

Ordered Partitions Avoiding a Permutation Pattern of Length 3

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Abstract

An ordered partition of $[n] = \{1, 2, \dots, n\}$ is a partition whose blocks are endowed with a linear order. Let $\mathcal{OP}_{n,k}$ be the set of ordered partitions of $[n]$ with k blocks and $\mathcal{OP}_{n,k}(\sigma)$ be the set of ordered partitions in $\mathcal{OP}_{n,k}$ that avoid a pattern σ . For any permutation pattern σ of length three, Godbole, Goyt, Herdan and Pudwell obtained formulas for the number of ordered partitions of $[n]$ with 3 blocks avoiding σ as well as the number of ordered partitions of $[n]$ with $n - 1$ blocks avoiding σ . They also showed that $|\mathcal{OP}_{n,k}(\sigma)| = |\mathcal{OP}_{n,k}(123)|$ for any permutation σ of length 3. Moreover, they raised a question concerning the enumeration of $\mathcal{OP}_{n,k}(123)$, and conjectured that the number of ordered partitions of $[2n]$ with blocks of size 2 avoiding σ satisfied a second order linear recurrence relation. In answer to the question of Godbole, et al., we establish a connection between $|\mathcal{OP}_{n,k}(123)|$ and the number $e_{n,d}$ of 123-avoiding permutations of $[n]$ with d descents. Using the bivariate generating function of $e_{n,d}$ given by Barnabei, Bonetti and Silimbani, we obtain the bivariate generating function of $|\mathcal{OP}_{n,k}(123)|$. Meanwhile, we confirm the conjecture of Godbole, et al. by deriving the generating function for the number of 123-avoiding ordered partitions of $[2n]$ with n blocks of size 2.

Keywords: pattern avoidance, ordered partition, descent

AMS Subject Classifications: 05A15, 05A18

1 Introduction

The notion of pattern avoiding permutations was introduced by Knuth [10], and it has been extensively studied. Klazar [7] initiated the study of pattern avoiding set partitions. Further studies of pattern avoiding set partitions can be found in [4, 5, 8, 9, 11].

Recently, Godbole, Goyt, Herdan and Pudwell [3] considered pattern avoiding ordered set partitions. Let $[n] = \{1, 2, \dots, n\}$. For a permutation σ of length 3, Godbole, et al. obtained a formula for the number of σ -avoiding ordered partitions of $[n]$ with 3 blocks and a formula for the number of σ -avoiding ordered partitions of $[n]$ with $n - 1$ blocks. Moreover, they raised a question of finding the number of σ -avoiding ordered partitions of $[n]$ with k blocks.

In answer to the above question, we establish a connection between the number of 123-avoiding ordered partitions of $[n]$ with k blocks and the number of 123-avoiding permutations of $[n]$ with d descents. This enables us to derive a bivariate generating function for the number of 123-avoiding ordered partitions of $[n]$ with k blocks. Meanwhile, we confirm the conjecture of Godbole, Goyt, Herdan and Pudwell [3] on a recurrence relation concerning the number of 123-avoiding ordered partitions of $[2n]$ with blocks of size 2.

Let us give an overview of notation and terminology. Let S_n be the set of permutations of $[n]$. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ and a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in S_k$, where $1 \leq k \leq n$, we say that π contains a pattern σ if there exists a subsequence $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ ($1 \leq i_1 < i_2 < \cdots < i_k \leq n$) of π that is order-isomorphic to σ , in other words, for all $l, m \in [k]$, we have $\pi_{i_l} < \pi_{i_m}$ if and only if $\sigma_l < \sigma_m$. Otherwise, we say that π avoids a pattern σ , or π is σ -avoiding. Let $S_n(\sigma)$ denote the set of permutations of S_n that avoid a pattern σ . For example, 41532 is 123-avoiding, while it contains a pattern 312 corresponding to the subsequence 412.

A partition π of a set $[n]$, written $\pi \vdash [n]$, is a family of nonempty, pairwise disjoint subsets B_1, B_2, \dots, B_k of $[n]$ such that $\cup_{i=1}^k B_i = [n]$, where each B_i ($1 \leq i \leq k$) is called a block. We write $\pi = B_1/B_2/\cdots/B_k$ and define the length of π , denoted $b(\pi)$, to be the number of blocks. An ordered partition of $[n]$ is a partition of $[n]$ whose blocks are endowed with a linear order. Let $\mathcal{OP}_{n,k}$ denote the set of ordered partitions of $[n]$ with k blocks, let \mathcal{OP}_n denote the set of ordered partitions of $[n]$, and let $\mathcal{OP}_{[b_1, b_2, \dots, b_k]}$ denote the set of ordered partitions of $[b_1 + b_2 + \cdots + b_k]$ such that the i -th block contains b_i elements. If $b_1 = \cdots = b_k = s$, we write $\mathcal{OP}_{[s^k]}$ for $\mathcal{OP}_{[b_1, b_2, \dots, b_k]}$. Let $\text{op}_{n,k} = |\mathcal{OP}_{n,k}|$, $\text{op}_n = |\mathcal{OP}_n|$, $\text{op}_{[b_1, b_2, \dots, b_k]} = |\mathcal{OP}_{[b_1, b_2, \dots, b_k]}|$ and $\text{op}_{[s^k]} = |\mathcal{OP}_{[s^k]}|$.

Given an ordered partition $\pi = B_1/B_2/\cdots/B_k \in \mathcal{OP}_{n,k}$ and a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_m \in S_m$, we say that π contains a pattern σ if there exist blocks $B_{i_1}, B_{i_2}, \dots, B_{i_m}$ with $1 \leq i_1 < i_2 < \cdots < i_m \leq k$ and elements $b_1 \in B_{i_1}, b_2 \in B_{i_2}, \dots, b_m \in B_{i_m}$ such that $b_1b_2\cdots b_m$ is order-isomorphic to σ . Otherwise, we say that π avoids a pattern σ . For example, the ordered partition $14/35/2 \in \mathcal{OP}_{5,3}$ is 123-avoiding, while it contains a pattern 132. Similarly, let $\mathcal{OP}_{n,k}(\sigma)$ denote the set of ordered partitions of $\mathcal{OP}_{n,k}$ that are σ -avoiding. Let $\text{op}_{n,k}(\sigma) = |\mathcal{OP}_{n,k}(\sigma)|$, $\text{op}_n(\sigma) = |\mathcal{OP}_n(\sigma)|$, $\text{op}_{[b_1, b_2, \dots, b_k]}(\sigma) = |\mathcal{OP}_{[b_1, b_2, \dots, b_k]}(\sigma)|$ and $\text{op}_{[s^k]}(\sigma) = |\mathcal{OP}_{[s^k]}(\sigma)|$.

Godbole, et al. [3] obtained the following formulas for $\text{op}_{n,3}(\sigma)$ and $\text{op}_{n,n-1}(\sigma)$ for any $\sigma \in S_3$.

Theorem 1.1 For $n \geq 1$, $1 \leq k \leq n$, and for any permutation σ of length 3, we have

$$\begin{aligned} \text{op}_{n,3}(\sigma) &= \left(\frac{n^2}{8} + \frac{3n}{8} - 2 \right) 2^n + 3, \\ \text{op}_{n,n-1}(\sigma) &= \frac{3(n-1)^2}{n(n+1)} \binom{2n-2}{n-1}. \end{aligned} \quad (1.1)$$

Godbole, et al. [3] also showed that

$$\text{op}_{n,k}(\sigma) = \text{op}_{n,k}(123), \quad (1.2)$$

$$\text{op}_{[b_1, b_2, \dots, b_k]}(\sigma) = \text{op}_{[b_1, b_2, \dots, b_k]}(123) \quad (1.3)$$

for any $\sigma \in S_3$. They raised a question concerning the enumeration of $\mathcal{OP}_{n,k}(123)$. Using Zeilberger's Maple package *FindRec* [12], they conjectured that $\text{op}_{[2^k]}(123)$ satisfied the following second order linear recurrence relation.

Conjecture 1.1 For $k \geq 0$, we have

$$\begin{aligned} \text{op}_{[2^{k+2}]}(123) &= \frac{329k^3 + 1215k^2 + 1426k + 528}{2(k+2)(2k+5)(7k+5)} \text{op}_{[2^{k+1}]}(123) \\ &\quad + \frac{3(k+1)(2k+1)(7k+12)}{(k+2)(2k+5)(7k+5)} \text{op}_{[2^k]}(123). \end{aligned} \quad (1.4)$$

In this paper, we provide an answer to the above question by deriving a bivariate generating function for $\text{op}_{n,k}(123)$ and we confirm the conjectured recurrence relation by computing the generating function of $\text{op}_{[2^k]}(123)$.

2 The generating function of $\text{op}_{n,k}(123)$

In this section, we obtain the bivariate generating function of $\text{op}_{n,k}(123)$. Let $F(x, y)$ be the generating function of $\text{op}_{n,k}(123)$, that is,

$$F(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} \text{op}_{n,k}(123) x^n y^k. \quad (2.1)$$

We show that $F(x, y)$ can be expressed in terms of the bivariate generating function $E(x, y)$ of 123-avoiding permutations of $[n]$ with respect to the number of descents. More precisely, for a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$, the descent set of σ is defined by

$$D(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$$

and the number of descents of σ is denoted by $des(\sigma) = |D(\sigma)|$. Barnabei, Bonetti and Silimbani [2] defined the generating function

$$E(x, y) = \sum_{n \geq 0} \sum_{\sigma \in S_n(123)} x^n y^{des(\sigma)} = \sum_{n \geq 0} \sum_{d \geq 0} e_{n,d} x^n y^d, \quad (2.2)$$

where

$$e_{n,d} = |\{\sigma \mid \sigma \in S_n(123), des(\sigma) = d\}|.$$

Furthermore, they obtained the following formula:

$$E(x, y) = \frac{-1 + 2xy + 2x^2y - 2xy^2 - 4x^2y^2 + 2x^2y^3 + \sqrt{1 - 4xy - 4x^2y + 4x^2y^2}}{2xy^2(xy - 1 - x)}. \quad (2.3)$$

The following theorem gives the generating function $F(x, y)$ in terms of $E(x, y)$.

Theorem 2.1 *We have*

$$F(x, y) = E(xy, 1 + y^{-1}),$$

which implies that

$$F(x, y) = \frac{-y - 2xy - 2x + 2x^2y + 2x^2 + y\sqrt{1 - 4xy - 4x + 4x^2y + 4x^2}}{2x(y+1)^2(x-1)}. \quad (2.4)$$

To prove the above theorem, we establish a connection between $op_{n,k}(123)$ and $e_{n,d}$.

Theorem 2.2 *For $n \geq 1$ and $1 \leq k \leq n$, we have*

$$op_{n,k}(123) = \sum_{d=n-k}^{n-1} \binom{d}{n-k} e_{n,d}. \quad (2.5)$$

Proof. Define a map $\varphi: \mathcal{OP}_{n,k}(123) \rightarrow S_n(123)$ as a canonical representation of an ordered partition. Given an ordered partition $\pi = B_1/B_2/\cdots/B_k \in \mathcal{OP}_{n,k}(123)$. If we list the elements of each block in decreasing order and ignore the symbol ‘/’ between two adjacent blocks, we get a permutation $\varphi(\pi) = \sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$. It can be shown that φ is well-defined, that is, $\sigma = \varphi(\pi)$ is a 123-avoiding permutation of S_n . Assume to the contrary that σ contains a 123-pattern, that is, there exist $i < j < l$ such that $\sigma_i\sigma_j\sigma_l$ is a 123-pattern in σ . By the construction of σ , we see that the elements σ_i, σ_j and σ_l are in different blocks in π . This implies that $\sigma_i\sigma_j\sigma_l$ is a 123-pattern of π , a contradiction. Thus $\sigma \in S_n(123)$. Moreover, according to the construction of σ , we find that

$$des(\sigma) \geq \sum_{s=1}^k (|B_s| - 1) = n - k. \quad (2.6)$$

Conversely, given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ in $S_n(123)$ with d descents, we aim to count the preimages π in $\mathcal{OP}_{n,k}(123)$ such that $\varphi(\pi) = \sigma$. If $d < n - k$, by inequality (2.6), it is impossible for any π in $\mathcal{OP}_{n,k}(123)$ to be a preimage of σ . So we may assume that $d \geq n - k$. Let $\pi' = \sigma_1/\sigma_2/\cdots/\sigma_n$. Clearly, $\varphi(\pi') = \sigma$. If $i \in D(\sigma)$, we may merge σ_i and σ_{i+1} of π' into a block to form a new ordered partition π'' . It is easily verified that $\varphi(\pi'') = \sigma$ and $b(\pi'') = n - 1$. Moreover, we may iterate this process if $\text{des}(\pi'') > 0$. Note that at each step we get a preimage of σ with one less block. To obtain the preimages π with k blocks, we need to repeat this process $n - k$ times. Observe that the resulting ordered partition depends only on the positions we choose in $D(\sigma)$. Hence we conclude that there are $\binom{d}{n-k}$ ordered partitions π in $\mathcal{OP}_{n,k}(123)$ such that $\varphi(\pi) = \sigma$. Hence the theorem follows from summing over d . \blacksquare

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Theorem 2.2, we have

$$\begin{aligned} \sum_{k=0}^n \text{op}_{n,k}(123)x^n y^k &= \sum_{k=0}^n \sum_{d=n-k}^{n-1} \binom{d}{n-k} e_{n,d} x^n y^k \\ &= \sum_{d=0}^{n-1} \sum_{k=n-d}^n \binom{d}{n-k} e_{n,d} x^n y^k \\ &= \sum_{d=0}^{n-1} \sum_{j=0}^d \binom{d}{j} e_{n,d} x^n y^{n-j} \\ &= \sum_{d=0}^{n-1} e_{n,d} (xy)^n (1 + y^{-1})^d. \end{aligned}$$

Summing over n , we obtain that $F(x, y) = E(xy, (1 + y^{-1}))$. \blacksquare

An alternative proof of the formula (2.4) for $F(x, y)$ was given by Kasraoui [6]. Setting $y = 1$ in the generating function $F(x, y)$, we are led to the generating function of $\text{op}_n(123)$.

Corollary 2.3 *Let $H(x)$ be the generating function of $\text{op}_n(123)$, that is*

$$H(x) = \sum_{n \geq 0} \text{op}_n(123)x^n.$$

Then we have

$$H(x) = \frac{1}{2} + \frac{1}{1 + \sqrt{1 - 8x + 8x^2}}.$$

The connection between $\text{op}_{n,k}(123)$ and $e_{n,d}$ can be used to derive the following generating function of $\text{op}_{n,n-1}(123)$.

Corollary 2.4 Let $G(x)$ be the generating function of $\text{op}_{n,n-1}(123)$, that is,

$$G(x) = \sum_{n \geq 1} \text{op}_{n,n-1}(123)x^n.$$

Then we have

$$G(x) = \frac{2x^2 - 7x + 2 + 3x\sqrt{1-4x} - 2\sqrt{1-4x}}{2x\sqrt{1-4x}}. \quad (2.7)$$

Proof. By Theorem 2.2, we have

$$\text{op}_{n,n-1}(123) = \sum_{d=1}^{n-1} de_{n,d}. \quad (2.8)$$

It follows that

$$\begin{aligned} G(x) &= \sum_{n \geq 1} \sum_{d=1}^{n-1} de_{n,d}x^n \\ &= \left. \frac{\partial E(x,y)}{\partial y} \right|_{y=1}. \end{aligned}$$

By expression (2.3) for $E(x,y)$, we obtain (2.7). ■

Notice that formula (1.1) for $\text{op}_{n,n-1}$ can be deduced from (2.7).

3 The generating function of $\text{op}_{[2^k]}(123)$

In this section, we compute the generating function of $\text{op}_{[2^k]}(123)$ which leads to the recurrence relation of $\text{op}_{[2^k]}(123)$ as in Conjecture 1.1

Theorem 3.1 Let $Q(x)$ be the generating function of $\text{op}_{[2^k]}(123)$, that is,

$$Q(x) = \sum_{k \geq 0} \text{op}_{[2^k]}(123)x^{2^k}.$$

Then we have

$$Q(x) = \sqrt{\frac{2}{1 + 2x^2 + \sqrt{1 - 12x^2}}}. \quad (3.1)$$

Let $Q'(x)$, $Q''(x)$ and $Q'''(x)$ denote the first derivative, second derivative and third derivative of $Q(x)$, respectively. The following theorem shows that $Q(x)$ satisfies a third order differential equation.

Theorem 3.2 *We have*

$$\begin{aligned} & \left(\frac{21}{2}x^7 + \frac{329}{8}x^5 - \frac{7}{2}x^3 \right) Q'''(x) + \left(99x^6 + \frac{1443}{8}x^4 - 5x^2 \right) Q''(x) \\ & + \left(207x^5 + \frac{717}{8}x^3 + 11x \right) Q'(x) + (72x^4 - 12x^2)Q(x) = 0. \end{aligned} \quad (3.2)$$

Equating coefficients of x^{2n+4} in (3.2), we obtain the recurrence relation (1.4) for $\text{op}_{[2^k]}(123)$.

To prove Theorem 3.1, we construct a bijection between ordered partitions and permutations on multisets. Given an ordered partition $\pi = B_1/B_2/\dots/B_k \in \mathcal{OP}_{n,k}$, its canonical sequence, denoted $\psi(\pi)$, is defined to be a sequence $\rho = \rho_1\rho_2 \dots \rho_n$ with $\rho_i = j$ if $i \in B_j$. Let $\mathcal{W}_{[1^{b_1}2^{b_2}\dots k^{b_k}]}$ denote the set of permutations on a multiset $\{1^{b_1}, 2^{b_2}, \dots, k^{b_k}\}$, where i^r means r occurrences of i . It is easily verified that ψ is a bijection between $\mathcal{OP}_{[b_1, b_2, \dots, b_k]}$ and $\mathcal{W}_{[1^{b_1}2^{b_2}\dots k^{b_k}]}$.

Any permutation $\sigma \in S_m$ corresponds naturally to a unique ordered partition of $[m]$ with each element in its own block. Define the canonical sequence of σ to be the canonical sequence of the corresponding ordered partition. It is not hard to see that the canonical sequence of σ is its inverse σ^{-1} . For example, the canonical sequence of 43512 is 45213.

By the definition of pattern avoiding ordered partitions, we see that an ordered partition π contains a pattern σ if and only if its canonical sequence $\psi(\pi)$ contains a pattern σ^{-1} . This implies that ψ is a bijection between $\mathcal{OP}_{[b_1, b_2, \dots, b_k]}(\sigma)$ and $\mathcal{W}_{[1^{b_1}2^{b_2}\dots k^{b_k}]}(\sigma^{-1})$, where $\mathcal{W}_{[1^{b_1}2^{b_2}\dots k^{b_k}]}(\tau)$ is the set of τ -avoiding permutations in $\mathcal{W}_{[1^{b_1}2^{b_2}\dots k^{b_k}]}$. Hence we have

$$\text{op}_{[b_1, b_2, \dots, b_k]}(\sigma) = |\mathcal{W}_{[1^{b_1}2^{b_2}\dots k^{b_k}]}(\sigma^{-1})|. \quad (3.3)$$

In order to establish the recurrence relation for $\text{op}_{[2^k]}(123)$, we need to use $\text{op}_{[2^k, 1]}(123)$ and $\text{op}_{[2^k, 1, 1]}(123)$. Combining (3.3) and (1.3), we obtain

$$\begin{aligned} \text{op}_{[2^n]}(123) &= |\mathcal{W}_{[1^2 2^2 \dots n^2]}(132)|, \\ \text{op}_{[2^n, 1]}(123) &= |\mathcal{W}_{[1^2 2^2 \dots n^2 (n+1)]}(132)|, \\ \text{op}_{[2^n, 1, 1]}(123) &= |\mathcal{W}_{[1^2 2^2 \dots n^2 (n+1)(n+2)]}(132)|. \end{aligned}$$

Let

$$\begin{aligned} u_{2n} &= |\mathcal{W}_{[1^2 2^2 \dots n^2]}(132)|, \\ u_{2n+1} &= |\mathcal{W}_{[1^2 2^2 \dots n^2 (n+1)]}(132)|, \\ v_{2n} &= |\mathcal{W}_{[1^2 2^2 \dots (n-1)^2 n (n+1)]}(132)|, \end{aligned}$$

where we set $u_0 = v_0 = 1$ and set $u_n = v_n = 0$ for $n < 0$.

We proceed to derive recurrence relations for u_{2n}, u_{2n+1} and v_{2n} that can be used to obtain a system of equations on the generating functions. In particular, we get the generating function of u_{2n} , that is, the generating function of $\text{op}_{[2^n]}(123)$.

Let $U_e(x)$, $U_o(x)$ and $V(x)$ denote the generating functions of u_{2n} , u_{2n+1} and v_{2n} , namely,

$$\begin{aligned} U_e(x) &= \sum_{n \geq 0} u_{2n} x^{2n}, \\ U_o(x) &= \sum_{n \geq 0} u_{2n+1} x^{2n+1}, \\ V(x) &= \sum_{n \geq 0} v_{2n} x^{2n}. \end{aligned}$$

We need the following lemma due to Atkinson, Walker and Linton [1].

Lemma 3.3 *Given two permutations $p = p_1 p_2 \cdots p_n$ and $q = q_1 q_2 \cdots q_n$ of the same multiset of $[n]$, we have*

$$|\mathcal{W}_{[1^{p_1} 2^{p_2} \cdots n^{p_n}]}(132)| = |\mathcal{W}_{[1^{q_1} 2^{q_2} \cdots n^{q_n}]}(132)|.$$

The following theorem gives a recurrence relation for u_{2n} and u_{2n+1} .

Theorem 3.4 *For $n \geq 0$, we have*

$$u_{2n+1} = \sum_{i+j=2n} u_i u_j, \tag{3.4}$$

which implies that

$$U_o(x) = x (U_o^2(x) + U_e^2(x)). \tag{3.5}$$

Proof. Assume that $\pi \in \mathcal{W}_{[1^{2^2} \cdots n^2 (n+1)]}(132)$. Write π in the form $\sigma(n+1)\tau$. Since π is 132-avoiding, both σ and τ are 132-avoiding. Moreover, for any element r in σ and any element s in τ , we have $r \geq s$. Let k be the maximum number in τ . It can be seen that τ contains all the numbers in the multiset $\{1^2, 2^2, \dots, n^2, (n+1)\}$ that are smaller than k , that is, τ contains all the elements in the multiset $\{1^2, 2^2, \dots, (k-1)^2\}$.

There are two cases. If $|\tau|$ is even, then τ contains two occurrences of k . Thus τ is in $\mathcal{W}_{[1^{2^2} \cdots k^2]}(132)$, which is counted by u_{2k} . Moreover, σ is in $\mathcal{W}_{[(k+1)^2 (k+2)^2 \cdots n^2]}(132)$. It is easily seen that $|\mathcal{W}_{[(k+1)^2 (k+2)^2 \cdots n^2]}(132)| = |\mathcal{W}_{[1^{2^2} \cdots (n-k)^2]}(132)|$, which is counted by u_{2n-2k} .

If $|\tau|$ is odd, then we have $\tau \in \mathcal{W}_{[1^2 2^2 \dots (k-1)^2 k]}(132)$ and $\sigma \in \mathcal{W}_{[k(k+1)^2 (k+2)^2 \dots n^2]}(132)$. In this case, $\mathcal{W}_{[1^2 2^2 \dots (k-1)^2 k]}(132)$ is counted by u_{2k-1} . By Lemma 3.3, we see that $|\mathcal{W}_{[k(k+1)^2 \dots n^2]}(132)| = |\mathcal{W}_{[k^2 (k+1)^2 \dots (n-1)^2 n]}(132)|$, which is counted by $u_{2n+1-2k}$. Combining the above two cases, we obtain (3.4).

Using (3.4), we obtain

$$\begin{aligned} U_o(x) &= \sum_{n \geq 0} u_{2n+1} x^{2n+1} \\ &= x \sum_{n \geq 0} \sum_{i+j=2n} u_i u_j x^{2n} \\ &= x \sum_{n \geq 0} \sum_{2i+2j=2n} u_{2i} u_{2j} x^{2n} + x \sum_{n \geq 0} \sum_{2i+1+2j+1=2n} u_{2i+1} u_{2j+1} x^{2n} \\ &= x (U_o^2(x) + U_e^2(x)), \end{aligned}$$

as claimed. ■

The following theorem shows that v_{2n} can be expressed in terms of u_{2n} and u_{2n-1} .

Theorem 3.5 *For $n \geq 0$, we have*

$$v_{2n} = u_{2n} + u_{2n-1}, \quad (3.6)$$

which implies that

$$V(x) = U_e(x) + xU_o(x). \quad (3.7)$$

Proof. Clearly, (3.6) holds for $n = 0$ under the assumptions that $u_{-1} = 0$ and $u_0 = v_0 = 1$. So we assume that $n \geq 1$, and assume that $\pi = \pi_1 \pi_2 \cdots \pi_{2n} \in \mathcal{W}_{[1^2 2^2 \dots (n-1)^2 n(n+1)]}(132)$. There are two cases. If $n+1$ precedes n in π , then we have $\pi_1 = n+1$. Otherwise, $\pi_1(n+1)n$ forms a 132-pattern in π , a contradiction. Using the fact that $\pi_1 = n+1$, it is clear that $\pi \in \mathcal{W}_{[1^2 2^2 \dots (n-1)^2 n(n+1)]}(132)$ if and only if $\pi_2 \pi_3 \cdots \pi_{2n} \in \mathcal{W}_{[1^2 2^2 \dots (n-1)^2 n]}(132)$. Notice that $\mathcal{W}_{[1^2 2^2 \dots (n-1)^2 n]}(132)$ is counted by u_{2n-1} .

If n precedes $n+1$ in π , then there does not exist any 132-pattern of π that contains both n and $n+1$. In this case, we may treat $n+1$ as n . Such permutations form the set $\mathcal{W}_{[1^2 2^2 \dots (n-1)^2 n^2]}(132)$, which is counted by u_{2n} . Combining the above two cases, we obtain (3.6), which yields (3.7). ■

To compute the generating functions $U_e(x)$, $U_o(x)$ and $V(x)$, we still need one more relation, which is given below.

Theorem 3.6 *For $n \geq 1$, we have*

$$u_{2n} = 2 \sum_{2i+j=2n-1} u_{2i} u_j + \sum_{2i+1+j=2n-2} u_{2i+1} u_j - u_{2n-1}, \quad (3.8)$$

which implies that

$$U_e(x) = 1 + 2xU_e(x)U_o(x) - x^2U_e^2(x). \quad (3.9)$$

Proof. Assume that $\pi \in W_{[1^2 2^2 \dots n^2]}(132)$. Write π in the form $\sigma n \tau$ such that n appears in σ . Since π is 132-avoiding, both σ and τ are 132-avoiding. Moreover, for any element r in σ and any element s in τ , we have $r \geq s$.

Let k be the maximum number in τ . There are two cases. If $|\tau|$ is even, using the same argument as in Theorem 3.4, we deduce that $\tau \in \mathcal{W}_{[1^2 2^2 \dots k^2]}(132)$ and $\sigma \in \mathcal{W}_{[(k+1)^2 \dots (n-1)^2 n]}(132)$. In this case, $\mathcal{W}_{[1^2 2^2 \dots (k-1)^2 k^2]}(132)$ is counted by u_{2k} and $\mathcal{W}_{[(k+1)^2 \dots (n-1)^2 n]}(132)$ is counted by $u_{2n-1-2k}$.

If $|\tau|$ is odd, it can be seen that τ is in $\mathcal{W}_{[1^2 2^2 \dots (k-1)^2 k]}(132)$, which is counted by u_{2k-1} , and σ is in $\mathcal{W}_{[k(k+1)^2 \dots (n-1)^2 n]}(132)$. By Lemma 3.3, we find that

$$|\mathcal{W}_{[k(k+1)^2 \dots (n-1)^2 n]}(132)| = |\mathcal{W}_{[k^2 \dots (n-2)^2 (n-1)n]}(132)|,$$

which is counted by v_{2n-2k} . Observing that σ is not empty, we have $2n - 2k > 0$.

Combining the above two cases, we get

$$u_{2n} = \sum_{2i+j=2n-1} u_{2i}u_j + \sum_{2i+1+j=2n-1} u_{2i+1}v_j - u_{2n-1}.$$

In view of relation (3.6), we obtain

$$\begin{aligned} u_{2n} &= \sum_{2i+j=2n-1} u_{2i}u_j + \sum_{2i+1+j=2n-1} u_{2i+1}u_j + \sum_{2i+1+j=2n-1} u_{2i+1}u_{j-1} - u_{2n-1} \\ &= 2 \sum_{2i+j=2n-1} u_{2i}u_j + \sum_{2i+1+j=2n-2} u_{2i+1}u_j - u_{2n-1}. \end{aligned}$$

It remains to prove relation (3.9). Using (3.8), we have

$$\begin{aligned} U_e(x) &= 1 + \sum_{n \geq 1} u_{2n}x^{2n} \\ &= 1 + \sum_{n \geq 1} \left(2 \sum_{2i+j=2n-1} u_{2i}u_j + \sum_{2i+1+j=2n-2} u_{2i+1}u_j - u_{2n-1} \right) x^{2n} \\ &= 1 + 2 \sum_{n \geq 1} \sum_{2i+j=2n-1} u_{2i}u_j x^{2n} + \sum_{n \geq 1} \sum_{2i+1+j=2n-2} u_{2i+1}u_j x^{2n} - \sum_{n \geq 1} u_{2n-1}x^{2n} \\ &= 1 + 2xU_e(x)U_o(x) + x^2U_o^2(x) - xU_o(x). \end{aligned} \quad (3.10)$$

Substituting (3.5) into (3.10), we obtain

$$U_e(x) = 1 + 2xU_e(x)U_o(x) + x^2U_o^2(x) - x^2(U_o^2(x) + U_e^2(x))$$

$$= 1 + 2xU_e(x)U_o(x) - x^2U_e^2(x),$$

as claimed. ■

We are now ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Note that $Q(x) = U_e(x)$. By (3.9), we get

$$U_o(x) = \frac{x^2U_e^2(x) + U_e(x) - 1}{2xU_e(x)}. \quad (3.11)$$

Plugging (3.11) into (3.5) yields the following equation

$$(x^4 + 4x^2)U_e^4(x) - (2x^2 + 1)U_e^2(x) + 1 = 0. \quad (3.12)$$

Given the initial values of u_{2n} , we are led the solution of $U_e(x)$ as given by (3.1). ■

To conclude, we note that the generating functions $U_o(x)$ and $V(x)$ are given as follows:

$$U_o(x) = \frac{1}{2x} - \frac{1 + \sqrt{1 - 12x^2}}{4x}U_e(x),$$

$$V(x) = \frac{1}{2} + \frac{3 - \sqrt{1 - 12x^2}}{4}U_e(x).$$

Acknowledgments. We wish to thank the referees for valuable suggestions. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

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