Congruences of Multipartition Functions Modulo Powers of Primes

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Abstract. Let $p_r(n)$ denote the number of r-component multipartitions of n, and let $S_{\gamma,\lambda}$ be the space spanned by $\eta(24z)^\gamma\phi(24z)$, where $\eta(z)$ is the Dedekind's eta function and $\phi(z)$ is a holomorphic modular form in $M_\lambda(\mathrm{SL}_2(\mathbb{Z}))$. In this paper, we show that the generating function of $p_r(\frac{m^kn+r}{24})$ with respect to n is congruent to a function in the space $S_{\gamma,\lambda}$ modulo m^k. As special cases, this relation leads to many well known congruences including the Ramanujan congruences of $p(n)$ modulo $5, 7, 11$ and Gandhi's congruences of $p_2(n)$ modulo 5 and $p_8(n)$ modulo 11. Furthermore, using the invariance property of $S_{\gamma,\lambda}$ under the Hecke operator T_{ℓ^2}, we obtain two classes of congruences pertaining to the m^k-adic property of $p_r(n)$.

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1 Introduction

The objective of this paper is to use the theory of modular forms to derive certain congruences of multipartitions modulo powers of primes.

Recall that an ordinary *partition* λ of a nonnegative integer n is a non-increasing sequence of positive integers whose sum is n, where n is called the weight of λ. The partition function $p(n)$ is defined to be the number of partitions of n. A *multipartition* of n with r components, as called by Andrews [2], also referred to as an r-colored partition, (see, for example [9, 11]) is an r-tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of partitions whose weights sum to n. The number of r-component multipartitions of n is denoted by $p_r(n)$.

Multipartitions arise in combinatorics, representation theory, and physics. As pointed out by Fayers [12], the representations of the Ariki–Koike algebra are naturally indexed by multiparititions. Bouwknegt [8] showed that the Durfee square formulas of multipartitions are useful in deriving expressions for the characters of modules of affine Lie algebras in terms of the universal chiral partition functions.

For the partition function $p(n)$, Ramanujan [25–28] proved that

$$p(An+B) \equiv 0 \pmod{M}, \tag{1.1}$$

for all nonnegative integers n and for $(A, B, M) = (5, 4, 5)$, $(7, 5, 7)$ and $(11, 6, 11)$. In general, congruences of form (1.1) are called Ramanujan-type congruences. For $m = 5$ and 7, Watson [31] proved that

$$p(m^k n + \beta_{m,k}) \equiv 0 \pmod{m^{r_k}}, \tag{1.2}$$

where $k \geq 1$, $\beta_{m,k} \equiv 1/24 \pmod{m^k}$, $r_k = k$ for $m = 5$ and $r_k = \lfloor k/2 \rfloor + 1$ for $m = 7$. The case $m = 5$ in (1.2) was considered by Ramanujan, see Berndt and Ono [7]. Atkin [3] showed that (1.2) is also valid for $m = 11$. When M is not a power of $5, 7$ or 11, Atkin and O'Brien [5] discovered the following congruence

$$p(11^3 \cdot 13n + 237) \equiv 0 \pmod{13}.$$

Using the theory of modular forms, Ono [23] proved that, for any prime $m \geq 5$ and positive integer k, there is a positive proportion of primes ℓ such that

$$p\left(\frac{m^k \ell^3 n + 1}{24}\right) \equiv 0 \pmod{m} \tag{1.3}$$

holds for every nonnegative integer n coprime to ℓ. Weaver [32] gave an algorithm for finding the values of ℓ in (1.3) for primes $13 \leq m \leq 31$. Recently, Folsom, Kent, and Ono [13] provided a very general theorem which gives new generalized partition congruences systematically. In this framework, they proved that if $5 \leq m \leq 31$ is a prime and k is a positive integer, then there exists an integer $A_m(b_1, b_2, k)$ such that

$$p\left(\frac{m^{b_1} n + 1}{24}\right) \equiv A_m(b_1, b_2, k) p\left(\frac{m^{b_2} n + 1}{24}\right) \pmod{m^k} \tag{1.4}$$

for all positive integers n and $b_1 \equiv b_2 \pmod{2}$ larger than some fixed integer.

Ramanujan-type congruences of $p_r(n)$ have been extensively studied, see, for example [2, 4, 14, 15, 17, 19, 22, 30]. Gandhi [14] derived the following congruences of $p_r(n)$ by applying the identities of Euler and Jacobi

$$p_2(5n+3) \equiv 0 \pmod{5}, \tag{1.5}$$

$$p_8(11n+4) \equiv 0 \pmod{11}. \tag{1.6}$$

With the aid of Sturm's theorem [29], Eichhorn and Ono [11] computed an upper bound $C(A, B, r, m^k)$ such that

$$p_r(An+B) \equiv 0 \pmod{m^k}$$

holds for all nonnegative integers n if and only if it is true for $n \leq C(A, B, r, m^k)$. For example, to prove (1.5), it suffices to check that it holds for $n \leq 3$. In the same vain, one can prove (1.6) by verifying that it holds for $n \leq 11$. Treneer [30] extended (1.3) to weakly holomorphic modular forms and showed that, for any prime $m \geq 5$ and positive integers k, there is a positive proportion of primes ℓ such that

$$p_r\left(\frac{m^k \ell^{\mu_r} n + r}{24}\right) \equiv 0 \pmod{m}$$

for every nonnegative integer n coprime to ℓ, where μ_r equals 1 if r is even and 3 if r is odd. Using the methods of Folsom, Kent, and Ono [13], Belmont et al. [6, Corollary 1.2] generalized congruence (1.4) to the cases of $p_r(n)$. They proved that if the rank of the corresponding space is no more than 1, then there exists an integer $C_\ell(r, b_1, b_2, k)$ such that

$$p_r\left(\frac{m^{b_1} + r}{24}\right) \equiv C_\ell(r, b_1, b_2, k) p_r\left(\frac{m^{b_2} + r}{24}\right) \pmod{m^k}, \tag{1.7}$$

where n is a positive integer and $b_1 \equiv b_2 \pmod{2}$ are large enough integers.

The aim of this paper is to study congruence properties of $p_r(n)$ modulo powers of primes. For example, we shall derive the following two classes of congruences

$$p_r\left(\frac{m^k \ell^{2\mu K - 1} n + r}{24}\right) \equiv 0 \pmod{m^k}, \tag{1.8}$$

$$p_r\left(\frac{m^k \ell^{i} n + r}{24}\right) \equiv p_r\left(\frac{m^k \ell^{2M+i} n + r}{24}\right) \pmod{m^k}, \tag{1.9}$$

where r is an odd integer, ℓ is any prime other than $2, 3$ and m, and μ is an arbitrary positive integer, K and M are fixed positive integers, and n is a positive integer coprime to ℓ.

To derive congruences of $p_r(n)$, one may consider the congruence properties of the generating functions of $p_r(n)$. For the case of ordinary partitions, i.e., $r = 1$, Chua [10] showed that

$$\sum_{mn \equiv -1 \pmod{24}} p\left(\frac{mn + 1}{24}\right) q^n \equiv \eta(24z)^{\gamma_m} \phi_m(24z) \pmod{m}, \tag{1.10}$$

where $\eta(z)$ is Dedekind's eta function, γ_m is an integer depending on m, and $\phi_m(z)$ is a holomorphic modular form. Ahlgren and Boylan [1] extended (1.10) to congruences modulo powers of primes, namely,

$$F_{m,k}(z) = \sum_{m^k n \equiv -1 \pmod{24}} p\left(\frac{m^k n + 1}{24}\right) q^n \equiv \eta(24z)^{\gamma_{m,k}} \phi_{m,k}(24z) \pmod{m^k}, \tag{1.11}$$

where $\gamma_{m,k}$ is an integer and $\phi_{m,k}(z)$ is a holomorphic modular form.

In order to prove the existence of congruences of $p_r(n)$ modulo powers of primes, Brown and Li [9] introduced the generating function

$$G_{m,k,r}(z) \equiv \sum_{\left(\frac{n}{m}\right)=-\left(\frac{-r}{m}\right)} p_r\left(\frac{n+r}{24}\right) q^n \pmod{m^k}, \tag{1.12}$$

and showed that $G_{m,k,r}(z)$ is a modular form of level $576m^3$. Kilbourn [18] used the generating function

$$H_{m,k,r}(z) \equiv \sum_{mn\equiv -r \pmod{24}} p_r\left(\frac{mn+r}{24}\right) q^n \pmod{m^k}, \tag{1.13}$$

and proved that $H_{m,k,r}(z)$ is a modular form of level $576m$. However, due to the large dimensions of the spaces $M_\lambda(\Gamma_0(576m^3))$ and $M_\lambda(\Gamma_0(576m))$, it does not seem to be a feasible task to compute explicit bases. In other words, to derive explicit congruence formulas of $p_r(n)$, it is desirable to find a generating function of $p_r(n)$ that can be expressed in terms of modular forms of a small level.

In this paper, we find the following extension of the generating function $F_{m,k}(z)$, namely,

$$F_{m,k,r}(z) = \sum_{m^k n\equiv -r \pmod{24}} p_r\left(\frac{m^k n+r}{24}\right) q^n, \tag{1.14}$$

where $q = e^{2\pi i z}$. We show that $F_{m,k,r}(z)$ is congruent to a weakly holomorphic function modulo m^k. More precisely, we find

$$F_{m,k,r}(z) \equiv \eta(24z)^{\gamma_{m,k,r}} \phi_{m,k,r}(24z) \pmod{m^k}, \tag{1.15}$$

where $\gamma_{m,k,r}$ is an integer and $\phi_{m,k,r}(z)$ is a holomorphic modular form in $M_{\lambda_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$. Noting that any element of $M_{\lambda_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$ can be expressed as a polynomial of the Eisenstein series $E_4(z)$ and $E_6(z)$, this enables us to derive explicit congruences of the generating function of $p_r(n)$ modulo m^k.

If $\phi_{m,k,r}(z) = 0$, then (1.15) yields a Ramanujan-type congruence as follows

$$p_r\left(\frac{m^k n+r}{24}\right) \equiv 0 \pmod{m^k}. \tag{1.16}$$

For example, it is easily checked that $\phi_{5,1,2}(z) = 0$ and $\phi_{11,1,2}(z) = 0$, hence Gandhi's congruences (1.5) and (1.6) are the consequences of (1.16). We also find

$$p_2(5^2 n + 23) \equiv 0 \pmod{5^2}, \tag{1.17}$$

$$p_8(11^2n+81) \equiv 0 \pmod{11^2}, \tag{1.18}$$

since $\phi_{5,2,2}(z) = 0$ and $\phi_{11,2,8}(z) = 0$. For more congruences of form (1.16), see Table 5.2.

On the other hand, if $\phi_{m,k,r}(z) \neq 0$ in (1.15), we may use Yang's method [33] to find congruences of form (1.8). For example, since $F_{5,2,3}(z)$ is congruent to a modular form in the invariant space $S_{21,48}$ of T_{5^2} modulo 5^2, we have

$$p_3\left(\frac{5^2 \cdot 13^{199}n+3}{24}\right) \equiv 0 \pmod{5^2}.$$

2 Preliminaries

To make this paper self-contained, we recall some definitions and facts on modular forms. In particular, we shall use the U-operator, the V-operator, the Hecke operator, and the twist operator on modular forms.

Let $k \in \frac{1}{2}\mathbb{Z}$ be an integer or a half-integer, N be a positive integer (with $4|N$ if $k \notin \mathbb{Z}$) and χ be a Nebentypus character. We use $M_k(\Gamma_0(N), \chi)$ to denote the space of holomorphic modular forms on $\Gamma_0(N)$ of weight k and character χ. The corresponding space of cusp forms is denoted by $S_k(\Gamma_0(N), \chi)$. If χ is the trivial character, we shall write $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ for $M_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$, respectively. Moreover, we write $\mathrm{SL}_2(\mathbb{Z})$ for $\Gamma_0(1)$.

Let $f(z) \in M_k(\Gamma_0(N), \chi)$ with the following Fourier expansion at ∞

$$f(z) = \sum_{n \geq 0} a(n)q^n,$$

where $q = e^{2\pi i z}$. Let us recall some operators acting on $f(z)$.

Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a 2×2 real matrix with positive determinant. The k *slash operator* $|_k$ is defined by

$$(f|_k\gamma)(z) = (\det\gamma)^{k/2}(cz+d)^{-k}f(\gamma z), \tag{2.1}$$

where

$$\gamma z = \frac{az+b}{cz+d}.$$

In particular, let ℓ be an integer and

$$\gamma_\ell = \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix}.$$

The *Fricke involution* W_ℓ is given by

$$f|W_\ell = f|_k\gamma_\ell. \tag{2.2}$$

The *U-operator* U_ℓ and *V-operator* V_ℓ are defined by

$$f(z)|U_\ell = \sum_{n\geq 0} a(\ell n)q^n \tag{2.3}$$

and

$$f(z)|V_\ell = \sum_{n\geq 0} a(n)q^{\ell n}. \tag{2.4}$$

It is known that

$$f(z)|_kU_\ell = \ell^{\frac{k}{2}-1}\sum_{\mu=0}^{\ell-1} f(z)\Big|_k \begin{pmatrix} 1 & \mu \\ 0 & \ell \end{pmatrix}. \tag{2.5}$$

Let ψ be a Dirichlet character. The ψ-twist of $f(z)$ is defined by

$$(f\otimes\psi)(z) = \sum_{n\geq 0}\psi(n)a(n)q^n.$$

Let ℓ be a prime and $f(z)\in M_{\lambda+\frac{1}{2}}(\Gamma_0(N),\chi)$ be a modular form of half-integral weight. The Hecke operator T_{ℓ^2} is defined by

$$f(z)|T_{\ell^2} = \sum_{n\geq 0}\left(a(\ell^2 n) + \chi(\ell)\left(\frac{(-1)^\lambda n}{\ell}\right)\ell^{\lambda-1}a(n) + \chi(\ell^2)\ell^{2\lambda-1}a\left(\frac{n}{\ell^2}\right)\right)q^n. \tag{2.6}$$

We will use the following level reduction properties of the operators U_ℓ and $\mathrm{Tr}_\ell = U_\ell + \ell^{\frac{k}{2}-1}W_\ell$ (see [20, Lemma 1] and [10, Lemma 2.2]).

Lemma 2.1 *Let $k\in\mathbb{Z}$, N be a positive integer, χ be a character modulo N, and $f(z)\in M_k(\Gamma_0(N),\chi)$. Assume that ℓ is a prime factor of N and χ is also a character modulo N/ℓ.*

1. *If $\ell^2 \mid N$, then $f|U_\ell \in M_k(\Gamma_0(N/\ell),\chi)$.*
2. *If $N=\ell$ and χ is the trivial character, then $f|\mathrm{Tr}_\ell \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.*

In the proof of congruence (1.15) on the generating function $F_{m,k,r}(z)$, we need the following relation

$$\eta(\gamma z) = \epsilon_{a,b,c,d}(cz+d)^{\frac{1}{2}}\eta(z), \tag{2.7}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $\epsilon_{a,b,c,d}$ is a 24th root of unity, and $\eta(z)$ is Dedekind's eta function as given by

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n). \tag{2.8}$$

As a special case, we have

$$\eta(-1/z) = \sqrt{z/i} \cdot \eta(z). \tag{2.9}$$

3 The generating function of $p_r(n)$ modulo m^k

In this section, we derive the congruence of the generating function $F_{m,k,r}(z)$ defined by (1.14), namely,

$$F_{m,k,r}(z) = \sum_{m^k n \equiv -r \pmod{24}} p_r\left(\frac{m^k n + r}{24}\right) q^n.$$

Theorem 3.1 *Let $m \geq 5$ be a prime, and let k and r be positive integers. Then there exists a modular form $\phi_{m,k,r}(z) \in M_{\lambda_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$ such that*

$$F_{m,k,r}(z) \equiv \eta(24z)^{\gamma_{m,k,r}} \phi_{m,k,r}(24z) \pmod{m^k}, \tag{3.1}$$

where

$$\lambda_{m,k,r} = \begin{cases} \frac{m^k - m^{k-1}}{2} r - \frac{\gamma_{m,k,r}+r}{2} & \text{if } k \text{ is odd,} \\ (m^k - m^{k-1}) r - \frac{\gamma_{m,k,r}+r}{2} & \text{if } k \text{ is even,} \end{cases} \tag{3.2}$$

$$\gamma_{m,k,r} = \frac{24\beta_{m,k,r} - r}{m^k}, \tag{3.3}$$

and $\beta_{m,k,r}$ is the unique integer in the range $0 \leq \beta_{m,k,r} < m^k$ congruent to $r/24$ modulo m^k.

The first step of the proof of Theorem 3.1 is to express $F_{m,k,r}(z)$ in terms of a modular form. Consider the η-quotient

$$f_{m,k,r}(z) = \left(\frac{\eta(m^k z)^{m^k}}{\eta(z)}\right)^r, \tag{3.4}$$

which is a modular form in $M_{\frac{(m^k-1)r}{2}}\left(\Gamma_0(m^k), \left(\frac{\cdot}{m}\right)^{kr}\right)$. The following lemma shows that $F_{m,k,r}(z)$ can be obtained from $f_{m,k,r}(z)$ by applying the U-operator and the V-operator.

Lemma 3.2 *Let $m \geq 5$ be a prime, and let k and r be positive integers. Then we have*

$$F_{m,k,r}(z) = \frac{(f_{m,k,r}(z)|U_{m^k}) \mid V_{24}}{\eta(24z)^{m^k r}}. \tag{3.5}$$

Proof. Since

$$\sum_{n=0}^{\infty} p_r(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^r},$$

we find that

$$\begin{aligned} f_{m,k,r}(z) &= q^{\frac{m^{2k}-1}{24}r} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^r} \cdot \prod_{n=1}^{\infty} (1-q^{m^k n})^{m^k r} \\ &= q^{\frac{m^{2k}-1}{24}r} \sum_{n=0}^{\infty} p_r(n)q^n \cdot \prod_{n=1}^{\infty} (1-q^{m^k n})^{m^k r}. \end{aligned}$$

Applying the operator U_{m^k} to the above relation, we obtain

$$f_{m,k,r}(z)|U_{m^k} = q^{\frac{m^{2k}-1}{24m^k}r} \sum_{n=0}^{\infty} p_r(m^k n)q^n \cdot \prod_{n=1}^{\infty} (1-q^n)^{m^k r}. \tag{3.6}$$

Let $0 \leq \beta_{m,k,r} \leq m^k - 1$ be the integer uniquely determined by the congruence $24\beta_{m,k,r} \equiv r \pmod{m^k}$. Substituting n by $n + \frac{\beta_{m,k,r}}{m^k}$ in the summation in (3.6), we find

$$f_{m,k,r}(z)|U_{m^k} = \sum_{n=0}^{\infty} p_r(m^k n + \beta_{m,k,r}) q^{n + \frac{r(m^{2k}-1)+24\beta_{m,k,r}}{24m^k}} \cdot \prod_{n=1}^{\infty} (1-q^n)^{m^k r},$$

which belongs to $\mathbb{Z}[[q]]$. So we deduce that

$$\sum_{n=0}^{\infty} p_r(m^k n + \beta_{m,k,r}) q^{n + \frac{r(m^{2k}-1)+24\beta_{m,k,r}}{24m^k}} = \frac{f_{m,k,r}(z)|U_{m^k}}{\prod_{n=1}^{\infty}(1-q^n)^{m^k r}}.$$

Applying the operator V_{24}, we get

$$\sum_{n=0}^{\infty} p_r(m^k n + \beta_{m,k,r}) q^{24n + \frac{24\beta_{m,k,r}-r}{m^k}} = \frac{(f_{m,k,r}(z)|U_{m^k}) \mid V_{24}}{\eta(24z)^{m^k r}}. \tag{3.7}$$

Substituting $24n + \frac{24\beta_{m,k,r}-r}{m^k}$ by n in (3.7), the sum on the left-hand side becomes $F_{m,k,r}(z)$. This completes the proof. ∎

The second step of the proof of Theorem 3.1 is to derive a congruence relation for $f_{m,k,r}(z)|U_{m^k}$ modulo m^k.

Theorem 3.3 *Let $m \geq 5$ be a prime, and let k and r be positive integers. Then there exists a modular form $G_{m,k,r}(z) \in M_{w_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$ such that*

$$f_{m,k,r}(z)|U_{m^k} \equiv G_{m,k,r}(z) \pmod{m^k},$$

where

$$w_{m,k,r} = \begin{cases} \frac{2m^k - m^{k-1} - 1}{2} r & \text{if } k \text{ is odd,} \\ \frac{3m^k - 2m^{k-1} - 1}{2} r & \text{if } k \text{ is even.} \end{cases}$$

Proof. Let

$$g_{m,k,r}(z) = \left(\frac{\eta(z)^m}{\eta(mz)} \right)^{c_k\, m^{k-1} r},$$

where

$$c_k = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even.} \end{cases}$$

Since $g_{m,k,r}(z)$ is an η-quotient, using the modular transformation property due to Gordon, Hughes, and Newman [16, 21], see also, [24, Theorem 1.64], we deduce that

$$g_{m,k,r}(z) \in M_{\frac{c_k\,(m^k - m^{k-1})r}{2}} \left(\Gamma_0(m), \left(\frac{\cdot}{m} \right)^{kr} \right).$$

Moreover, since $(1-q^n)^m \equiv 1 - q^{mn} \pmod{m}$, we see that

$$\eta(z)^m \equiv \eta(mz) \pmod{m},$$

which implies that

$$g_{m,k,r}(z) \equiv 1 \pmod{m^k}. \tag{3.8}$$

Since $f_{m,k,r}(z) \in S_{\frac{(m^k-1)r}{2}} \left(\Gamma_0(m^k), \left(\frac{\cdot}{m}\right)^{kr} \right)$, using Lemma 2.1 repeatedly, we obtain that

$$f_{m,k,r}(z)|U_{m^{k-1}} \in S_{\frac{(m^k-1)r}{2}} \left(\Gamma_0(m), \left(\frac{\cdot}{m} \right)^{kr} \right).$$

Thus, $f_{m,k,r}(z)|U_{m^{k-1}} \cdot g_{m,k,r}(z)$ is a modular form on $\Gamma_0(m)$ of the trivial character and of weight

$$w_{m,k,r} = \frac{c_k\,(m^k - m^{k-1})r}{2} + \frac{(m^k-1)r}{2}.$$

Invoking Lemma 2.1, we find that

$$G_{m,k,r}(z) = \left(f_{m,k,r}(z)|U_{m^{k-1}} \cdot g_{m,k,r}(z) \right) |Tr_m \tag{3.9}$$

is a modular form in $M_{w_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$.

To complete the proof of Theorem 3.3, it remains to show that

$$(f_{m,k,r}(z)|U_{m^{k-1}}\cdot g_{m,k,r}(z))\,|\mathrm{Tr}_m \equiv f_{m,k,r}(z)|U_{m^k} \pmod{m^k}, \tag{3.10}$$

where

$$\mathrm{Tr}_m = U_m + m^{\frac{w_{m,k,r}}{2}-1}W_m,$$

and the operator W_m is given by (2.2). By congruence (3.8), we see that the left-hand side of (3.10) equals

$$f_{m,k,r}(z)|U_{m^k} + m^{\frac{w_{m,k,r}}{2}-1}\,(f_{m,k,r}(z)|U_{m^{k-1}}\cdot g_{m,k,r}(z))\,|W_m \pmod{m^k}.$$

To prove (3.10), it suffices to show that

$$m^{\frac{w_{m,k,r}}{2}-1}\,(f_{m,k,r}(z)|U_{m^{k-1}}\cdot g_{m,k,r}(z))\,|W_m \equiv 0 \pmod{m^k}. \tag{3.11}$$

We only consider the case when k is odd. The case when k is even can be dealt with in the same manner. In light of the transformation formula (2.9) of the eta function, we find that

$$\begin{aligned} g_{m,k,r}(z)|W_m &= m^{\frac{(m^k-m^{k-1})r}{4}}(mz)^{-\frac{(m^k-m^{k-1})r}{2}}g_{m,k,r}\left(-\frac{1}{mz}\right)\\ &= m^{-\frac{(m^k-m^{k-1})r}{4}}z^{-\frac{(m^k-m^{k-1})r}{2}}\left(\frac{(\sqrt{mz/i}\,\eta(mz))^m}{\sqrt{z/i}\,\eta(z)}\right)^{m^{k-1}r}\\ &= m^{\frac{(m+1)m^{k-1}r}{4}}(-i)^{\frac{(m-1)m^{k-1}r}{2}}\left(\frac{\eta(mz)^m}{\eta(z)}\right)^{m^{k-1}r}. \end{aligned}$$

Therefore, (3.11) can be deduced from the following congruence

$$m^{\frac{(3m^k-1)r}{4}-1}\,(\,f_{m,k,r}(z)|\,U_{m^{k-1}})|\,W_m \equiv 0 \pmod{m^k}. \tag{3.12}$$

By the property of the U-operator as in (2.5), we have

$$\begin{aligned} &m^{\frac{(3m^k-1)r}{4}-1}\;f_{m,k,r}(z)|\,U_{m^{k-1}}|\,W_m\\ =\;& m^{\frac{(k+2)m^kr-(r+4)k}{4}}\sum_{\mu=0}^{m^{k-1}-1} f_{m,k,r}(z)\Big|_{\frac{(m^k-1)r}{2}}\begin{pmatrix}1 & \mu\\ 0 & m^{k-1}\end{pmatrix}\Big|W_m\\ =\;& m^{\frac{(k+2)m^kr-(r+4)k}{4}}\sum_{\mu=0}^{m^{k-1}-1} f_{m,k,r}(z)|_{\frac{(m^k-1)r}{2}}\begin{pmatrix}\mu m & -1\\ m^k & 0\end{pmatrix}. \end{aligned} \tag{3.13}$$

Using the transformation formula (2.9) of the eta function, (3.13) can be written as

$$m^{\frac{m^k r}{2}-k} z^{-\frac{(m^k-1)r}{2}} \sum_{\mu=0}^{m^{k-1}-1} \left(\frac{\eta(m\mu - \frac{1}{z})^{m^k}}{\eta(\frac{m\mu z-1}{m^k z})} \right)^r$$

$$= m^{\frac{m^k r}{2}-k} z^{\frac{r}{2}} \eta(z)^{m^k r} \sum_{\mu=0}^{m^{k-1}-1} \frac{\alpha_\mu}{\eta(\frac{m\mu z-1}{m^k z})^r}, \tag{3.14}$$

where α_μ is a 24th root of unity.

For $\mu \neq 0$, we write $\mu = m^s t$ where $m \nmid t$. For $\mu = 0$, we set $s = k-1$ and $t = 0$. In either case, there exist integers b and d such that $bt + dm^{k-s-1} = -1$. It follows that

$$\begin{pmatrix} m\mu & -1 \\ m^k & 0 \end{pmatrix} = \begin{pmatrix} t & d \\ m^{k-s-1} & -b \end{pmatrix} \begin{pmatrix} m^{s+1} & b \\ 0 & m^{k-s-1} \end{pmatrix}.$$

Applying the corresponding slash operator to $\eta(z)$, we obtain that

$$\eta\left(\frac{m\mu z-1}{m^k z}\right) = \epsilon_\mu m^{\frac{s+1}{2}} z^{\frac{1}{2}} \eta\left(\frac{m^{s+1}z+b}{m^{k-s-1}}\right),$$

where ϵ_μ is a 24th root of unity. Since the coefficients of the Fourier expansion of $\eta(z)$ at ∞ are integers and the coefficient of the term with the lowest degree is 1, the Fourier coefficients of each term in (3.14) are divisible by $m^{\frac{m^k-s-1}{2}r-k}$ in the ring $\mathbb{Z}[\zeta_{24}]$. Clearly, $0 \leq s \leq k-1$. Thus we have

$$\frac{m^k-s-1}{2}r - k \geq \frac{m^k-k}{2}r - k \geq \frac{m^k-k}{2} - k \geq k$$

for $m \geq 5$ and $k \geq 1$. Hence the Fourier coefficients of each term in (3.14) are divisible by m^k. So we arrive at (3.12). This completes the proof. ∎

We are now in a position to finish the proof of Theorem 3.1.

Proof of Theorem 3.1. By Theorem 3.3, there exists a modular form $G_{m,k,r}(z) \in M_{w_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$ such that

$$f_{m,k,r}(z)|U_{m^k} \equiv G_{m,k,r}(z) \pmod{m^k}. \tag{3.15}$$

Let

$$\phi_{m,k,r}(z) = \frac{G_{m,k,r}(z)}{\Delta(z)^{\frac{m^k r+\gamma_{m,k,r}}{24}}},$$

where $\Delta(z) = \eta(z)^{24}$ is Ramanujan's Δ-function. In the proof of Lemma 3.2, we have shown that

$$f_{m,k,r}(z)|U_{m^k} = \sum_{n=0}^{\infty} p_r(m^k n + \beta_{m,k,r}) q^{n+\frac{r(m^{2k}-1)+24\beta_{m,k,r}}{24m^k}} \cdot \prod_{n=1}^{\infty}(1-q^n)^{m^k r},$$

which implies that the order of the Fourier expansion of $f_{m,k,r}(z)|U_{m^k}$ at ∞ is at least

$$\frac{r(m^{2k}-1)+24\beta_{m,k,r}}{24m^k} = \frac{m^k r + \gamma_{m,k,r}}{24}.$$

Thus $\phi_{m,k,r}(z)$ is a modular form in $M_{\lambda_{m,k,r}}(\mathrm{SL}_2(\mathbb{Z}))$. Combining (3.15) and Lemma 3.2, we conclude that

$$\begin{aligned} F_{m,k,r}(z) &\equiv \frac{\left(\Delta(z)^{\frac{m^k r+\gamma_{m,k,r}}{24}}\phi_{m,k,r}(z)\right)\Big| V_{24}}{\eta(24z)^{m^k r}} \\ &= \eta(24z)^{\gamma_{m,k,r}}\phi_{m,k,r}(24z) \pmod{m^k}, \end{aligned}$$

as required. ∎

4 Congruences of $p_r(n)$ modulo m^k

In this section, we apply Theorem 3.1 on the congruence relation for the generating function $F_{m,r,k}(z)$ and Yang's method [33] to derive two classes of congruences of $p_r(n)$ modulo m^k.

Let

$$S_{\gamma,\lambda} = \{\eta(24z)^{\gamma}\phi(24z)\colon \phi(z) \in M_{\lambda}(\mathrm{SL}_2(\mathbb{Z}))\}.$$

Yang [33] showed that when γ is an odd integer such that $0 < \gamma < 24$ and λ is a nonnegative even integer, $S_{\gamma,\lambda}$ is an invariant subspace of $S_{\lambda+\gamma/2}(\Gamma_0(576), \chi_{12})$ under the action of the Hecke algebra. More precisely, for all primes $\ell \neq 2, 3$ and all $f \in S_{\gamma,\lambda}$, we have $f|T_{\ell^2} \in S_{\gamma,\lambda}$. By the invariant property of $S_{\gamma,\lambda}$, we obtain two classes of congruences of $p_r(n)$ modulo m^k.

Theorem 4.1 *Let $m \geq 5$ be a prime, k be a positive integer, r be an odd positive integer less than m^k, and ℓ be a prime different from $2, 3,$ and m. Then there exists an explicitly computable positive integer K such that*

$$p_r\left(\frac{m^k\ell^{2\mu K-1}n + r}{24}\right) \equiv 0 \pmod{m^k} \tag{4.1}$$

for all positive integers μ and all positive integers n relatively prime to ℓ. There is also a positive integer M such that

$$p_r\left(\frac{m^k\ell^{i}n + r}{24}\right) \equiv p_r\left(\frac{m^k\ell^{2M+i}n + r}{24}\right) \pmod{m^k} \tag{4.2}$$

for all nonnegative integers i and n.

Proof. According to congruence relation (3.1), the generating function $F_{m,k,r}(z)$ is congruent to a modular form in $S_{\gamma_{m,k,r},\lambda_{m,k,r}} \cap \mathbb{Z}[[q]]$, where $\lambda_{m,k,r}$ and $\gamma_{m,k,r}$ are integers as given in (3.2) and (3.3). It is known that $S_{\gamma_{m,k,r},\lambda_{m,k,r}} \cap \mathbb{Z}[[q]]$ has a basis $\{f_1(z), \ldots, f_d(z)\}$ of the form

$$f_i(z) = E_4(z)^{u_i} E_6(z)^{v_i} \Delta(z)^{w_i},$$

where u_i, v_i and w_i are nonnegative integers satisfying $4u_i + 6v_i + 12w_i = \lambda_{m,k,r} + \gamma_{m,k,r}/2$ (for more details, see [24]). Suppose that

$$f_i(z) = \sum_{n \geq 0} a_i(n) q^n,$$

where $i = 1, 2, \ldots, d$.

To prove (4.1), it suffices to show that there exists a positive integer K such that for any $1 \leq i \leq d$,

$$a_i(\ell^{2\mu K - 1} n) \equiv 0 \pmod{m^k} \tag{4.3}$$

for all n coprime to ℓ.

From the relation $\gamma_{m,k,r} m^k = 24\beta_{m,k,r} - r$, one sees that $\gamma_{m,k,r}$ and r have the same parity. Since $r < m^k$ is odd, we have $0 < \gamma_{m,k,r} < 24$, and hence $S_{\gamma_{m,k,r},\lambda_{m,k,r}}$ is invariant under the Hecke operator T_{ℓ^2}. So there exists a $d \times d$ matrix A such that

$$\left. \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \right| T_{\ell^2} = A \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}. \tag{4.4}$$

Let

$$X = \begin{pmatrix} A & I_d \\ -\ell^{\gamma_{m,k,r} + 2\lambda_{m,k,r} - 2} I_d & 0 \end{pmatrix}.$$

Using the property of the basis $\{f_1(z), \ldots, f_d(z)\}$ under the action of the U-operator as given by Yang [33, Corollary 3.4], we obtain

$$\left. \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \right| U_{\ell^2}^s = A_s \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} + B_s \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} + C_s \left. \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \right| V_{\ell^2}, \tag{4.5}$$

where s is a positive integer, $g_i = f_i \otimes \left(\frac{\cdot}{\ell}\right)$, and A_s, B_s, and C_s are $d \times d$ matrices given by

$$\begin{pmatrix} A_s & A_{s-1} \end{pmatrix} = \begin{pmatrix} I_d & 0 \end{pmatrix} X^s, \tag{4.6}$$

$$B_s = -\ell^{\lambda_{m,k,r} + (\gamma_{m,k,r} - 3)/2} \left(\frac{(-1)^{(\gamma_{m,k,r} - 1)/2} 12}{\ell} \right) A_{s-1},$$

$$C_s = -\ell^{\gamma_{m,k,r}+2\lambda_{m,k,r}-2} A_{s-1}.$$

Since $\gcd(m, \ell) = 1$, the matrix $X \pmod{m^k}$ is invertible in the ring $\mathcal{M}$ consisting of $2d \times 2d$ matrices over $\mathbb{Z}_{m^k}$. By the finiteness of $\mathcal{M}$, we see that there exist integers $a > b$ such that X^a and X^b are linearly dependent over $\mathbb{Z}_{m^k}$, i.e., there exists a constant $c \in \mathbb{Z}_{m^k}$ such that $X^a \equiv cX^b \pmod{m^k}$. Thus $X^K \equiv cI_{2d} \pmod{m^k}$, where $K = a - b$. In view of the relation

$$\begin{pmatrix} A_{\mu K} & A_{\mu K-1} \end{pmatrix} \equiv c^\mu \begin{pmatrix} I_d & 0 \end{pmatrix} \pmod{m^k},$$

we find that $A_{\mu K-1} \equiv 0 \pmod{m^k}$. Hence, from (4.5) it follows that

$$\left.\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}\right| U_{\ell^2}^{\mu K-1} \equiv B_{\mu K-1} \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} + C_{\mu K-1} \left.\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}\right| V_{\ell^2} \pmod{m^k}.$$

Applying the U-operator U_ℓ and observing that

$$g_i | U_\ell = f_i \otimes \left.\left(\frac{\cdot}{\ell}\right)\right| U_\ell = 0,$$

relation (4.5) leads to the following congruence

$$\left.\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}\right| U_{\ell^2}^{\mu K-1} U_\ell \equiv C_{\mu K-1} \left.\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}\right| V_\ell \pmod{m^k},$$

namely,

$$\sum_{n \geq 0} a_i(\ell^{2\mu K-1} n) q^n \equiv \sum_{n \geq 0} a_i(n) q^{\ell n} \pmod{m^k},$$

which implies (4.3).

We now turn to the proof of congruence (4.2). By the finiteness of $\mathcal{M}$, we see that there exists a positive integer M such that $X^M \equiv I_{2d} \pmod{m^k}$. Thus matrix equation (4.6) reduces to the following congruence

$$\begin{pmatrix} A_M & A_{M-1} \end{pmatrix} \equiv \begin{pmatrix} I_d & 0 \end{pmatrix} \pmod{m^k}.$$

It follows that $A_M \equiv I_d \pmod{m^k}$ and $B_M \equiv C_M \equiv 0 \pmod{m^k}$. Thus, relation (4.5) implies

$$\left.\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}\right| U_{\ell^2}^{M} \equiv \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \pmod{m^k}.$$

So the coefficient of q^n is congruent to the coefficient of $q^{\ell^{2M} n}$in $f_i(z)$ modulo m^k for all i and n. Since $F_{m,k,r}(z)$ is a linear combination of $f_i(z)$ with integer coefficients, we obtain congruence (4.2). This completes the proof. ∎

5 Examples

In this section, we present some consequences of Theorem 3.1 and Theorem 4.1. We first give some examples for the congruences of the generating function $F_{m,k,r}(z)$ of $p_r(n)$.

Example 5.1 *By Theorem 3.1, we find*

$$F_{m,k,r}(z) \equiv \eta(24z)^{\gamma_{m,k,r}} \phi_{m,k,r}(24z) \pmod{m^k},$$

where $\gamma_{m,k,r}$ is an integer, $\phi_{m,k,r}(z)$ is a polynomial of $\Delta(z)$ and the Eisenstein series $E_4(z)$ and $E_6(z)$. Table 5.1 gives the list of explicit expressions of $\eta(z)^{\gamma_{m,1,r}}\phi_{m,1,r}(z)$ for $m \le 19$ and $2 \le r \le 7$.

r	m	$\eta(z)^{\gamma_{m,1,r}}\phi_{m,1,r}(z)$
2	5	0
	7	$3\eta(z)^{10}$
	11	$2\eta(z)^2E_4(z)^2$
	13	$8\eta(z)^{22}$
	17	$5\eta(z)^{14}E_4(z)^2$
	19	$\eta(z)^{10}(14E_4(z)^3 + 12\Delta(z))$
3	5	$4\eta(z)^9$
	7	$3\eta(z)^3E_6(z)$
	11	0
	13	$\eta(z)^9(4E_4(z)^3 + 6\Delta(z))$
	17	0
	19	$\eta(z)^{15}(2E_6(z)^3 + 3E_6(z)\Delta(z))$
4	5	$4\eta(z)^4E_4(z)$
	7	0
	11	$\eta(z)^4(3E_4(z)^4 + 8E_4(z)\Delta(z))$
	13	$\eta(z)^{20}(7E_4(z)^3 + 4\Delta(z))$
	17	$\eta(z)^4(6E_4(z)^7 + 11E_4(z)^4\Delta(z) + 4E_4(z)\Delta(z)^2)$
	19	$\eta(z)^{20}(16E_4(z)^6 + 18E_4(z)^3\Delta(z) + 2\Delta(z)^2)$
5	5	$\eta(z)^{-1}E_4(z)^2$
	7	$\eta(z)^{13}E_6(z)$
	11	0
	13	$\eta(z)^7(8E_4(z)^6 + 11E_4(z)^3\Delta(z) + 5\Delta(z)^2)$
	17	$\eta(z)^{11}(16E_4(z)^8 + 16E_4(z)^5\Delta(z) + 4E_4(z)^2\Delta(z)^2)$
	19	$\eta(z)(5E_6(z)^7 + 15E_6(z)^5\Delta(z) + 16E_6(z)^3\Delta(z)^2)$
6	5	0
	7	$\eta(z)^6(6E_4(z)^3 + 6\Delta(z))$
	11	$\eta(z)^6(10E_4(z)^6 + E_4(z)^3\Delta(z))$
	13	$\eta(z)^{18}(7E_4(z)^6 + 8E_4(z)^3\Delta(z) + 6\Delta(z)^2)$

	17	$\eta(z)^{18}(3E_4(z)^9+3E_4(z)^6\Delta(z)+5E_4(z)^3\Delta(z)^2)$
	19	$\eta(z)^6(6E_4(z)^{12}+E_4(z)^9\Delta(z)+14\Delta(z)^4)$
7	5	0
	7	$\eta(z)^{-1}E_6(z)^3$
	11	0
	13	$\eta(z)^5(10E_4(z)^9+6E_4(z)^6\Delta(z)+9E_4(z)^3\Delta(z)^2+11\Delta(z)^3)$
	17	$\eta(z)(7E_4(z)^{13}+2E_4(z)^{10}\Delta(z)+E_4(z)^7\Delta(z)^2+3E_4(z)^4\Delta(z)^3)$
	19	0

Table 5.1: Explicit congruences derived from Theorem 3.1.

Example 5.2 *Let* $0 \le \beta < m^k$ *be an integer with* $\beta \equiv r/24 \pmod{m^k}$. *If* $\phi_{m,k,r}(z) \equiv 0 \pmod{m^k}$, *using Theorem 3.1, we obtain the following Ramanujan-type congruences of multipartition functions*

$$p_r(m^k n+\beta) \equiv 0 \pmod{m^k}. \tag{5.1}$$

The values of m *and* β *for* $r \le 9$ *and* $k = 1, 2$ *are given in Table 5.2.*

r	(m,β)	(m^2,β)
1	$(5,4),(7,5),(11,6)$	$(25,24),(49,47),(121,116)$
2	$(5,3)$	$(25,23)^*$
3	$(11,7),(17,15)$	$(121,106)^*$
4	$(7,6)$	$(49,41)^*$
5	$(11,8),(23,5)$	$(121,96)^*$
6	$(5,4)$	$(25,19)$
7	$(5,3),(11,9),(19,9)$	$(25,18),(121,86)$
8	$(7,5),(11,4)$	$(121,81)^*$
9	$(17,11),(19,17),(23,9)$	—

Table 5.2: Ramanujan-type congruences of multipartitions.

It can be seen that Table 5.2 contains the Ramanujan congruences (1.1) of $p(n)$ modulo $5, 7$ and 11, as well as Gandhi's congruences (1.5) for $p_2(n)$ and (1.6) for $p_8(n)$. The congruences marked by $*$ in the table seem to be new.

The following examples demonstrate how to derive certain congruences of $p_r(n)$ with the aid of Theorem 4.1.

Example 5.3 *For the values of* ℓ *and* K_ℓ *as given in Table 5.3, we have*

$$p_3\left(\frac{7\cdot \ell^{2\mu K_\ell-1}n+3}{24}\right) \equiv 0 \pmod{7} \tag{5.2}$$

for all positive integers μ *and all positive integers* n *not divisible by* ℓ.

ℓ	5	11	13	17	19	23	29	31	37	41	43	47	53	59
a_ℓ	6	4	0	4	3	6	2	5	3	0	0	3	5	5
K_ℓ	6	7	2	6	8	7	7	8	3	2	2	8	3	8

Table 5.3: Eigenvalues a_ℓ of $F_{7,1,3}(z)$ acted by T_{ℓ^2} and the corresponding K_ℓ.

Proof. By Theorem 3.1, we find

$$F_{7,1,3}(z) \equiv 3\eta(24z)^3 E_6(24z) \pmod 7.$$

Since $\eta(24z)^3 E_6(24z)$ belongs to the 1-dimensional space $S_{3,6}$, for any prime $\ell \neq 2, 3, 7$, there exists an integer a_ℓ such that

$$F_{7,1,3}(z)|T_{\ell^2} \equiv a_\ell F_{7,1,3}(z) \pmod 7.$$

Inspecting the proof of Theorem 4.1, we obtain the corresponding orders K_ℓ for which congruence (5.2) holds. ∎

Example 5.4 *We have*

$$p_3\left(\frac{5^2 \cdot 13^{199} n + 3}{24}\right) \equiv 0 \pmod{5^2}$$

for all integers n coprime to 13 *and*

$$p_3\left(\frac{5^2 \cdot 13^{i} n + 3}{24}\right) \equiv p_3\left(\frac{5^2 \cdot 13^{200+i} n + 3}{24}\right) \pmod{5^2}$$

for all nonnegative integers n and i.

Proof. By Theorem 3.1, $F_{5,2,3}(z)$ is congruent to a modular form in the space $S_{21,48}$ of dimension 5. Setting

$$f_i = \eta(24z)^{21} E_4(24z)^{3(5-i)} \Delta(24z)^{i-1}$$

for $1 \leq i \leq 5$. Clearly, $f_1, f_2, \ldots, f_5$ form a $\mathbb{Z}$-basis of $S_{21,48} \cap \mathbb{Z}[[q]]$. Let A be the matrix of T_{ℓ^2} with respect to this basis. By computing the first five Fourier coefficients of f_i and $f_i|T_{13^2}$ and equating the Fourier coefficients of both sides of (4.4), we find

$$A \equiv \begin{pmatrix} 17 & 21 & 18 & 3 & 3 \\ 0 & 19 & 5 & 5 & 5 \\ 0 & 0 & 22 & 4 & 19 \\ 0 & 0 & 0 & 22 & 10 \\ 0 & 0 & 0 & 0 & 12 \end{pmatrix} \pmod{5^2},$$

with the corresponding orders $K = M = 100$. Setting $\mu = 1$ in Theorem 4.1, we complete the proof. ∎

Below are two more examples for $p_3(n)$ and $p_5(n)$ modulo 7^2. The proofs are analogous to the proof of the above example, and hence are omitted.

Example 5.5 *We have*

$$p_3\left(\frac{7^2\cdot 11^{2351}n+3}{24}\right)\equiv 0 \pmod{7^2}$$

for all positive integers n coprime to 7 and

$$p_3\left(\frac{7^2\cdot 11^{i}n+3}{24}\right)\equiv p_3\left(\frac{7^2\cdot 11^{1176+i}n+3}{24}\right) \pmod{7^2}$$

for all nonnegative integers n and i.

Example 5.6 *We have*

$$p_5\left(\frac{7^2\cdot 17^{195}n+5}{24}\right)\equiv 0 \pmod{7^2}$$

for all positive integers n coprime to 17 and

$$p_5\left(\frac{7^2\cdot 17^{i}n+5}{24}\right)\equiv p_5\left(\frac{7^2\cdot 17^{588+i}n+5}{24}\right) \pmod{7^2}$$

for all nonnegative integers n and i.

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