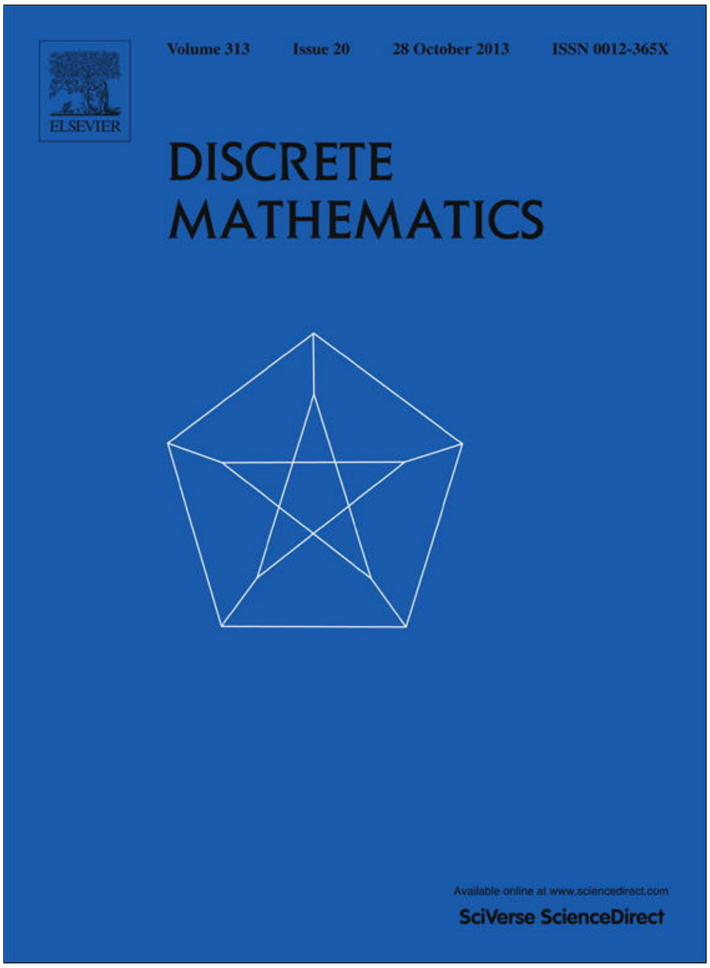
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Note

Eulerian pairs on Fibonacci words

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ABSTRACT

Recently, Sagan and Savage introduced the notion of Eulerian pairs. In this note, we find Eulerian pairs on Fibonacci words based on Foata's first transformation or Han's bijection and a map in the spirit of a bijection of Steingrímsson.

1. Introduction

This paper is motivated by the notion of Eulerian pairs introduced by Sagan and Savage [6] in their study of Mahonian pairs. Let $\mathbb{P}$ be the set of positive integers and let $\mathbb{P}^*$ be the set of words on $\mathbb{P}$. For two finite subsets $S, T \subset \mathbb{P}^*$, the pair (S, T) is called a Mahonian pair if the distribution of the major index over S is the same as the distribution of the inversion number over T. Similarly, (S, T) is said to be an Eulerian pair if the distribution of the descent number over S is the same as the distribution of the excedance number over T.

The well-known theorem of MacMahon [5] can be rephrased as the fact that (S_n, S_n) is a Mahonian pair, where S_n is the set of permutations on $[n] = \{1, 2, \ldots, n\}$. Foata [3] found a combinatorial proof of this fact by establishing a correspondence which has been called the second fundamental transformation, denoted as Φ_2. With the aid of the map Φ_2, Sagan and Savage found Mahonian pairs $(S, \Phi_2(S))$, where S is a set of ballot sequences or a set of Fibonacci words. By a Fibonacci word we mean a word on $\{1, 2\}$ containing no consecutive 1s. Dokos et al. [1] studied Mahonian pairs on permutations avoiding some patterns. In this paper, we find Eulerian pairs on Fibonacci words based on bijections of Foata [2], Han [4] and Steingrímsson [7].

We adopt some common notation on words. For a word $\omega = a_1 a_2 \cdots a_n$, the descent number $\mathrm{des}(\omega)$, the inversion number $\mathrm{inv}(\omega)$ and the major index $\mathrm{maj}(\omega)$ are defined by

$$\mathrm{des}(\omega) = \#\{i | a_i > a_{i+1}, 1 \le i \le n-1\},$$

$$\mathrm{inv}(\omega) = \#\{(i, j) | a_i > a_j, 1 \le i < j \le n\},$$

$$\mathrm{maj}(\omega) = \sum_{\substack{a_i > a_{i+1}, \\ 1 \le i \le n-1}} i,$$

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where # indicates the cardinality of a set. Writing ω in the two-line form

$$\omega = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}, \tag{1.1}$$

where $x_1, x_2, \ldots, x_n$ is the nondecreasing rearrangement of $a_1a_2\cdots a_n$, one can define the excedance number $\mathrm{exc}(\omega)$ as follows:

$$\mathrm{exc}(\omega) = \#\{i | a_i > x_i, 1 \le i \le n\}.$$

Usually, we say that (a_i, a_{i+1}) is a descent in ω if $a_i > a_{i+1}$ and (a_i, x_i) is an excedance if $a_i > x_i$.

2. Eulerian pairs derived from Φ_1^{-1}

In this section, we construct Eulerian pairs on Fibonacci words by using Foata's first fundamental transformation [2]. It is worth mentioning that Foata's first fundamental transformation Φ_1 coincides with Han's bijection [4] when restricted to words on $\{1, 2\}$. From now on, we shall still use Φ_1 to denote Foata's first fundamental transformation (or Han's bijection) when restricted to $\{1, 2\}^*$.

Throughout this paper, by a binary word we mean a word on $\{1, 2\}$. Let $\{1, 2\}_n^*$ denote the set of binary words of length n. Clearly, a word $\omega \in \{1, 2\}^*$ with d descents can be uniquely written as

$$\omega = 1^{m_0}2^{n_0}1^{m_1}2^{n_1}\cdots 1^{m_d}2^{n_d}, \tag{2.1}$$

where $m_0, n_d \ge 0$, and $m_i, n_j > 0$ for $1 \le i \le d$ and $0 \le j \le d-1$. It can be easily checked that $\Phi_1^{-1}(\omega)$ takes the following form:

$$\Phi_1^{-1}(\omega) = 1^{m_0}21^{m_1-1}2\cdots 21^{m_d-1}2^{n_0-1}12^{n_1-1}\cdots 2^{n_{d-1}-1}12^{n_d}. \tag{2.2}$$

The expression (2.2) enables us to describe the Eulerian pairs $(S, \Phi_1^{-1}(S))$ when $S = F_n$ and $S = F_n'$, where F_n is the set of Fibonacci words of length n and F_n' is the set of Fibonacci words of length n ending with 1. We shall use the correspondence between binary words and integer partitions analogously to the description of the Mahonian pairs obtained by Sagan and Savage [6]. For convenience, we let $\lambda(\omega)$ be the partition corresponding to the binary word ω. Making use of this connection, $\Phi_1^{-1}(F_n)$ and $\Phi_1^{-1}(F_n')$ can be described in terms of statistics on integer partitions.

The following theorem gives Eulerian pairs involving F_n and F_n', where we use $N_\omega(1)$ to denote the number of 1s in a word ω. For any partition λ, we denote by $l(\lambda)$ the number of parts of λ. Recall that the Durfee square $D(\lambda)$ of λ is the square partition (d^d), where d is the largest integer $i \le l(\lambda)$ such that $\lambda_1 \ge i, \ldots, \lambda_i \ge i$. Denote by $d(\lambda)$ the size d of $D(\lambda)$, and let $B(\lambda) = (\lambda_{d+1}, \ldots, \lambda_k)$.

Theorem 2.1. *Let*

$$R_n = \{\omega \in \{1, 2\}_n^* \mid \lambda = \lambda(\omega),\ N_\omega(1) - 1 \le d(\lambda) \le N_\omega(1), B(\lambda) = \emptyset\},$$

and let

$$R_n' = \{\omega \in \{1, 2\}_n^* \mid \lambda = \lambda(\omega), \lambda_1 = n - N_\omega(1), N_\omega(1) - 1 \le d(\lambda) \le N_\omega(1), B(\lambda) = \emptyset\}.$$

Then (F_n, R_n) and (F_n', R_n') are Eulerian pairs.

Proof. Keep in mind that Φ_1 is a bijection on words which maps the excedance number to the descent number; for more details, see [2]. Thus for any set S, $(S, \Phi_1^{-1}(S))$ is an Eulerian pair. So it suffices to show that $R_n = \Phi_1^{-1}(F_n)$ and $R_n' = \Phi_1^{-1}(F_n')$.

Suppose that $\omega = 1^{m_0}2^{n_0}1^{m_1}\cdots 2^{n_d} \in F_n$, where $m_0 = 0$ or 1. Notice that $d = N_\omega(1) - m_0$. From (2.2) it follows that

$$\Phi_1^{-1}(\omega) = 1^{m_0}2^{d+n_0-1}12^{n_1-1}\cdots 12^{n_{d-2}-1}12^{n_{d-1}-1}12^{n_d}. \tag{2.3}$$

Let $\lambda = \lambda(\Phi_1^{-1}(\omega))$. From the correspondence between binary words and partitions, we see that λ has exactly d parts. Moreover, we have

$$\lambda_d = d + n_0 - 1 \ge d.$$

Hence $B(\lambda) = \emptyset$ and $D(\lambda) = (d^d)$. It follows from (2.3) that the size of the Durfee square of λ is given by

$$d(\lambda) = \begin{cases} N_\omega(1) - 1, & \text{if } m_0 = 1; \\ N_\omega(1), & \text{if } m_0 = 0. \end{cases}$$

So we see that $\Phi_1^{-1}(\omega) \in R_n$, which yields that $\Phi_1^{-1}(F_n) \subseteq R_n$.

Conversely, let $\sigma = a_1a_2\cdots a_n \in R_n$. We wish to show that there is a word $\rho \in F_n$ such that $\Phi_1^{-1}(\rho) = \sigma$. Let $k = N_\sigma(1)$ and $\mu = \lambda(\sigma)$. By the definition of R_n, we have $k - 1 \le d(\mu) \le k$ and $B(\mu) = \emptyset$. By the construction of μ, we see that there exists some nonnegative integer t such that $\mu_1 = n - k - t$. In fact, t is the largest integer i such that σ ends with 2^i. If

$d(\mu) = k$, then we have $\mu_k \geq k$ and $l(\mu) = k$. Hence σ takes the form $2^k a_{k+1} \cdots a_{n-t-1} 12^t$. Since $n - t - k = \mu_1 \geq \mu_k \geq k$, there exists a sequence of k positive integers $n_0, n_1, \ldots, n_{k-1}$ such that σ has the form $2^{k+n_0-1}12^{n_1-1}\cdots 2^{n_{k-1}-1}12^t$. Let $\rho = 2^{n_0}12^{n_1}1\cdots 2^{n_{k-1}}12^t$. Obviously, $\rho \in F_n$. In view of (2.2), we find that $\Phi_1^{-1}(\rho) = \sigma$. For the case $d(\mu) = k-1$, by a similar argument it can be shown that there exists a word ρ' in F_n such that $\Phi_1^{-1}(\rho') = \sigma$. So we have shown that $R_n \subseteq \Phi_1^{-1}(F_n)$. Consequently, we arrive at the conclusion that $R_n = \Phi_1^{-1}(F_n)$.

We now proceed to show that $R'_n = \Phi_1^{-1}(F'_n)$. Let ω be a binary word of length n. In view of (2.2), we see that ω ends with 1 if and only if $\Phi_1^{-1}(\omega)$ ends with 1. So we deduce that

$$\Phi_1^{-1}(F'_n) = \{\omega \in \Phi_1^{-1}(F_n) \mid \omega \text{ ends with } 1\}.$$

On the other hand, by the construction of the correspondence between binary words and partitions, it can be checked that ω ends with 1 if and only if $\lambda_1 = n - N_\omega(1)$, where $\lambda = \lambda(\omega)$. Since $R_n = \Phi_1^{-1}(F_n)$, we obtain that

$$\begin{aligned}\Phi_1^{-1}(F'_n) &= \{\omega \in R_n \mid \omega \text{ends with} 1\}\\ &= \{\omega \in R_n \mid \lambda = \lambda(\omega), \lambda_1 = n - N_\omega(1)\},\end{aligned}$$

that is, $R'_n = \Phi_1^{-1}(F'_n)$. This completes the proof. □

3. An Eulerian pair derived from Γ

In this section, we extend the bijection of Steingrímsson ϕ [7] on permutations to a map Γ on words. While the extended map is not a bijection, it still transforms the descent number to the excedance number. As far as F_n is concerned, the map Γ is not injective, but it turns out to be injective on F'_n. Therefore, we obtain an Eulerian pair $(F'_n, \Gamma(F'_n))$.

We begin with an overview of Steingrímsson's bijection ϕ on permutations. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. For notational convenience, let $\phi(\pi) = f(1)f(2)\cdots f(n)$. Set $\pi_0 = 0$ and $\pi_{n+1} = n+1$. For $1 \leq k \leq n$,

(1) If there exists an integer m such that $k < m \leq n$ and $\pi_m < \pi_k$, then we set $f(\pi_{k+1}) = \pi_k$.
(2) If $\pi_k > \pi_m$ for $k < m \leq n$, then we set $f(\pi_{j+1}) = \pi_k$, where j is the largest number such that $\pi_j < \pi_k$.

Steingrímsson proved that the map ϕ is a bijection which maps the descent number to the excedance number.

Proposition 3.1 *([7, Remark 4.7]). Let π be a permutation on $[n]$. Then for $1 \leq k \leq n$, $\pi_k > \pi_{k+1}$ if and only if (π_k, π_{k+1}) is an excedance in $\phi(\pi)$.*

Steingrímsson's bijection can be extended to a map Γ on words. Recall that the standardization of a word $\omega = a_1a_2\cdots a_n$ can be expressed as $\pi = \beta_\omega(1)\beta_\omega(2)\cdots\beta_\omega(n)$ on $[n]$, where $\beta_\omega(i)$ is given by

$$\beta_\omega(i) = \#\{j \mid 1 \leq j \leq n, a_j < a_i\} + \#\{j \mid j \leq i, a_j = a_i\}. \tag{3.1}$$

Let $\omega = a_1a_2\cdots a_n$ be a word. The map Γ is defined as follows. Assume that $\pi = \beta_\omega(1)\beta_\omega(2)\cdots\beta_\omega(n)$ is the standardization of ω. Let $\phi(\pi) = f(1)f(2)\cdots f(n)$. For $1 \leq i \leq n$, there exists a unique integer j_i such that $\beta_\omega(j_i) = f(i)$. Then $\Gamma(\omega)$ is defined to be the word $a_{j_1}a_{j_2}\cdots a_{j_n}$. For example, let $\omega = 132232131$. Then the standardization of ω is $\pi = 174586293$ and $\phi(\pi) = 169748253$. So we have $\Gamma(\omega) = 123323121$.

The following theorem shows that the map Γ also transforms the descent number to the excedance number.

Theorem 3.2. *For any word ω, we have*

$$\mathrm{des}(\omega) = \mathrm{exc}(\Gamma(\omega)).$$

Proof. Assume that $\omega = a_1a_2\cdots a_n$ is a word. Let $\pi = \sigma_1\sigma_2\cdots\sigma_n$ be the standardization of ω. It is obvious that (a_i, a_{i+1}) is a descent in ω if and only if (σ_i, σ_{i+1}) is a descent in π. By Proposition 3.1, we see that (σ_i, σ_{i+1}) is a descent in π if and only if (σ_i, σ_{i+1}) forms an excedance in $\phi(\pi)$. With the aid of the construction of Γ, it can be seen that (σ_i, σ_{i+1}) forms an excedance in $\phi(\pi)$ if and only if (a_i, a_{i+1}) is an excedance in $\Gamma(\omega)$. Thus, we have $\mathrm{des}(\omega) = \mathrm{exc}(\Gamma(\omega))$. This completes the proof. □

Next we consider the restriction of Γ to words on $\{1, 2\}$. In this case, it is easy to verify that $\Gamma(\omega 2^m) = \Gamma(\omega)2^m$ for $m \geq 1$. The following lemma shows how to compute $\Gamma(\omega 1^m)$ on the basis of $\Gamma(\omega)$.

Lemma 3.3. *Suppose that ω is a binary word of length n that contains k 1s. Let $\Gamma(\omega) = b_1b_2\cdots b_n$. Assume that t is the largest integer i such that ω ends with 2^i. Set $U = b_1b_2\cdots b_k$ and $V = b_{k+1}b_{k+2}\cdots b_{n-t}$. Then we have the following recurrence relations:*

(1) If $t = 0$, then $\Gamma(\omega 1) = U1V$. In general, if $t = 0$, then $\Gamma(\omega 1^m) = U1^mV$ for any $m \geq 1$.
(2) If $t > 0$, then $\Gamma(\omega 1) = U2V12^{t-1}$. In general, if $t > 0$, then we have $\Gamma(\omega 1^m) = U21^{m-1}V12^{t-1}$ for any $m \geq 1$.

Proof. Let $\omega = a_1a_2\cdots a_n$ and $a_{n+1} = 1$. Suppose that $\Gamma(\omega 1) = c_1c_2\cdots c_{n+1}$. To determine $\Gamma(\omega 1)$, we consider occurrences of 1s in $\Gamma(\omega 1)$. Assume that $a_{s_1}, a_{s_2}, \ldots, a_{s_k}$ are the 1s in ω, where $s_1 < s_2 < \cdots < s_k$. Let us define $\beta(i) = \beta_\omega(i)$ and $\beta'(j) = \beta_{\omega 1}(j)$ for $1 \le i \le n$ and $1 \le j \le n+1$. It can be seen that $\beta'(n+1) = k+1$ and for $i \le n$,

$$\beta'(i) = \begin{cases} \beta(i), & \text{if } a_i = 1; \\ \beta(i)+1, & \text{otherwise.} \end{cases}$$

Thus we have

$$\{\beta'(s_1) < \beta'(s_2) < \cdots < \beta'(s_k)\} = \{1, 2, \ldots, k\}$$

and

$$\{\beta'(i) | 1 \le i \le n, a_i = 2\} = \{k+2, \ldots, n+1\}.$$

By the construction of Γ, it is not hard to see that $b_{\beta(s_i+1)} = a_{s_{(i+1)}} = 1$ and $c_{\beta'(s_i+1)} = a_{s_{(i+1)}} = 1$ for $0 \le i \le k-1$, where $s_0 = 0$. For $0 \le i \le k-1$, it is clear that $\beta'(s_i+1) \le k$ if and only if $\beta'(s_i+1) = \beta(s_i+1)$. This means that the 1s in $c_1c_2\cdots c_k$ appear in the same positions as in U. Moreover, for the case $\beta'(s_i+1) \ge k+2$, we see that $\beta'(s_i+1) = \beta(s_i+1)+1$. In other words, a 1 appearing in the jth position in V corresponds to a 1 in the jth position in $c_{k+2}c_{k+3}\cdots c_{n-t+1}$.

Let us further consider the position of a_{n+1} in $\Gamma(\omega 1)$. Observe that $s_k = n-t$. By the construction of Γ, we find that $c_{\beta'(n-t+1)} = a_{n+1} = 1$. If $t = 0$, then $c_{k+1} = c_{\beta'(n+1)} = a_{n+1}$, which means that a_{n+1} is in the $(k+1)$th position in $\Gamma(\omega 1)$. When $t > 0$, since $\beta'(n-t+1) = n-t+2$, we find that $c_{n-t+2} = c_{\beta'(n-t+1)} = a_{n+1}$. Thus a_{n+1} is in the $(n-t+2)$th position in $\Gamma(\omega 1)$. In summary, we deduce that

$$\Gamma(\omega 1) = \begin{cases} U1V, & \text{if } t = 0; \\ U2V12^{t-1}, & \text{if } t > 0. \end{cases} \tag{3.2}$$

So the lemma holds for $m = 1$. By iterating the above process, it can be seen that the lemma holds for $m > 1$. This completes the proof. $\square$

By Lemma 3.3, for any word ω in form (2.1), $\Gamma(\omega)$ is of the following form:

$$\Gamma(\omega) = 1^{m_0}21^{m_1-1}\cdots 21^{m_{d-1}-1}21^{m_d}2^{n_0-1}12^{n_1-1}\cdots 2^{n_{d-2}-1}12^{n_{d-1}-1+n_d}. \tag{3.3}$$

The following theorem gives a description of $\Gamma(F_n')$.

Theorem 3.4. *Let*

$$T_n = \{\omega \in \{1,2\}_n^* \mid \lambda = \lambda(\omega),\ N_\omega(1) - 1 \le l(\lambda) = \lambda_{l(\lambda)} \le N_\omega(1)\}.$$

Then we have $\Gamma(F_n') = T_n$. Moreover, (F_n', T_n) is an Eulerian pair.

Proof. Using an argument similar to that in the proof of Theorem 2.1, it can be shown that $\Gamma(F_n') = T_n$. To prove that (F_n', T_n) is an Eulerian pair, it suffices to verify that Γ is injective on F_n'. Assume that $\omega = 1^{m_0}2^{n_0}12^{n_1}\cdots 12^{n_{d-2}}12^{n_{d-1}}1$ and $\omega' = 1^{m_0'}2^{n_0'}12^{n_1'}\cdots 12^{n'_{d'-2}}12^{n'_{d'-1}}1$ are two words in F_n' such that $\Gamma(\omega) = \Gamma(\omega')$. It follows from (3.3) that $\Gamma(\omega) = 1^{m_0}2^d12^{n_0-1}\cdots 2^{n_{d-2}-1}1$ and $\Gamma(\omega') = 1^{m_0'}2^{d'}12^{n_0'-1}\cdots 2^{n'_{d'-2}-1}1$. So we have $d = d'$, $m_0 = m_0'$ and $n_i = n_i'$ for any $0 \le i \le d-1$. This implies that $\omega = \omega'$. Hence Γ is injective on F_n'. This completes the proof. $\square$

It should be noted that Γ is neither surjective nor injective on F_n. For example, there is no ω satisfying $\Gamma(\omega) = 2121$. On the other hand, we have

$$\Gamma(2^212^212^31) = \Gamma(2^212^212^212) = \Gamma(2^212^21212^2) = 2^3121212^2.$$

We conclude this section with a remark that $\Gamma(F_n) = \Gamma(F_n')$. In fact, for any word $\omega = 1^{m_0}2^{n_0}12^{n_1}\cdots 12^{n_{d-1}}12^{n_d} \in F_n$, let $\sigma = 1^{m_0}2^{n_0}12^{n_1}\cdots 12^{n_{d-1}+n_d}1$ in F_n'. Then we have $\Gamma(\omega) = \Gamma(\sigma)$.

4. Concluding remarks

In this section, we make some remarks on Euler–Mahonian pairs on binary words, which are related to the bijections Φ_1, Φ_2 and Γ.

For any word $\omega = 1^{m_0}2^{n_0}\cdots 1^{m_d}2^{n_d}$, Sagan and Savage have shown that

$$\Phi_2(\omega) = 1^{m_d-1}21^{m_{d-1}-1}2\cdots 1^{m_1-1}21^{m_0}2^{n_0-1}12^{n_1-1}1\cdots 2^{n_{d-1}-1}12^{n_d}. \tag{4.1}$$

It is clear from (4.1) that $\mathrm{des}(\omega) = \mathrm{exc}(\Phi_2(\omega))$. So we deduce that the Mahonian pairs (S, T) given by Sagan and Savage [6] are Euler–Mahonian pairs in the sense that

$$\sum_{\omega\in S} p^{\mathrm{des}(\omega)}q^{\mathrm{maj}(\omega)} = \sum_{\omega\in T} p^{\mathrm{exc}(\omega)}q^{\mathrm{inv}(\omega)}.$$

It should be noted that in general $\Phi_2(F_n) \neq \Phi_1^{-1}(F_n)$, $\Phi_2(F_n') \neq \Phi_1^{-1}(F_n')$ and $\Phi_2(F_n') \neq \Gamma(F_n')$. However, there exists a set G_n such that $(G_n, \Phi_1^{-1}(G_n))$, $(G_n, \Phi_2(G_n))$ and $(G_n, \Gamma(G_n))$ are the same Eulerian pairs. Meanwhile, we find a set H of binary words for which $\Phi_1^{-1} = \Phi_2$.

Theorem 4.1. *Let G_n be the set of words in $\{1, 2\}_n^*$ with no consecutive 2s and let*

$$H = \{\omega = 1^{m_0}2^{n_0}1^{m_1}2^{n_1} \cdots 1^{m_d}2^{n_d} \mid m_0 = m_d - 1, m_i = m_{d-i} \text{ for } 1 \leq i \leq d-1\}.$$

Then we have $\Phi_2(G_n) = \Phi_1^{-1}(G_n) = \Gamma(G_n)$ and $\Phi_1^{-1}(\omega) = \Phi_2(\omega)$ for any $\omega \in H$.

Proof. Given a word $\omega \in G_n$ with d descents, it can be written uniquely as

$$1^{m_0}21^{m_1}2 \cdots 1^{m_d}2^{n_d},$$

where $m_0 \geq 0$, $n_d = 0$ or 1, and $m_i > 0$ for $1 \leq i \leq d$. By (2.2) and (3.3), we find that $\Phi_1^{-1}(\omega) = \Gamma(\omega)$ for all $\omega \in G_n$. Therefore, we have $\Phi_1^{-1}(G_n) = \Gamma(G_n)$. To show that $\Phi_1^{-1}(G_n) = \Phi_2(G_n)$, we define a map φ on binary words

$$\varphi(1^{m_0}2^{n_0}1^{m_1}2^{n_1} \cdots 1^{m_{d-1}}2^{n_{d-1}}1^{m_d}2^{n_d}) = 1^{m_d-1}2^{n_0}1^{m_{d-1}}2^{n_1} \cdots 2^{n_{d-2}}1^{m_1}2^{n_{d-1}}1^{m_0+1}2^{n_d}.$$

It is easy to check that φ is an involution on $\{1, 2\}^*$. Observing that $\varphi(G_n) = G_n$, by (4.1) and (2.2), we obtain that $\Phi_2(\omega) = \Phi_1^{-1}(\varphi(\omega))$ for any $\omega \in G_n$. Thus we have $\Phi_1^{-1}(G_n) = \Phi_2(G_n)$.

By the definition of φ, we find that $H = \{\omega \in \{1, 2\}^* \mid \varphi(\omega) = \omega\}$. Since $\Phi_2(\omega) = \Phi_1^{-1}(\varphi(\omega))$ for any binary word ω, we conclude that $\Phi_1^{-1}(\omega) = \Phi_2(\omega)$ for any $\omega \in H$. This completes the proof. □

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