Affine primitive groups and Semisymmetric graphs

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Abstract

In this paper, we investigate semisymmetric graphs which arise from affine primitive permutation groups. We give a characterization of such graphs, and then construct an infinite family of semisymmetric graphs which contains the Gray graph as the third smallest member. Then, as a consequence, we obtain a factorization of the complete bipartite graph $\mathsf{K}_{p^{sp^t},p^{sp^t}}$ into connected semisymmetric graphs, where p is an prime, $1 \leqslant t \leqslant s$ with $s \geqslant 2$ while $p = 2$.

Keywords: semisymmetric graph; normal quotient; primitive permutation group

1 Introduction

All graphs considered in this paper are assumed to be finite and simple with non-empty edge sets.

For a graph Γ, denote by $V\Gamma$, $E\Gamma$ and $\mathsf{Aut}\Gamma$ the vertex set, edge set and automorphism group, respectively. A graph Γ is said to be *vertex-transitive* or *edge-transitive* if $\mathsf{Aut}\Gamma$ acts transitively on $V\Gamma$ or $E\Gamma$, respectively. A regular graph is called *semisymmetric* if it is edge-transitive but not vertex-transitive. For a graph Γ, an arc of Γ is an ordered pair (α, β) of two adjacency vertices. A graph Γ is called *symmetric* if it has no isolated vertices and $\mathsf{Aut}\Gamma$ acts transitively on the set of arcs of Γ.

The class of semisymmetric graphs was first studied by Folkman [9], who possed several open problem. Afterwards, many authors have done much work on this topic, see [1, 2, 3, 7, 8, 10, 11, 12, 15, 16, 17, 18, 19] for references. In particular, lots of interesting examples of such graphs were found. For example, the Folkman graph on 20 vertices, the smallest semisymmetric graph, was constructed by Folkman [9]; the Gray graph, a

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cubic graph of order 54, was first observed to be semisymmetric by Bouwer [1]. In 1985, Iofinova and Ivanov [11] classified all bi-primitive cubic semisymmetric graphs and they proved that there are only five such graphs.

In this paper, we consider the semisymmetric graphs whose automorphism group contains a subgroup inducing an affine primitive permutation group. For more information about groups of this kind, see [5, 6]. To state our result, we need to introduce several concepts and some notation.

Let p be a prime and $\mathbb{F}_p$ the field of order p. Then, for an integer $l \geqslant 1$ and an irreducible subgroup H of the general linear group $\mathrm{GL}(l,p)$, all affine transformations $\tau_{h,v}$ form a primitive subgroup X of $\mathrm{AGL}(l,p)$ (acting on the vectors of $\mathbb{F}_p^l$), where $h \in H$, $v \in \mathbb{F}_p^l$ and $\tau_{h,v}$ is defined as $\mathbb{F}_p^l \to \mathbb{F}_p^l$, $u \mapsto u^h + v$. The above group is a split extension of a regular normal subgroup $\{\tau_{1,v} \mid v \in \mathbb{F}_p^l\}$ by the subgroup H. For convenience, for a subspace V of $\mathbb{F}_p^l$, we set $\mathrm{T}(V) = \{\tau_{1,v} \mid v \in V\}$ and denote by $N_H(V)$ the subgroup of H fixing V set-wise. Then $X = \mathrm{T}(\mathbb{F}_p^l){:}H$, and $N_H(V) = \mathrm{N}_H(\mathrm{T}(V))$, the normalizer of $\mathrm{T}(V)$ in H.

For a graph Γ and a subgroup $G \leqslant \mathsf{Aut}\Gamma$, Γ is said to be *G-vertex-transitive* or *G-edge-transitive* if G is transitive on $V\Gamma$ or $E\Gamma$, respectively. The graph Γ is called *G-semisymmetric* graphs if it is regular, G-edge-transitive but not G-vertex-transitive.

Let Γ be a connected G-edge-transitive graph, where $G \leqslant \mathsf{Aut}\Gamma$. Assume that G is not transitive on $V\Gamma$. Then Γ is a bipartite graph with two parts, say U and W, which are the G-orbits on $V\Gamma$. Denote respectively by G^U and G^W the permutation groups induced by G on U and on W. Our main result deals with the case where one of G^U and G^W is an affine primitive permutation group.

Theorem 1.1. *Let $X = \mathrm{T}(\mathbb{F}_p^l){:}H$, where p is a prime, $l \geqslant 1$ and H is an irreducible subgroup of the general linear group $\mathrm{GL}(l,p)$. Then the following two statements are equivalent.*

(1) $\mathbb{F}_p^l$ has an s-dimensional subspace V for some integer $1 \leqslant s < l$ such that $|H : \mathrm{N}_H(V)| = p^t$ for some integer $1 \leqslant t \leqslant s$.

(2) There exists a semisymmetric graph Γ with bipartition $V\Gamma = U \cup W$ with one of G^U and G^W is permutation isomorphic to X for some edge-transitive subgroup G of $\mathsf{Aut}\Gamma$.

Remark. Theorem 1.1 suggests the following interesting problem.

Problem 1. *Characterize irreducible subgroups of the general linear group* $\mathrm{GL}(l,p)$ *satisfying* Theorem 1.1 (1).

It is well-known that there are no semisymmetric graph of orders 16, $2p$ and $2p^2$, see [9]. Thus, for Theorem 1.1 (1), we have $l \geqslant 3$ and $(p,l) \neq (2,3)$.

2 Proof of Theorem 1.1

Assume that Γ is a G-edge-transitive but not G-vertex-transitive graph, where $G \leqslant \mathsf{Aut}\Gamma$. Then Γ is a bipartite graph with two parts, say U and W, which are the G-orbits on $V\Gamma$.

It follows that Γ is semiregular, that is, the vertices in a same bipartition subset have the same valency. For a given vertex $\alpha \in V\Gamma$, the stabilizer acts transitively on $\Gamma(\alpha)$. Thus, if Γ is vertex-transitive then it must be symmetric. Take $\beta \in \Gamma(\alpha)$. Then each vertex of Γ can be written as α^g or β^g for some $g \in G$. Then, for two arbitrary vertices α^g and β^h, they are adjacent in Γ if and only if α and $\beta^{hg^{-1}}$ are adjacent, i.e., $hg^{-1} \in G_\beta G_\alpha$. Moreover, it is well-known and easily shown that Γ is connected if and only if $\langle G_\alpha, G_\beta\rangle = G$.

Let Γ be a G-semisymmetric graph with two bipartition subsets U and W. Suppose that G has a subgroup R which is regular on both U and W. Take an edge $\{\alpha, \beta\} \in E\Gamma$. Then each vertex in U (W, resp.) can be written uniquely as α^x (β^x, resp.) for some $x \in R$. Set $S = \{s \in R \mid \beta^s \in \Gamma(\alpha)\}$. Then α^x and β^y are adjacent if and only if $yx^{-1} \in S$. If R is abelian, then it is easily shown that $\alpha^x \mapsto \beta^{x^{-1}}, \beta^x \mapsto \alpha^{x^{-1}}, \forall\, x \in R$ is an automorphism of Γ, which leads to the vertex-transitivity of Γ, refer to [8, 14].

Lemma 2.1. *Let Γ be a G-semisymmetric graph. Assume that G has an abelian subgroup which is regular on both parts of Γ. Then Γ is symmetric.*

Let Γ be a G-semisymmetric graph. Suppose that G has a normal subgroup N which acts intransitively on at least one of the bipartition subsets of Γ. Then we define the *quotient graph* Γ_N to have vertices the N-orbits on $V\Gamma$, and two N-orbits Δ and Δ' are adjacent in Γ_N if and only if some $\alpha \in \Delta$ and some $\beta \in \Delta'$ are adjacent in Γ. It is easy to see that the quotient Γ_N is a regular graph if and only if all N-orbits have the same length.

Let Γ be a finite connected G-semisymmetric graph with $G \leqslant \mathsf{Aut}\Gamma$. Take an edge $\{\alpha, \beta\} \in E\Gamma$ and let $U = \alpha^G$ and $W = \beta^G$ be the two G-orbits on $V\Gamma$. Assume that G is unfaithful on U, and let K be the kernel of G acting on U. Then K is faithful on W, and each K-orbits on W contains at least two vertices. It follows that there are two distinct vertices in W which have the same neighborhood in Γ. Thus, as observed in [8], if any two distinct vertices in U have different neighborhoods in the quotient Γ_K then Γ is semisymmetric and can be reconstructed from Γ_K as follows.

Construction 2.2. Let Σ be a bipartite graph with two bipartition subset U and $\bar{W}$ such that $m|\bar{W}| = |U|$ for integer $m > 1$. Define a bipartite graph $\Sigma^{1,m}$ with vertex set $U \cup (\bar{W} \times \mathbb{Z}_m)$ such that α and (β, i) are adjacent if and only if $\{\alpha, \beta\} \in E\Gamma$. For convenience, we set $\Sigma^{1,1} = \Sigma$.

For a group X, the socle of X, denoted by $\mathsf{soc}(X)$, is generated by all minimal normal subgroups of X. A permutation group is called *quasiprimitive* if each of its minimal normal subgroups is transitive.

Lemma 2.3. *Let Γ be a finite connected G-semisymmetric graph with $G \leqslant \mathsf{Aut}\Gamma$. Take an edge $\{\alpha, \beta\} \in E\Gamma$ and set $U = \alpha^G$ and $W = \beta^G$. Assume that G^U is quasiprimitive. Then either G is faithful on both U and W, or one of the following statements hold.*

(1) Γ is isomorphic to the complete bipartite graph $\mathsf{K}_{|U|,|U|}$;

(2) G is faithful on W but not faithful on U, $G^U \cong G^{\bar{W}}$, and Γ is semisymmetric if further G^U is primitive, where K is the kernel of G on U and $\bar{W}$ is the set of K-orbits on W.

Proof. To prove this lemma, we assume next that G is unfaithful on at least one of U and W and that $\Gamma \not\cong \mathsf{K}_{|U|,|U|}$.

Since $G \leqslant \mathsf{Aut}\Gamma$, the kernel of G on one bipartition subset must be faithful on the other one. Then the above assumption implies that G is faithful on W. Thus G is unfaithful on U. Let K and $\bar{W}$ be as in the lemma. Then $G^U \cong G/K$ and G induces a subgroup $\bar{G} \cong G/K$ of $\mathsf{Aut}\Gamma_K$. Suppose that $\bar{G}$ is unfaithful on $\bar{W}$. Then it follows that $\Gamma_K \cong \mathsf{K}_{|U|,|\bar{W}|}$, and so $\Gamma \cong \mathsf{K}_{|U|,|U|}$ by noting that $\beta^K \subseteq \Gamma(\alpha)$ if $\beta \in \Gamma(\alpha)$, a contradiction. Thus $\bar{G}$ is faithful on $\bar{W}$, so K is the kernel of G acting on $\bar{W}$, and hence $G^U \cong G/K \cong G^{\bar{W}}$.

Note that each K-orbit on W has size at least 2, and that two vertices in a same K-orbit have the same neighborhood in Γ. Assume that G^U is primitive. Then, since $\Gamma \not\cong \mathsf{K}_{|U|,|U|}$, any two distinct vertices in U have different neighborhood. Thus there is no $x \in \mathsf{Aut}\Gamma$ with $U^x = W$, so Γ must be semisymmetric. ∎

Corollary 2.4. *Let G a finite group which acts faithfully and transitively on both nonempty sets U and $\bar{W}$ with $|U| = m|\bar{W}|$ for an integer $m > 1$. For $\alpha \in U$ and a G_α-orbit Θ on $\bar{W}$, define a bipartite graph Σ on $U \cup \bar{W}$ such that $\alpha^g \in U$ and $\bar{\beta} \in \bar{W}$ are adjacent if and only if $\bar{\beta} \in \Theta^g$. If G^U is primitive, then $\Sigma^{1,m}$ is a semisymmetric graph unless $\Theta = \bar{W}$.*

Proof. Assume that G^U is primitive and $\Theta \neq \bar{W}$. By Lemma 2.3, it suffices to show that $\mathsf{Aut}\Sigma^{1,m}$ has an edge-transitive subgroup which fixes U and induces a permutation group on U permutation isomorphic to G^U. Let $Y = G \times \mathbb{Z}_m$. Define an action of Y on $V\Sigma^{1,m}$ as follows:

$$(\alpha^g)^{(x,i)} = \alpha^{gx},\ (\bar{\beta}, j)^{(x,i)} = (\bar{\beta}^x, i+j),\ \forall g, x \in G,\, i, j \in \mathbb{Z}_m, \bar{\beta} \in \bar{W}.$$

Then, under the above action, Y is a subgroup of $\mathsf{Aut}\Sigma^{1,m}$ as desired. ∎

Remark. A graph is called *edge-primitive* if its automorphism group is primitive on its edge set. Using Corollary 2.4, we can construct examples of semisymmetric graphs from an edge-primitive graph of even valency, such as the complete graph K_{2l+1}, the Perkel graph and etc., by taking U, $\bar{W}$ and Θ respectively the edge set, vertex set and an orbit on $\bar{W}$ of some edge-stabilizer of the edge-primitive graph.

Here we pose the next interesting problem.

Problem 2. *Characterize or classify the primitive subgroups of S_n which have transitive permutation representations of degree properly dividing n.*

Lemma 2.5. *Let Γ be a connected G-semisymmetric graph with $G \leqslant \mathsf{Aut}\Gamma$. Take an edge $\{\alpha, \beta\} \in E\Gamma$ and set $U = \alpha^G$ and $W = \beta^G$. Assume that G is faithful on both U and W. Assume that G^U is an affine primitive permutation group and Γ is not a complete bipartite graph. Then either G is primitive on W, or Γ is semisymmetric.*

Proof. Set $N = \mathsf{soc}(G)$. Then $N \cong \mathbb{Z}_p^l$ for a prime p and integer $l \geqslant 1$. It is easily shown that G is primitive on W if and only if N is transitive on W. To finish the proof, in the following, we prove that Γ is semisymmetric if N is intransitive on W.

Suppose that N is intransitive on W and Γ is symmetric. Then $N_\gamma \neq 1$ for any $\gamma \in W$. Let X be the set-wise stabilizer of U in $\mathsf{Aut}\Gamma$. Then $G \leqslant X$ and $|\mathsf{Aut}\Gamma : X| = 2$. Note that X^U is a primitive permutation group. By Lemma 2.3, since Γ is not a complete bipartite graph, we may assume that X is faithful on both U and W. Let $M = \mathsf{soc}(X)$. Take $x \in \mathsf{Aut}\Gamma$ with $\alpha^x = \beta$ and $\beta^x = \alpha$. Then $U^x = W$ and $W^x = U$. Note that X is a primitive permutation group (on U) of degree p^l. Then M is the unique minimal normal subgroup of X by the O'Nan-Scott Theorem, refer to [4, Theorem 4.1A]. Thus $M^x = M$, it follows that M is transitive on both U and W. If X is of affine type then $M = N$ by [20, Proposition 5.1], so N is transitive on U, a contradiction. Then, by [13] and [20, Proposition 5.1], $N < M$ and M is listed as follows:

(i) $M \cong T^l$, where (p, T) is one of $(11, \mathrm{PSL}(2, 11))$, $(11, \mathrm{M}_{11})$, $(23, \mathrm{M}_{23})$ and $(\frac{q^d-1}{q-1}, \mathrm{PSL}(d, q))$; or

(ii) $M = T_1 \times \cdots \times T_t$ and $N = (N \cap T_1) \times \cdots \times (N \cap T_t)$, where $T_i = \mathrm{A}_{p^s}$ and $N \cap T_i \cong \mathbb{Z}_p^s$ for $1 \leqslant i \leqslant t$, $p^s \geqslant 5$ and $st = l$.

For (i) and (ii) with $s = 1$, the stabilizer M_α has order coprime to p, and so does for $M_\alpha^x = M_\beta$, hence $N \cap M_\beta = 1$, which contradicts that $N_\beta \neq 1$. Thus (ii) occurs and $s \geqslant 2$, so $p^s > 5$. It is easily shown that $(T_i)_\alpha \cong \mathrm{A}_{p^s-1}$ and $M_\alpha = (T_1)_\alpha \times \cdots \times (T_t)_\alpha$. Then $M_\beta = M_\alpha^x = ((T_1)_\alpha \times \cdots \times (T_t)_\alpha)^x = ((T_1 \cap M_\alpha) \times \cdots \times (T_t \cap M_\alpha))^x = (T_1)_\beta \times \cdots \times (T_t)_\beta$. It implies that $\{(T_1)_\alpha, \cdots, (T_t)_\alpha\}^x = \{(T_1)_\beta, \cdots, (T_t)_\beta\}$ as all $(T_i)_\alpha$ are simple and nonabelian. In particular, $(T_i)_\beta \cong \mathrm{A}_{p^s-1}$, so $(T_i)_\alpha$ and $(T_i)_\beta$ are conjugate in T_i for $1 \leqslant i \leqslant t$. Thus M_α and M_β are conjugate in M. Since M is transitive on W, there is $\gamma \in W$ with $M_\alpha = M_\gamma$. Then $N_\gamma = N \cap M_\gamma = N \cap M_\alpha = 1$ as N is regular on U, again a contradiction. This completes the proof. ∎

Remark It is well-known that there are no symmetric graphs of order $2p$ and $2p^2$. Thus, for Lemma 2.5, if N is intransitive on W then $N \cong \mathbb{Z}_p^l$ for $l \geqslant 3$, which also follows from checking the irreducible subgroups of $\mathrm{GL}(2, p)$.

Proof of Theorem 1.1. Let Γ be a semisymmetric graph with bipartition $V\Gamma = U \cup W$ satisfying Theorem 1.1 (2). Then Γ must be connected. Without loss of generality, we assume that G^U is permutation isomorphic to X. By Lemma 2.3, G is faithful on W. Let K be the kernel of G acting on U and $\bar{W}$ be the set of K-orbits on W, while $K = 1$ and $\bar{W} = W$ if G is faithful on U. Then, by Lemmas 2.3 and 2.5, Γ_K is edge-transitive but not vertex-transitive. Let $\bar{G}$ be the subgroup of $\mathsf{Aut}\Gamma_K$ induced by G. Then $\bar{G} \cong G/K$, and $\bar{G}^U$ is permutation isomorphic to X.

Set $N = \mathsf{soc}(\bar{G})$. Suppose that N is transitive on $\bar{W}$. Then, since N is faithful on $\bar{W}$, we have $|\bar{W}| = |N| = |U|$. Thus $K = 1$ and $\Gamma = \Gamma_K$, Γ is symmetric by Lemma 2.1, a contradiction. Then N is intransitive on $\bar{W}$. Take an edge $\{\alpha, \bar{\beta}\}$ of Γ_K with $\alpha \in U$ and $\bar{\beta} \in \bar{W}$. Then each N-orbit on $\bar{W}$ has size $|N{:}N_{\bar{\beta}}| \neq 1$, and N has exactly $\frac{|\bar{W}|}{|N:N_{\bar{\beta}}|}$ orbits on $\bar{W}$. Set $|N_{\bar{\beta}}| = p^s$ and $\frac{|\bar{W}|}{|N:N_{\bar{\beta}}|} = p^r$. Then $1 \leqslant r \leqslant s < l$. Since

$\bar{G}$ is transitive on $\bar{W}$, we know that $\bar{G}_\alpha$ is transitive on the set of N-orbits. Let B be an N-orbit containing $\bar{\beta}$. Then $|\bar{G}_\alpha : (\bar{G}_\alpha)_B| = p^r$. Since $\bar{G}_\alpha$ is maximal in $\bar{G}$, we have $\bar{G}_\alpha \not\leqslant \mathrm{N}_{\bar{G}}(N_{\bar{\beta}})$. So $|\bar{G}_\alpha : \mathrm{N}_{\bar{G}_\alpha}(N_{\bar{\beta}})| > 1$. Consider the set-wise stabilizer $(\bar{G}_\alpha)_B$. For $g \in (\bar{G}_\alpha)_B$, noting that $N_{\bar{\beta}}$ is the kernel of N acting on B, we have $\bar{\beta}^{gx} = \bar{\beta}^g$ for $x \in N_{\bar{\beta}}$, so $gxg^{-1} \in N \cap \bar{G}_{\bar{\beta}} = N_{\bar{\beta}}$. Thus $(\bar{G}_\alpha)_B \leqslant \mathrm{N}_{\bar{G}_\alpha}(N_{\bar{\beta}})$. Set $|\bar{G}_\alpha : \mathrm{N}_{\bar{G}_\alpha}(N_{\bar{\beta}})| = p^t$. Then $t \geqslant 1$, and p^t divides $|\bar{G}_\alpha : (\bar{G}_\alpha)_B| = p^r$, so $t \leqslant r \leqslant s$. Noting that $\bar{G}^U$ is permutation isomorphic to X, (1) of Theorem 1.1 follows.

Now we assume that Theorem 1.1 (1) holds. To show (2), it suffices to construct a suitable semisymmetric graph. Set $R = \mathrm{T}(V)\mathrm{N}_H(V)$. Let $\bar{W}$ be the set of right cosets of R in X, and let U be the set of vectors in $\mathbb{F}_p^l$. Then $|\bar{W}| = p^{l-s+t}$. Extend X to a permutation group on $U \cup \bar{W}$ such that $X^U = X$ and X acts on $\bar{W}$ by the right multiplication on the right cosets of R in X. It is easily shown that X is faithful on both U and $\bar{W}$. Let $\Theta = \{Rh \mid h \in H\}$. Then $|\Theta| = \frac{|V||H|}{|R|} = |H : \mathrm{N}_H(V)| = p^t < |\bar{W}|$. Define a bipartite graph Σ on $U \cup \bar{W}$ such that $u \in U$ and $\bar{\beta} \in \bar{W}$ are adjacent if and only if $\bar{\beta} \in \Theta^{\tau_{1,u}}$. Then Σ is X-edge-transitive. Let $m = p^{s-t}$. If $m = 1$ then, by Lemma 2.5, Σ is a semisymmetric graph as desired. If $m > 1$ then, by Corollary 2.4, $\Sigma^{1,m}$ is a semisymmetric graph as desired. This completes the proof. ∎

3 Some examples

Let $X = \mathrm{T}(\mathbb{F}_p^l){:}H$ be a primitive permutation group satisfying Theorem 1.1 (1). Then, up to isomorphism, each graph satisfying Theorem 1.1 (2) can be constructed from an X-edge-transitive graph which is not a complete bipartite graph and has one bipartition subset coinciding with the underlaying set of $\mathbb{F}_p^l$. Let Σ be such an X-edge-transitive graph with two bipartition subsets $U = \mathbb{F}_p^l$ and W. Then, for each $\beta \in W$, the stabilizer X_β normalizes $(\mathrm{T}(\mathbb{F}_p^l))_\beta = \mathrm{T}(V)$, where V is an s-dimensional subspace of U and $X_\beta \cap \mathrm{T}(\mathbb{F}_p^l) = \mathrm{T}(V)$. Thus $X_\beta \leqslant \mathrm{N}_X(\mathrm{T}(V)) = \mathrm{T}(\mathbb{F}_p^l)\mathrm{N}_H(V)$. Assume that $\mathrm{T}(\mathbb{F}_p^l)$ has p^r orbits on W, and set $|H : \mathrm{N}_H(V)| = p^t$. Then $1 \leqslant t \leqslant r \leqslant s < l$, and $|X_\beta| = p^{s-r}|H|$. We next consider one extreme case.

Assume that $X_\beta = \mathrm{T}(V){:}\mathrm{N}_H(V)$. Then $r = t$, and W may be identified with $\cup_{h\in H}\{u + V^h \mid u \in \mathbb{F}_p^l\}$ with the action of X on W as follows:

$$(u + V^h)^{\tau_{1,u'}h'} = (u + u')^{h'} + V^{hh'} \ \forall\, u, u' \in \mathbb{F}_p^l,\ h, h' \in H.$$

For each $u \in U = \mathbb{F}_p^l$, set $\Theta(u) = \{u^h + V^h \mid h \in H\}$. Then $\{\Theta(u) \mid u \in \mathbb{F}_p^l\}$ is the set of H-orbits on W. Moreover, $|\Theta(0)| = p^t < |W| = p^{l-s+t}$, and so $|\Theta(u)| < |W|$ for each $u \in U$. Thus Σ is isomorphic to one of the graphs constructed as follows.

Construction 3.1. Let p, l, H and V be as in Theorem 1.1 (1). Let $U = \mathbb{F}_p^l$ and $W = \cup_{h\in H}\{u + V^h \mid u \in \mathbb{F}_p^l\}$. For each $u_0 \in U$, define a bipartite graph $\mathit{\Gamma}(p, l, s, t; H, u_0)$ on $U \cup W$ such that $u \in U$ and $u' + V^{h'} \in W$ are adjacent if and only if

$$u' + V^{h'} = u + u_0^h + V^h \text{ for some } h \in H,$$

i.e.,

$$u_0 + (u - u')^{h^{-1}} \in V \text{ and } h'h^{-1} \in N_H(V) \text{ for some } h \in H.$$

The next result follows from Lemmas 2.3 and 2.5, Corollary 2.4 and the above argument.

Corollary 3.2. *If there is a graph* $\Sigma = \Gamma(p, l, s, t; H, u_0)$ *as in* Construction 3.1, *then* $\Sigma^{1,p^{s-t}}$ *is semisymmetric.*

Now we construct an infinite family of semisymmetric graphs.

Example 3.3. Let p be a prime, and let s and t be two integers with $1 \leqslant t \leqslant s$ such that $s \geqslant 2$ if further $p = 2$. Let $l = sp^t$. Write $\mathbb{F}_p^l$ in a direct sum

$$\mathbb{F}_p^l = \bigoplus_{i=1}^{p^t} U_i$$

of s-dimensional subspaces. Without loss of generality, for each i, we take $\{e_{ij} \mid 1 \leqslant j \leqslant s\}$ as a basis of U, where e_{ij} is the column vector with the $((i-1)s + j)$-th entry equal to 1 and the other entries equal to zero.

Let H be the subgroup of $\mathrm{GL}(l, p)$ fixes the above decomposition. Then $H \cong \mathrm{GL}(s, p) \wr \mathrm{S}_{p^t}$. Let $V = U_1$ and $\Sigma = \Gamma(p, l, s, t; H, u)$. Set

$$G(p, s, t; u) = \Sigma^{1,m}, \text{ where } m = p^{s-t}$$

with $\Sigma^{1,m} = 1$ while $m = 1$. Then we get a family

$$\mathcal{G} = \{G(p, s, t; u) \mid 1 \leqslant t \leqslant s, p \text{ is a prime}, (p, s) \neq (2, 1), u \in \mathbb{F}_p^{sp^t}\}$$

of semisymmetric graphs, and the following statements hold:

(i) $G(p, s, t; 0)$ has valency $p^s = p^t p^{s-t}$;

(ii) $G(p, s, t; u) = G(p, s, t; u')$ if $u - u' \in U_i$ for some $1 \leqslant i \leqslant p^t$;

(iii) $G(p, s, t; e_{i1}) = G(p, s, t; e_{i'1})$ for $i, i' \geqslant 2$;

(iv) $G(p, s, t; e_{21})$ has valency $p^s(p^s - 1)(p^t - 1)$;

(v) $G(p, s, t; \sum_{a=1}^{k} u_{i_a}) = G(p, s, t; \sum_{a=1}^{k} e_{i_a 1})$ for $2 \leqslant i_1 < i_2 < \cdots i_k \leqslant p^t$ and $u_i \in U_i \setminus \{0\}$;

(vi) $G(p, s, t; \sum_{i=2}^{k} e_{i1})$ has valency $p^s(p^s - 1)^{k-1}\binom{p^t-1}{k-1}$, where $2 \leqslant k \leqslant p^t$.

Therefore the complete graph $K_{p^{sp^t}, p^{sp^t}}$ can be factorized into p^t connected semisymmetric graphs. In particular, $K_{27,27}$ is the edge-disjoint union of three semisymmetric graphs of valency 3, 12 and 12, say, $G(3, 1, 1; 0)$, $G(3, 1, 1; e_{21})$ and $G(3, 1, 1; e_{21} + e_{31})$.

By [3], there is a unique cubic semisymmetric graph of order 54. Thus $G(3, 1, 1; 0)$ is in fact the Gray graph. The smallest members of $\mathcal{G}$ have order 32, which are $G(2, 2, 1; 0)$ and $G(2, 2, 1; e_{21})$ and have valency 4 and 12, respectively.

It is easily shown that, for $s \geqslant 2$, we can get the same graphs as in Example 3.3 if replace H by $H \cap \mathrm{SL}(l,p)$. This is also true for $s = 1$ unless $p = 3$.

Example 3.4. Let p be an odd prime. Write $\mathbb{F}_p^p$ in a direct sum $\mathbb{F}_p^p = \bigoplus_{i=1}^p U_i$ of 1-dimensional subspaces. Assume that $e_i \in U_i$ for each i, where e_i is the column vector with the i-th entry equal to 1 and the other entries equal to zero. Let H be the subgroup of $\mathrm{SL}(p,p)$ fixes the above decomposition. Then $H \cong \mathbb{Z}_{p-1}^{p-1}{:}\mathrm{A}_p$. Let $V = U_1$ and

$$S(p,p;u) = \Gamma(p,p,1,1;H,u).$$

Then each $S(p,p;u)$ is semisymmetric graph, and the following statements hold:

(i) $S(p,p;0) = G(p,1,1;0)$ has valency p;

(ii) $S(p,p;e_i) = G(p,1,1;e_i)$ has valency $p(p-1)^2$, for $p \geqslant 5$ and $2 \leqslant i \leqslant p$;

(iii) $S(3,3;e_2)$ and $S(3,3;e_3)$ have valency 6, and $G(3,1,1,e_2)$ is the edge-disjoint union of these two graphs;

(iv) $S(p,p;\sum_{i=2}^k e_i) = G(p,1,1;\sum_{i=2}^k e_i)$ has valency $p(p-1)^{k-1}\binom{p-1}{k-1}$ for $k \geqslant 3$.

References

[1] I. Z. Bouwer. An edge but not vertex transitive cubic graphs. *Canad Math. Bull.*, 11:533–535, 1968.

[2] I. Z. Bouwer. On edge but not vertex transitive graphs. *J. Combin. Theory Ser. B*, 12:32–40, 1972.

[3] M. Conder, A. Malnič, D. Marušič and P. Potočnik. A census of semisymmetric cubic graphs on up to 768 vertices. *J. Algebr. Comb.*, 23:255–294, 2006.

[4] J. D. Dixon and B. Mortimer. *Permutation Groups.* Springer-Verlag New York Berlin Heidelberg, 1996.

[5] R. Lyndon. *Groups and Geometry.* Cambridge University Press, Cambridge, 1985.

[6] R. Artzy. *Linear Geometry.* Addison–Wesley, 1965.

[7] S. F. Du and D. Marušič. Biprimitive Graphs of Smallest Order. *J Algebr. Comb.*, 9:151–156, 1999.

[8] S. F. Du and M. Y. Xu. A classification of semisymmetric graphs of order $2pq$. *Comm. Algebra*, 28(6):2685–2714, 2000.

[9] J. Folkman. Regular line-symmetric graphs. *J. Combin.Theory Ser. B*, 3:215–232, 1967.

[10] H. Han and Z. P. Lu. Semisymmetric graphs of order $6p^2$ and prime valency. *Sci. China Math.*, 55:2579–2592, 2012.

[11] M. E. Iofinova and A. A. Ivanov. Biprimitive cubic graphs (Russian), in: *Investigation in Algebraic Theory of Combinatorial Objects, Proc. of the Seminar, Institute for System Studies, Moscow*, pp.124–134, 1985.

[12] A. V. Ivanov. On edge but not vertex transitive regular graphs. *Ann. Discrete Math.*, 34:273–286, 1987.

[13] C. H. Li. The finite primitive permutation groups containing an abelian regular subgroup, *Proc. London Math. Soc.*, 87(3):725–747, 2003.

[14] Z. P. Lu. On the automorphism groups of bi-Cayley graphs. *Beijing Daxue Xuebao* 39:1–5, 2003.

[15] Z. P. Lu, C. Q. Wang and M. Y. Xu. On Semisymmetric Cubic Graphs of Order $6p^2$. *Sci.China Ser.A* , 47:1–17, 2004.

[16] A. Malnič, D. Marušič and C. Q. Wang. Cubic edge-transitive graphs of order $2p^3$. *Discrete Math.*, 274:187–198, 2004.

[17] D. Marušič and P. Potočnik. Semisymmetry of generalized Folkman graphs. *European J. Combin.*, 22:333–349, 2001.

[18] B. Monson, T. Pisanski, E. Schulte and A. I. Weiss. Semisymmetric graphs from polytopes. *J. Combin. Theory Ser. A*, 114:421–435, 2007.

[19] C. W. Parker. Semisymmetric cubic graphs of twice odd order. *European J. Combin.*, 28:572–591, 2007.

[20] C. E. Praeger. The inclusion problem for finite primitive permutation groups. *Proc. London Math. Soc.*, 60(3):68–88, 1990.