which consists of G_1, G_2, G_3, Q_4, all graphs in $\mathcal{G}_i$ and all graphs in $\mathcal{H}_j$. Combining Theorem 3.1 with Theorem 3.2, we conclude one of our main results as follows.

Theorem 3.3 *Let G^σ be a 4-regular oriented graph with optimum skew energy. Then the underlying graph G is a graph in $\mathcal{F}$.*

Now the question naturally arises: whether there exists an orientation for each graph of $\mathcal{F}$ such that the resultant oriented graph attains optimum skew energy. The following results tell us that for each graph of $\mathcal{F}$ such orientation indeed exists.

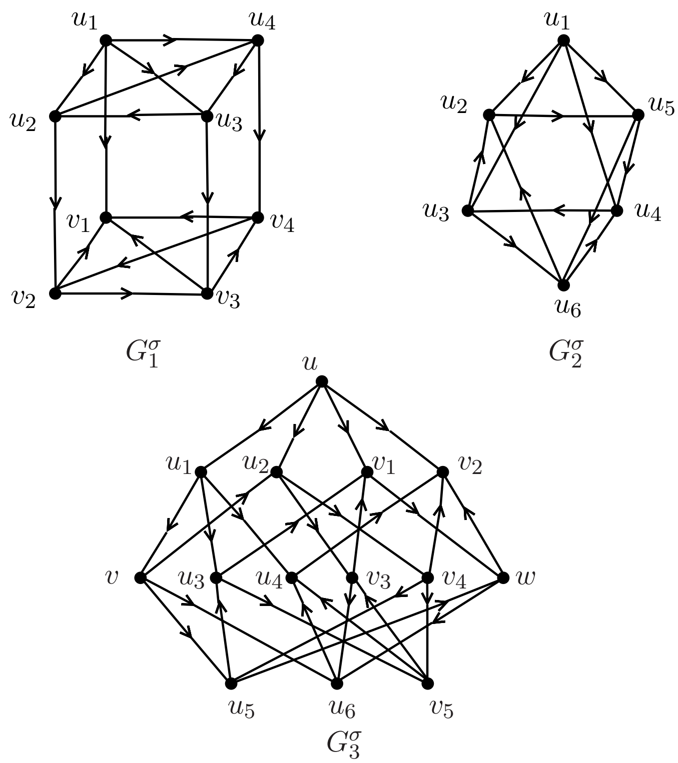


Figure 3.5: The optimum orientations for G_1, G_2 and G_3

Theorem 3.4 *Let G_1^σ, G_2^σ and G_3^σ be the oriented graphs of G_1, G_2 and G_3, respectively, given in Figure 3.5. Then each of them has the optimum shew energy.*

Proof. Let the rows of the skew-adjacency matrix $S(G_1^\sigma)$ correspond successively the vertices u_1, u_2, u_3, u_4, v_1, v_2, v_3 and v_4. It follows that

$$S(G_1^\sigma) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \end{bmatrix}$$

Let the rows of the skew-adjacency matrix $S(G_2^\sigma)$ correspond successively the vertices u_1, u_2, u_3, u_4, u_5 and u_6. Then

$$S(G_2^\sigma) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{bmatrix}$$

Similarly, let the rows of the skew-adjacency matrix $S(G_3^\sigma)$ correspond successively the vertices u, u_1, u_2, v_1, v_2, v, u_3, u_4, v_3, v_4, w, u_5, u_6 and v_5. Then

$$S(G_3^\sigma) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is not difficult to check that $S(G_1^\sigma)^T S(G_1^\sigma) = 4I_8$, $S(G_2^\sigma)^T S(G_2^\sigma) = 4I_6$ and $S(G_3^\sigma)^T S(G_3^\sigma) = 4I_{14}$. We can also verify these equalities by proving that different row vectors of each of $S(G_1^\sigma)$, $S(G_2^\sigma)$ and $S(G_3^\sigma)$ are pairwise orthogonal. The theorem is thus proved. ∎

We have known from [11] that there exists an orientation σ of Q_4 such that the resultant oriented graph Q_4^σ has optimum skew energy. The following two algorithms recursively describe optimum orientations of $\mathcal{G}_i$ and $\mathcal{H}_j$, respectively.

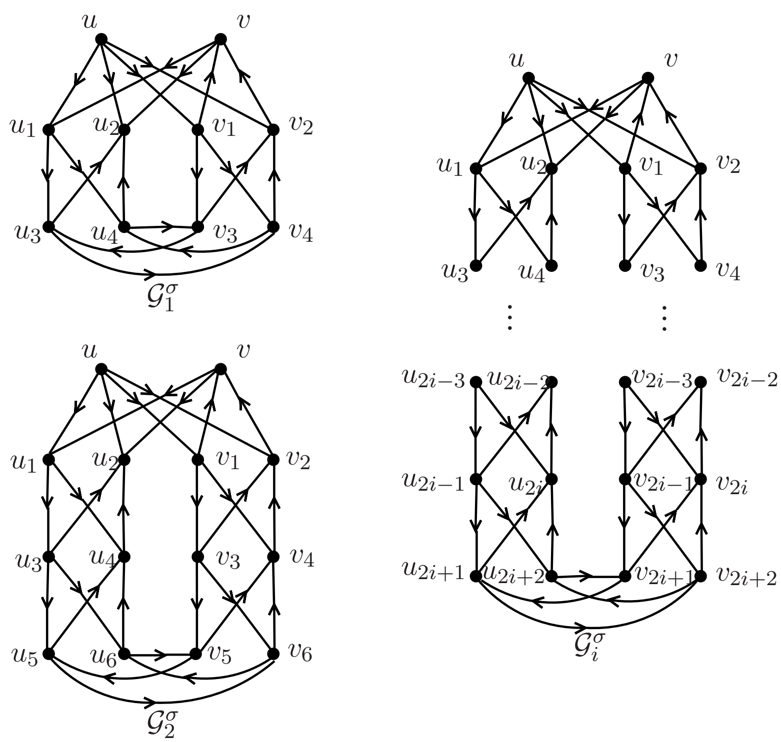


Figure 3.6: The optimum orientation for $\mathcal{G}_i$

Algorithm 1.

Step 1. Give $\mathcal{G}_1$ an orientation as shown in Figure 3.6.

Step 2. Assume that $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{t-1}$ have been oriented into $\mathcal{G}_1^\sigma, \mathcal{G}_2^\sigma, \ldots, \mathcal{G}_{t-1}^\sigma$. Then we orient $\mathcal{G}_t$ with the following method:

(i) Keep the orientations of all edges in $E(\mathcal{G}_{t-1}) \cap E(\mathcal{G}_t)$.

(ii) Give the remaining edges orientations such that $\langle u_{2t-1}, u_{2t+1}\rangle$, $\langle u_{2t-1}, u_{2t+2}\rangle$, $\langle u_{2t+1}, u_{2t}\rangle$, $\langle u_{2t+2}, u_{2t}\rangle$, $\langle v_{2t-1}, v_{2t+1}\rangle$, $\langle v_{2t-1}, v_{2t+2}\rangle$, $\langle v_{2t+1}, v_{2t}\rangle$, $\langle v_{2t+2}, v_{2t}\rangle$, $\langle u_{2t+1}, v_{2t+2}\rangle$, $\langle v_{2t+2}, u_{2t+2}\rangle$, $\langle u_{2t+2}, v_{2t+1}\rangle$ and $\langle v_{2t+1}, u_{2t+1}\rangle$ belong to $\Gamma(\mathcal{G}_t^\sigma)$.

Step 3. If $t = i$, stop; else take $t - 1 := t$, return to Step 2.

Algorithm 2.

Step 1. Give $\mathcal{H}_1$ an orientation as shown in Figure 3.7.

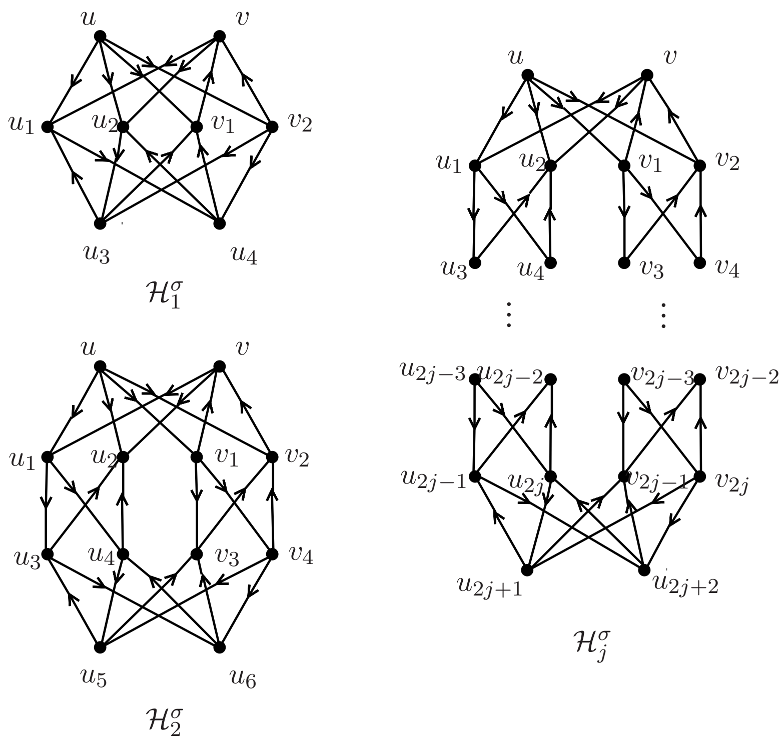


Figure 3.7: The optimum orientation for $\mathcal{H}_j$

Step 2. Assume that $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{t-1}$ have been oriented into $\mathcal{H}_1^\sigma, \mathcal{H}_2^\sigma, \ldots, \mathcal{H}_{t-1}^\sigma$. Then we orient $\mathcal{H}_t$ with the following method:

(i) Keep the orientations of all edges in $E(\mathcal{H}_{t-1}) \cap E(\mathcal{H}_t) \setminus \{(u_{2t-3}, u_{2t-1}), (u_{2t-3}, u_{2t}), (u_{2t-2}, u_{2t-1}), (u_{2t-2}, u_{2t})\}$.

(ii) Give the remaining edges orientations such that $\langle u_{2t-3}, u_{2t-1}\rangle$, $\langle u_{2t-3}, u_{2t}\rangle$, $\langle u_{2t-1}, u_{2t-2}\rangle$, $\langle u_{2t}, u_{2t-2}\rangle$, $\langle v_{2t-3}, v_{2t-1}\rangle$, $\langle v_{2t-3}, v_{2t}\rangle$, $\langle v_{2t-1}, v_{2t-2}\rangle$, $\langle v_{2t}, v_{2t-2}\rangle$, $\langle u_{2t+1}, u_{2t-1}\rangle$, $\langle u_{2t-1}, u_{2t+2}\rangle$, $\langle u_{2t}, u_{2t+1}\rangle$, $\langle u_{2t+2}, u_{2t}\rangle$, $\langle u_{2t+1}, v_{2t-1}\rangle$, $\langle u_{2t+2}, v_{2t-1}\rangle$ $\langle v_{2t}, u_{2t+1}\rangle$ and $\langle v_{2t}, u_{2t+2}\rangle$ belong to $\Gamma(\mathcal{H}_t^\sigma)$.

Step 3. If $t = i$, stop; else take $t - 1 := t$, return to Step 2.

Next, we shall prove that $\mathcal{G}_i^\sigma$ and $\mathcal{H}_j^\sigma$ derived from Algorithm 1 and Algorithm 2, respectively, have optimum skew energy, that is, their skew-adjacency matrices satisfy $S(\mathcal{G}_i^\sigma)^T S(\mathcal{G}_i^\sigma) = 4I$ and $S(\mathcal{H}_j^\sigma)^T S(\mathcal{H}_j^\sigma) = 4I$. In order to illustrate clearly the skew-adjacency matrices $S(\mathcal{G}_i^\sigma)$ and

$S(\mathcal{H}_j^\sigma)$, we here define some small matrix blocks.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}$$

Theorem 3.5 *Let $S(\mathcal{G}_i^\sigma)$ be the skew-adjacency matrix of $\mathcal{G}_i^\sigma$ obtained from Algorithm 1. Then $S(\mathcal{G}_i^\sigma)^T S(\mathcal{G}_i^\sigma) = 4I$.*

Proof. Let the rows of the skew-adjacency matrix $S(\mathcal{G}_i)$ correspond successively the vertices u, v, u_1, u_2, v_1, v_2, $\ldots$, u_{2i+1}, u_{2i+2}, v_{2i+1}, v_{2i+2}. Then from Algorithm 1, $S(\mathcal{G}_i)$ can be written as the block matrix for each positive integer i.

For $i = 1$ and $i = 2$,

$$S(\mathcal{G}_1^\sigma) = \begin{bmatrix} 0 & A & 0 \\ -A^T & 0 & B \\ 0 & -B^T & C \end{bmatrix} \qquad S(\mathcal{G}_2^\sigma) = \begin{bmatrix} 0 & A & 0 & 0 \\ -A^T & 0 & B & 0 \\ 0 & -B^T & 0 & B \\ 0 & 0 & -B^T & C \end{bmatrix}$$

By applying multiplication of block matrix, it is easy to compute that

$$S(\mathcal{G}_1^\sigma)^T S(\mathcal{G}_1^\sigma) = \begin{bmatrix} AA^T & 0 & -AB \\ 0 & A^TA + BB^T & -BC \\ -B^TA^T & -C^TB^T & B^TB + C^TC \end{bmatrix}$$

$$S(\mathcal{G}_2^\sigma)^T S(\mathcal{G}_2^\sigma) = \begin{bmatrix} AA^T & 0 & -AB & 0 \\ 0 & A^TA + BB^T & 0 & -B^2 \\ -B^TA^T & 0 & B^TB + BB^T & -BC \\ 0 & -(B^T)^2 & -C^TB^T & B^TB + C^TC \end{bmatrix}$$

In order to prove $S(\mathcal{G}_1^\sigma)^T S(\mathcal{G}_1^\sigma) = 4I$ and $S(\mathcal{G}_2^\sigma)^T S(\mathcal{G}_2^\sigma) = 4I$, it suffices to prove that the following equalities meanwhile hold.

$$\begin{aligned} &AA^T = 4I_2,\ A^TA + BB^T = 4I_4,\ B^TB + BB^T = 4I_4, \\ &B^TB + C^TC = 4I_4,\ AB = 0,\ B^2 = 0,\ BC = 0. \end{aligned} \tag{3.1}$$

By the definitions of A, B and C, it is easy to verify that all equalities 3.1 indeed hold. In fact, these equalities can further guarantee $S(\mathcal{G}_i^\sigma)^T S(\mathcal{G}_i^\sigma) = 4I$, because $S(\mathcal{G}_i)$ can be formulated as

$$S(\mathcal{G}_i^\sigma) = \begin{bmatrix} 0 & A & 0 & 0 & \cdots & 0 & 0 \\ -A^T & 0 & B & 0 & \cdots & 0 & 0 \\ 0 & -B^T & 0 & B & \cdots & 0 & 0 \\ 0 & 0 & -B^T & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & B \\ 0 & 0 & 0 & 0 & \cdots & -B^T & C \end{bmatrix}.$$

It should be pointed out that if one only considers $\mathcal{G}_1$, then it is enough to check that parts of the equalities hold. The proof is now complete. ∎

Theorem 3.6 *Let $S(\mathcal{H}_j^\sigma)$ be the skew-adjacency matrix of $\mathcal{H}_j^\sigma$ obtained from Algorithm 2. Then $S(\mathcal{H}_j^\sigma)^T S(\mathcal{H}_j^\sigma) = 4I$.*

Proof. Similar to the proof of Theorem 3.5, let the rows of the skew-adjacency matrix $S(\mathcal{H}_j)$ correspond successively the vertices u, v, u_1, u_2, v_1, v_2, $\ldots$, u_{2i-1}, u_{2i}, v_{2i-1}, v_{2i}, u_{2i+1} and u_{2i+2}.

$$S(\mathcal{H}_1^\sigma) = \begin{bmatrix} 0 & A & 0 \\ -A^T & 0 & D \\ 0 & -D^T & 0 \end{bmatrix} \qquad S(\mathcal{H}_2^\sigma) = \begin{bmatrix} 0 & A & 0 & 0 \\ -A^T & 0 & B & 0 \\ 0 & -B^T & 0 & D \\ 0 & 0 & -D^T & 0 \end{bmatrix}$$

$$S(\mathcal{H}_j^\sigma) = \begin{bmatrix} 0 & A & 0 & 0 & \cdots & 0 & 0 & 0 \\ -A^T & 0 & B & 0 & \cdots & 0 & 0 & 0 \\ 0 & -B^T & 0 & B & \cdots & 0 & 0 & 0 \\ 0 & 0 & -B^T & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & B & 0 \\ 0 & 0 & 0 & 0 & \cdots & -B^T & 0 & D \\ 0 & 0 & 0 & 0 & \cdots & 0 & -D^T & 0 \end{bmatrix}$$

We can verify that $S(\mathcal{H}_1)^T S(\mathcal{H}_1) = 4I$ if and only if the equalities below hold,

$$AA^T = 4I_2,\ A^T A + DD^T = 4I_4,\ D^T D = 4I_2,\ AD = 0. \tag{3.2}$$

while $S(\mathcal{H}_2)^T S(\mathcal{H}_2) = 4I$ if and only if the following equalities hold,

$$\begin{aligned} &AA^T = 4I_2,\ A^TA + BB^T = 4I_4,\ B^TB + D^TD = 4I_4, \\ &D^TD = 4I_2,\ AB = 0,\ BD = 0. \end{aligned} \tag{3.3}$$

For $j \geq 3$, combining equalities in (3.3) with equalities $B^TB + BB^T = 4I_4$ and $B^2 = 0$, it is enough to ensure that the equality $S(\mathcal{H}_j^\sigma)^T S(\mathcal{H}_j^\sigma) = 4I$ holds.

By the definitions of A, B and D, it can be directly checked that the all equalities above indeed hold. This completes the proof. ∎

We can summarize all results above as the following theorem.

Theorem 3.7 *Let G be a 4-regular graph. Then G has an optimum orientation if and only if G is a graph of $\mathcal{F}$.*

Remark 1. For arbitrary matrices A', B', C' and D' with entries 0, 1 and -1, if they have the same orders and the same number of $0'$s with A, B, C and D, respectively, and meanwhile they satisfy all the equalities of Theorem 3.5 and Theorem 3.6, then we can substitute A, B, C and D, respectively by A', B', C' and D' in the skew-adjacency matrices $S(\mathcal{G}_i)$ and $S(\mathcal{H}_j)$, and the corresponding oriented graphs still have optimum skew energy.

Remark 2. The proofs of Theorem 3.5 and Theorem 3.6 are based on matrix computations by proving that the skew-adjacency matrix S satisfies $S^TS = nI$. Besides, we can apply Proposition 2.4 to prove that for any two distinct vertices u and v, the number of all positive walks equals that of all negative walks from u to v with length 2.

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