Note on the generalized connectivity *

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Abstract

For a vertex set S with cardinality at least 2 in a graph G, we need a tree in order to connect the set, where this tree is usually called a *Steiner tree* connecting S (or *an S-tree*). Two S-trees T and T' are said to be *internally disjoint* if $V(T) \cap V(T') = S$ and $E(T) \cap E(T') = \emptyset$. Let $\kappa_G(S)$ denote the maximum number of internally disjoint Steiner trees connecting S in G. *The generalized k-connectivity* $\kappa_k(G)$ of a graph G, which was introduced by Chartrand et al., is defined as $\min_{S\subseteq V(G),|S|=k} \kappa_G(S)$. In this paper, we get a sharp upper bound of generalized k-connectivity. Moreover, graphs with order n and $\kappa_3(G) = n-2$, $n-3$ are characterized.

Keywords: connectivity, Steiner tree, internally disjoint trees, generalized connectivity, networks.

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1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to book [2] for graph theoretical notation and terminology not described here. In the world, there are numerous networks as, for example, transport networks, road networks, electrical networks, telecommunication systems or networks of servers. All networks can be modeled by a graph or a digraph whose vertices and edges represent, respectively, the processing elements (nodes of the network) and the communication links between them. Many attempts have been made to study reliability of such a network. Several classical measures are the edge-connectivity, the vertex-connectivity (or simply the connectivity) and super connectivity. Thousands of

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articles on connectivity have been published, such as [1, 5, 6, 7, 8, 13, 15]. There is another well measure, the generalized connectivity.

For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a such subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Two Steiner trees T and T' connecting S are said to be *internally disjoint* if $E(T) \cap E(T') = \varnothing$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa(S)$ is the maximum number of internally disjoint trees connecting S in G. Note that when $|S| = 2$ a Steiner tree connecting S is just a path connecting the two vertices of S. For an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* $\kappa_k(G)$ of G, introduced by Chartrand et al. in [3], is defined as $\kappa_k(G) = min\{\kappa(S) : S \subseteq V(G)\ and\ |S| = k\}$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is nothing new but the connectivity $\kappa(G)$ of G, that is, $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of G. So the generalized k-connectivity is a natural and nice generalization of the concept of vertex-connectivity. There have appeared many results on the generalized connectivity (see [3, 4, 12, 9, 10, 13]).

In addition to being a natural combinatorial measure, the generalized connectivity can be motivated by its interesting interpretation in practice. Suppose that G represents a network. If one considers to connect a pair of vertices of G, then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them unless the vertices of S lie on a common path. This kind of tree with minimum order for connecting a set of vertices is usually called a Steiner tree, and popularly used in the physical design of Very Large Scale Integration (see [14]). For a set S of vertices, usually the number of totally independent ways to connect S is a local measure for the reliability of a network. Then the generalized k-connectivity can serve for measuring the global capability of a network G to connect any k vertices in G.

Chartrand et al. in [4] obtained the following result.

Theorem 1. *[4] For every two integers n and k with $2 \leq k \leq n$, $\kappa_k(K_n) = n - \lceil k/2 \rceil$.*

The following result was given by Li et al. in [10], which will be used later.

Theorem 2. *[10] For any connected graph G, $\kappa_3(G) \leq \kappa(G)$. Moreover, the upper bound is sharp.*

2 Main results

For a graph G, let $V(G)$, $E(G)$, $\overline{G}$ be the set of vertices, the set of edges, the complement of G, respectively. As usual, the *union* of two graphs G and

H is the graph, denoted by $G \cup H$, with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Let mH be the disjoint union of m copies of a graph H. A subset M of $E(G)$ is called a *matching* in G if no two edges in M are adjacent in G. A matching M saturates a vertex v, or v is said to be *M-saturated*, if some edge of M is incident with v; otherwise, v is *M-unsaturated*. M is a *maximum matching* if G has no matching M' with $|M'| > |M|$.

To start with, we give the bounds of $\kappa_3(G)$.

Proposition 1. *For a connected graph G of order n $(n \geq 3)$, $1 \leq \kappa_3(G) \leq n-2$. Moreover, the upper and lower bounds are sharp.*

Proof. It is easy to see that $\kappa_3(G) \leq \kappa_3(K_n)$. From this together with Theorem 1, we have $\kappa_3(G) \leq n-2$. Since G is connected, $\kappa_3(G) \geq 1$. The result holds.

It is easy to check that the complete graph K_n attains the upper bound and the complete bipartite graph $K_{1,n-1}$ attains the lower bound. □

From Theorem 2, one may think that the monotone property of κ_k, namely, $\kappa_n \leq \kappa_{n-1} \leq \cdots \kappa_4 \leq \kappa_3 \leq \kappa_2 = \kappa$ is true for $2 \leq k \leq n$. Unfortunately, some counterexamples have been found to show that there exist some integers i, j such that $2 \leq i < j \leq n$ but $\kappa_i > \kappa_j$.

Let us now introduce one such example. Let G_1, G_2 be two copies of the complete graph K_r $(r \geq 4)$, and G be a graph obtained from G_1, G_2 by identifying one vertex in each of them. Clearly, $|V(G)| = 2r-1$ and each $G_i = K_r$ $(i = 1, 2)$ contains $\lfloor \frac{r}{2} \rfloor$ edge-disjoint spanning trees, say $T_{i,1}, T_{i,2}, \cdots, T_{i,\lfloor \frac{r}{2} \rfloor}$. Then the trees $T_j = T_{1,j} \cup T_{2,j}$ $(1 \leq j \leq \lfloor \frac{r}{2} \rfloor)$ are $\lfloor \frac{r}{2} \rfloor$ edge-disjoint spanning trees of G, which are also $\lfloor \frac{r}{2} \rfloor$ trees connecting $S = V(G)$. Therefore, $\kappa_{2r-1}(G) \geq \lfloor \frac{r}{2} \rfloor \geq 2$. Since G has a cut vertex, it follows that $\kappa_2(G) = \kappa(G) = 1$. So $\kappa_{2r-1}(G) > \kappa_2(G) = \kappa(G)$. In fact, Li et al. in [10, 11] already gave two such examples. One of them is the graph $G = K_t \vee (K_{\lfloor \frac{k-1}{2} \rfloor} \cup K_{\lceil \frac{k-1}{2} \rceil})$ $(k \geq 7, t \geq 1)$, which satisfies that $\kappa_k(G) = t+1$ but $\kappa(G) = t$. Clearly, $\kappa_k(G) > \kappa_2(G) = \kappa(G)$. Unlike the above example, in this example it is not necessary that $V(G) = S$. Another example is the graph G given in [10] (page 2154, Figure 9) that has $\kappa(G) = 4k+2$, $\kappa_3(G) = 3k+1$ and later in [11] Li showed that $\kappa_4(G) = 3k+2$, and so $\kappa_4(G) > \kappa_3(G)$. But for every two integers i and j with $i < j$, examples are needed to show that $\kappa_i > \kappa_j$. In any case, the monotone property cannot be guaranteed.

Let S be a set of k vertices of a connected graph G and $\mathcal{T}$ be a set of internally disjoint Steiner trees connecting S. A Steiner tree T connecting S is of $type\ I$ if $V(T) = S$ and T is of type II if $V(T) \setminus S \neq \emptyset$. Then $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, where the trees of $\mathcal{T}_1$ are $type\ I$, and the trees of $\mathcal{T}_2$ are $type\ II$ (Throughout this paper, $\mathcal{T}$, $\mathcal{T}_1$, $\mathcal{T}_2$ are always defined as this).

The following lemma is immediate by the above definitions.

Observation 1. *Let k, n be two integers with $3 \leq k \leq n$, and G be a connected graph of order n, and $S \subseteq V(G)$ with $|S| = k$. For each $T \in \mathcal{T}_1$, $|E(T) \cap (E[S] \cup E[S, \bar{S}])| = k - 1$; for $T \in \mathcal{T}_2$, $|E(T) \cap (E[S] \cup E[S, \bar{S}])| \geq k$, where $\bar{S} = V(G) \setminus S$.*

As we all know, a graph with large connectivity must have good edge distribution (almost uniform distribution). Now, we show that generalized k-connectivity has a similar property.

Theorem 3. *For any graph G with order at least k,*

$$\kappa_k(G) \leq \min_{S \subseteq V(G), |S|=k} \Big\lfloor \frac{1}{k-1}|E[S]| + \frac{1}{k}|E[S, \bar{S}]| \Big\rfloor,$$

where $S \subseteq V(G)$ with $|S| = k$, and $\bar{S} = V(G) \setminus S$. Moreover, the bound is sharp.

Proof. It suffices to show that $|\mathcal{T}_1| + |\mathcal{T}_2| \leq \frac{1}{k-1}|E[S]| + \frac{1}{k}|E[S, \bar{S}]|$. From Observation 1, $|E(T) \cap (E[S] \cup E[S, \bar{S}])| = k - 1$ for each $T \in \mathcal{T}_1$. Thus, $(k-1)|\mathcal{T}_1| \leq |E[S]|$, that is, $|\mathcal{T}_1| \leq \frac{|E[S]|}{k-1}$.

For $T \in \mathcal{T}_2$, $|E(T) \cap (E[S] \cup E[S, \bar{S}])| \geq k$ by Observation 1. On the one hand, $\Sigma_{T \in \mathcal{T}}|E[T] \cap (E[S] \cup E[S, \bar{S}])| = \Sigma_{T \in \mathcal{T}_1}|E[T] \cap (E[S] \cup E[S, \bar{S}])| + \Sigma_{T \in \mathcal{T}_2}|E[T] \cap (E[S] \cup E[S, \bar{S}])| \geq (k-1)|\mathcal{T}_1| + k|\mathcal{T}_2|$. On the other hand, $\Sigma_{T \in \mathcal{T}}|E[T] \cap (E[S] \cup E[S, \bar{S}])| = \Sigma_{T \in \mathcal{T}_1}|E[T] \cap (E[S] \cup E[S, \bar{S}])| + \Sigma_{T \in \mathcal{T}_2}|E[T] \cap (E[S] \cup E[S, \bar{S}])| \leq |E[S]| + |E[S, \bar{S}]|$. Thus, $(k-1)|\mathcal{T}_1| + k|\mathcal{T}_2| \leq |E[S]| + |E[S, \bar{S}]|$. Combining this with $|\mathcal{T}_1| \leq \frac{|E[S]|}{k-1}$, we have $|\mathcal{T}_1| + |\mathcal{T}_2| \leq \frac{1}{k-1}|E[S]| + \frac{1}{k}|E[S, \bar{S}]|$.

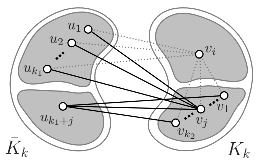

Figure 1: The edges of a tree are shown by the same type of lines.

To show sharp the sharpness of the bound, we consider the graph $G = K_k \vee \bar{K_k}$. Let $V(K_k) = \{v_1, v_2, \cdots, v_k\}$ and $V(\bar{K_k}) = \{u_1, u_2, \cdots, u_k\}$. From this theorem, set $S = V(\bar{K_k})$, we have $\kappa_k(G) \leq \frac{1}{k-1}|E[S]| + \frac{1}{k}|E[S, \bar{S}]| = \frac{1}{k-1} \cdot 0 + \frac{1}{k} \cdot k^2 = k$. It suffices to show that $\kappa_k(G) \geq k$. Without loss of generality, let $S = \{u_1, u_2, \cdots, v_{k_1}, v_1, v_2, \cdots, v_{k_2}\}$ where $k_1 + k_2 = k$. Then the trees $T_i = u_1v_i \cup u_2v_i \cup \cdots \cup u_{k_1}v_i \cup v_1v_i \cup v_2v_i \cup \cdots \cup v_{k_2}v_i (k_2 + 1 \leq i \leq k)$ and $T_j = u_{k_1+j}v_1 \cup u_{k_1+j}v_2 \cup \cdots \cup u_{k_1+j}v_{k_2} \cup u_1v_j \cup u_2v_j \cup \cdots \cup u_{k_1}v_j (1 \leq j \leq k_2)$ form k pairwise internally disjoint S-trees (see Figure 1), which implies that $\kappa_k(G) = k$. So the bound of this theorem is sharp. □

Remark 1. For a regular graph G, if G contains a clique of order k, then it must has small generalized k-connectivity by Theorem 3. Thus, in a sense, to obtain large generalized k-connectivity, a graph G must have almost uniform edge distribution.

Theorem 4. *For a connected graph G of order n,* $\kappa_3(G) = n - 2$ *if and only if* $G = K_n$ *or* $G = K_n \setminus e$.

Proof. Sufficiency. If $G = K_n$, then we have $\kappa_3(G) = n - 2$ by Theorem 1. If $G = K_n \setminus e$, then $\kappa_3(G) \leq n - 2$ by Proposition 1. We will show that $\kappa_3(G) \geq n - 2$. It suffices to show that for any $S \subseteq V(G)$ such that $|S| = 3$, there exist $n - 2$ internally disjoint S-trees in G.

Let $e = uv$, and $W = V(G) \setminus \{u, v\} = \{w_1, w_2, \cdots, w_{n-2}\}$. Clearly, $G[W]$ is a complete graph of order $n-2$. If $|\{u, v\} \cap S| = 1$ (see Figure 2 (a)), without loss of generality, let $S = \{u, w_1, w_2\}$, then the trees $T_i = w_iu \cup w_iw_1 \cup w_iw_2$ together with $T_1 = uw_1 \cup w_1w_2$, $T_2 = uw_2 \cup vw_2 \cup vw_1$ form $n - 2$ pairwise internally disjoint S-trees, where $i = 3, \cdots, n - 2$. If $|\{u, v\} \cap S| = 2$ (see Figure 2 (b)), without loss of generality, let $S = \{u, v, w_1\}$, then the trees $T_i = w_iu \cup w_iv \cup w_iw_1$ together with $T_1 = uw_1 \cup w_1v$ for $n - 2$ pairwise internally disjoint S-trees, where $i = 2, \cdots, n - 2$. Otherwise, suppose $S \subseteq W$ (see Figure 2 (c)). Without loss of generality, let $S = \{w_1, w_2, w_3\}$. The trees $T_i = w_iw_1 \cup w_iw_2 \cup w_iw_3$ $(i = 4, 5, \cdots, n-2)$ together with $T_1 = w_2w_1 \cup w_2w_3$ and $T_2 = uw_1 \cup uw_2 \cup uw_3$ and $T_3 = vw_1 \cup vw_2 \cup vw_3$ form $n-2$ pairwise internally disjoint S-trees. From the arguments above, we conclude that $\kappa_3(K_n \setminus e) \geq n-2$. From this together with Proposition 1, $\kappa(K_n \setminus e) = n - 2$.

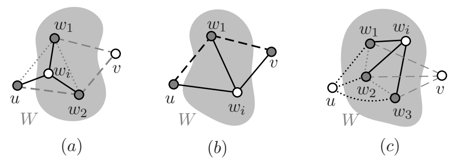

Figure 2: The edges of a tree are by the same type of lines.

Necessity. Next we show that if $G \neq K_n, K_n \setminus e$, then $\kappa_3(G) \leq n-3$, where G is a connected graph. Actually, we only need to show that $\kappa_3(G) \leq n - 3$ for a graph G obtained from the complete graph K_n by deleting any two edges. Let $G = K_n \setminus \{e_1, e_2\}$, where $e_1, e_2 \in E(K_n)$. It is easy to see that e_1 and e_2 form a path of order 3 (see Figure 3 (a)), or e_1 and e_2 are two independent edges (see Figure 3 (b)). First, we consider the former case. Let $P_3 = xyz$, $S = \{x, y, z\}$. Then $|E(G[S]) \cup E_G[S, \bar{S}]| = 3(n - 3) + 1$. Since $xy, yz \notin E(G)$, there exists no tree of type I. So each tree connecting S must belong to type II. From Observation 1, each tree of type II uses at least 3 edges in $E(G[S]) \cup E_G[S, \bar{S}]$.

So $3(n-3)+1$ edges form at most $\frac{3(n-3)+1}{3}$ trees. Thus $\kappa_3(G) \leq |\mathcal{T}| = |\mathcal{T}_1| + |\mathcal{T}_2| = 0 + |\mathcal{T}_2| \leq \frac{3(n-3)+1}{3}$ and $\kappa_3(G) \leq n-3$ since $\kappa_3(G)$ is an integer.

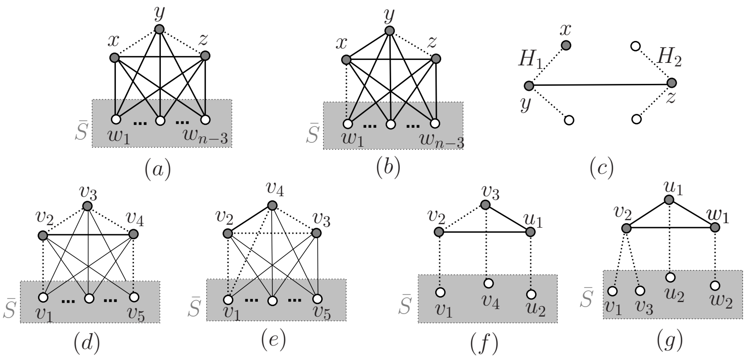

Figure 3: Graphs for Theorem 4 and 5.

Next, we consider the latter case. Set $e_1 = xw_1$, $e_2 = yz$, $S = \{x, y, z\}$. So $w_1 \in \bar{S}$ and $|E(G[S]) \cup E_G[S, \bar{S}]| = 3(n-3)+1$. If xy and xz form a tree of type I, then the tree use two edges in $E(G[S]) \cup E_G[S, \bar{S}]$ and the remaining $3(n-3)-1$ edges in $E(G[S]) \cup E_G[S, \bar{S}]$ form at most $\frac{3(n-3)-1}{3}$ trees of type II. So $|\mathcal{T}| = |\mathcal{T}_1| + |\mathcal{T}_2| \leq 1 + \frac{3(n-3)-1}{3}$ and $\kappa_3(G) \leq |\mathcal{T}| \leq n-3$ since $\kappa_3(G)$ is an integer. If xy and xz do not form a tree in $\mathcal{T}_1$, then all the edges of $E(G[S]) \cup E_G[S, \bar{S}]$ can only form trees of type II. From Observation 1, each tree of type II uses at least 3 edges in $E(G[S]) \cup E_G[S, \bar{S}]$. Thus $\kappa_3(G) \leq |\mathcal{T}| = |\mathcal{T}_2| \leq \frac{3(n-3)+1}{3}$ and $\kappa_3(G) \leq n-3$.

□

Li et al. obtained the following result in [10].

Lemma 1. *[10] Let G be a connected graph with minimum degree δ. Then $\kappa_3(G) \leq \delta$. In particular, if there are two adjacent vertices of degree δ, then $\kappa_3(G) \leq \delta - 1$.*

Recall that $\overline{G}$ denotes the complement of a graph G. Let us now give our main result.

Theorem 5. *Let G be a connected graph of order n ($n \geq 3$). $\kappa_3(G) = n-3$ if and only if $\overline{G} = P_4 \cup (n-4)K_1$ or $\overline{G} = P_3 \cup iP_2 \cup (n-2i-3)K_1$ ($i = 0, 1$) or $\overline{G} = C_3 \cup iP_2 \cup (n-2i-3)K_1$ ($i = 0, 1$) or $\overline{G} = rP_2 \cup (n-2r)K_1$ ($2 \leq r \leq \lfloor \frac{n}{2} \rfloor$).*

Proof. Sufficiency. Assume that $\kappa_3(G) = n-3$. From Lemm 1, $\delta(G) \geq \kappa_3(G) = n-3$ and hence $\delta(\overline{G}) \leq n-1-\delta(G) \leq 2$. So each component of $\overline{G}$ is a path or a cycle. We will show that the following claims hold.

Claim 1. $\overline{G}$ has at most one component of order larger than 2.

Proof of Claim 1. Suppose, to the contrary, that $\overline{G}$ has two components of order larger than 2, denoted by H_1 and H_2 (see Figure 3 (c)). Let $x, y \in V(H_1)$ and $z \in V(H_2)$ such that $d_{H_1}(y) = d_{H_2}(z) = 2$ and x is adjacent to y in H_1. Pick $S = \{x, y, z\}$. Clearly, $d_G(y) = n - 1 - d_{\overline{G}}(y) = n - 1 - d_{H_1}(y) = n - 3$. The same is true for z, that is, $d_G(z) = n - 3$. This implies that $\delta(G) \leq n - 3$. Since all components of $\overline{G}$ are paths or cycles, $\delta(G) \geq n - 3$. So $\delta(G) = n - 3$ and $d_G(y) = d_G(z) = \delta(G)$. Since $yz \in E(G)$, by Lemma 1 it follows that $\kappa_3(G) \leq \delta(G) - 1 = n - 4$, a contradiction.

Claim 2. If H is a component of $\overline{G}$ of order larger than three, then $\overline{G} = P_4 \cup (n-4)K_1$.

Proof of Claim 2. Assume, to the contrary, that H is a path or a cycle of order larger than 4, or a cycle of order 4, or H is a path of order 4 and there exists another nontrivial component in $\overline{G}$.

Suppose that H is a path or a cycle of order larger than 4. We can pick a P_5 in H. Let $P_5 = v_1, v_2, v_3, v_4, v_5$, $S = \{v_2, v_3, v_4\}$ (see Figure 3 (d)). Since $v_2v_3, v_3v_4 \notin E(G[S])$, there exists no tree of type I connecting S. From Observation 1, each tree of type II uses at least 3 edges. Since $|E(G[S]) \cup E_G[S, \bar{S}]| = 3(n-3) - 1$, we have $|\mathcal{T}_2| \leq \frac{3(n-3)-1}{3}$ and $|\mathcal{T}| = |\mathcal{T}_2| = n - 4$ since $\kappa_3(G)$ is an integer. This contradicts to $\kappa_3(G) = n - 3$.

Suppose that H is a cycle of order 4. Set $H = v_1, v_2, v_3, v_4$ be a cycle, and $S = \{v_2, v_3, v_4\}$ (see Figure 3 (e)). Since $v_2v_3, v_3v_4 \notin E(G[S])$, there exists no tree of type I. Since each tree of type II uses at least 3 edges by Observation 1 and $|E(G[S]) \cup E_G[S, \bar{S}]| = 3(n-3) - 1$, we have $|\mathcal{T}_2| \leq \frac{3(n-3)-1}{3}$ and $|\mathcal{T}| = |\mathcal{T}_2| = n - 4$, which also contradicts to $\kappa_3(G) = n - 3$.

From the above arguments, we assume that $H = P_4 = v_1v_2v_3v_4$ is a path of order 4 and there exists at least one edge in $\overline{G}$, say $e = u_1u_2$. Choose $S = \{v_2, v_3, u_1\}$ (see Figure 3 (f)). We claim that there exists one tree of type I. Otherwise, all trees are trees of type II. Since each tree of type II uses at least 3 edges by Observation 1, $|E(G[S]) \cup E_G[S, \bar{S}]| = 3(n-3) - 1$, we have $|\mathcal{T}| = |\mathcal{T}_2| \leq \frac{3(n-3)-1}{3}$. Then $\kappa_3(G) \leq n - 4$, a contradiction. So $T_1 = v_2u_1 \cup v_3u_1$ is a tree of type I. Since $\kappa_3(G) = n - 3$, there are $n - 4$ trees of type II connecting S. Set $G_1 = G \setminus E(T_1)$. Then $d_{G_1}(v_2) = d_{G_1}(v_3) = d_{G_1}(u_1) = n - 4$ and each edge incident to v_2 or v_3 or u_1 must belong to a tree of type II. By the definition of internally disjoint trees, each tree of type II uses at least one vertex of $\bar{S}$. One can see that there exist at most $n - 6$ trees such that each tree uses exact one vertex of $\bar{S}$. Then each remaining tree uses at least two vertices of $\bar{S}$. So there exist at most $n - 5$ trees connecting S, a contradiction.

Claim 3. If H is a component of $\overline{G}$ of order 3, then $\overline{G} = C_3 \cup iP_2 \cup (n - 2i - 3)K_1$ $(i = 0, 1)$ or $\overline{G} = P_3 \cup iP_2 \cup (n - 2i - 3)K_1$ $(i = 0, 1)$.

We only consider the case that $H = P_3$. If $H = P_3 = v_1v_2v_3$, then each of the other components are independent edges. We claim that $\overline{G}$ contains at most 2 independent edges except H, say $e_1 = u_1u_2$ and $e_2 = w_1w_2$. Choose $S = \{v_2, u_1, w_1\}$ (see Figure 3 (g)). Similar to the proof of Claim 2, there exists one tree of type I, say T_1. If $T_1 = v_2u_1 \cup v_2w_1$, then $d_{G_1}(v_2) = n-5$ where $G_1 = G \setminus E(T_1)$ and there exist at most $n-5$ trees connecting S in G_1, which implies that $\kappa_3(G) \leq n-4$, a contradiction. So $T_1 = w_1u_1 \cup v_2w_1$ or $T_2 = v_2u_1 \cup u_1w_1$. Without loss of generality, let $T_1 = w_1u_1 \cup v_2w_1$. Set $G_2 = G \setminus E(T_1)$. Then $d_{G_2}(w_1) = d_{G_2}(v_2) = n-4$ and each edge incident to w_1 or v_2 must belong to a tree of type II. By the definition of internally disjoint trees, each tree of type II uses at least one vertex of $\overline{S}$. One can see that there exists at most $n-7$ trees such that each tree uses exact one vertex of $\overline{S}$. Then each remaining tree uses at least two vertices of $\overline{S}$. So there exist at most $n-5$ trees connecting S, which contradicts to $\kappa_3(G) = n-3$.

From the above arguments, we can conclude that $\overline{G} = P_4 \cup (n-4)K_1$ or $\overline{G} = P_3 \cup iP_2 \cup (n-2i-3)K_1$ $(i = 0, 1)$ or $\overline{G} = C_3 \cup iP_2 \cup (n-2i-3)K_1$ $(i = 0, 1)$ or $\overline{G} = rP_2 \cup (n-2r)K_1$ $(2 \leq r \leq \lfloor \frac{n}{2} \rfloor)$.

Necessity. We show that $\kappa_3(G) \geq n-3$ if G is a graph such that $\overline{G} = P_4 \cup (n-4)K_1$ or $\overline{G} = P_3 \cup iP_2 \cup (n-2i-3)K_1$ $(i = 0, 1)$ or $\overline{G} = C_3 \cup iP_2 \cup (n-2i-3)K_1$ $(i = 0, 1)$ or $\overline{G} = rP_2 \cup (n-2r)K_1$ $(2 \leq r \leq \lfloor \frac{n}{2} \rfloor)$. We have the following cases to consider.

Case 1. $\overline{G} = rP_2 \cup (n-2r)K_1$ $(2 \leq r \leq \lfloor \frac{n}{2} \rfloor)$.

We can regard the graph G as a graph obtained from the complete graph K_n by deleting an edge set M, where M is a matching of K_n. We only need to prove that $\kappa_3(G) \geq n-3$ when M is a maximum matching of K_n. Let $S = \{x, y, z\}$. Since $|S| = 3$, S contains at most a pair of adjacent vertices under M.

If S contains a pair of adjacent vertices x and y under M, then the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_{n-3} = xy \cup yz$ form $n-3$ pairwise internally disjoint trees connecting S, where $\{w_1, w_2, \cdots, w_{n-4}\} = V(G) \setminus \{x, y, z, z'\}$ such that z' is the adjacent vertex of z under M if z is M-saturated, or z' is any vertex in $V(G) \setminus \{x, y, z\}$ if z is M-unsaturated. If S contains no pair of adjacent vertices under M, then the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_{n-5} = yx \cup xy' \cup y'z$ and $T_{n-4} = yx' \cup zx' \cup zx$ and $T_{n-3} = zy \cup yz' \cup z'x$ form $n-3$ pairwise edge-disjoint S-trees, where $\{w_1, w_2, \cdots, w_{n-6}\} = V(G) \setminus \{x, y, z, x', y', z'\}$, x', y', z' are the adjacent vertices of x, y, z under M, respectively, if x, y, z are all M-saturated, or x', y' are the adjacent vertices of x, y under M, respectively, and z' is any vertex in $V(G) \setminus \{x, y, z, x', y'\}$ if z is M-unsaturated.

From the arguments above , we know that $\kappa(S) \geq n-3$ for $S \subseteq V(G)$. Thus $\kappa_3(G) \geq n-3$. From this together with Theorem 4, we know that $\kappa_3(G) = n-3$.

Case 2. $\overline{G} = C_3 \cup iP_2 \cup (n-2i-3)K_1$ $(i = 0, 1)$ or $\overline{G} = P_3 \cup iP_2 \cup (n-2i-3)K_1$ $(i = 0, 1)$.

We only need to check that $\kappa_3(G) \geq n-3$ for $\overline{G} = C_3 \cup P_2 \cup (n-5)K_1$.

Let $C_3 = v_1, v_2, v_3$ and $P_2 = u_1u_2$, and let $S = \{x, y, z\}$ be a 3-subset of G. If $S = V(C_3)$, then there exist $n-3$ pairwise internally disjoint S-trees since each vertex in S is adjacent to each vertex in $G \setminus S$. Suppose $S \neq V(C_3)$. If $|S \cap V(C_3)| = 2$, without loss of generality, assume that $x = v_1$ and $y = v_2$. When $S \cap V(P_2) \neq \emptyset$, say $z = u_1$, the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_{n-4} = xz \cup yz$ and $T_{n-3} = xu_2 \cup u_2v_3 \cup zv_3 \cup u_2y$ form $n-3$ pairwise internally disjoint trees connecting S, where $\{w_1, w_2, \cdots, w_{n-5}\} = V(G) \setminus \{x, y, z, u_2, v_3\}$. When $S \cap V(P_2) = \emptyset$, the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_{n-3} = xz \cup zy$ are $n-3$ pairwise internally disjoint trees connecting S, where $\{w_1, w_2, \cdots, w_4\} = V(G) \setminus \{x, y, z, v_3\}$. If $|S \cap V(C_3)| = 1$, without loss of generality, assume $x = v_1$. When $|S \cap V(P_2)| = 2$, say $y = u_1$ and $z = u_2$, the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_{n-4} = xz \cup v_2z \cup v_2y$ and $T_{n-3} = xy \cup yv_3 \cup zv_3$ form $n-3$ pairwise internally disjoint trees connecting S, where $\{w_1, w_2, \cdots, w_{n-5}\} = V(G) \setminus \{x, y, z, v_2, v_3\}$. When $S \cap V(P_2) = 1$, say $u_1 = y$, the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_{n-5} = xz \cup zy$ and $T_{n-4} = xu_2 \cup u_2v_2 \cup v_2y \cup v_2z$ and $T_{n-3} = xy \cup yv_3 \cup v_3z$ are $n-3$ pairwise internally disjoint trees connecting S, where $\{w_1, w_2, \cdots, w_{n-6}\} = V(G) \setminus \{x, y, z, v_2, v_3, u_2\}$. When $|S \cap V(P_2)| = \emptyset$, the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_{n-4} = xz \cup zy$ and $T_{n-3} = xy \cup yv_3 \cup zv_3$ form $n-3$ pairwise internally disjoint S-trees, where $\{w_1, w_2, \cdots, w_{n-5}\} = V(G) \setminus \{x, y, z, v_2, v_3\}$. If $S \cap V(C_3) = \emptyset$, when $|S \cap V(P_2)| = 0$ or $|S \cap V(P_2)| = 2$, the trees $T_i = w_ix \cup w_iy \cup w_iz$ form $n-3$ pairwise internally disjoint S-trees, where $\{w_1, w_2, \cdots, w_{n-3}\} = V(G) \setminus \{x, y, z\}$. When $S \cap V(P_2) = 1$, say $u_1 = x$, the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_{n-3} = xz \cup zy$ form $n-3$ pairwise internally disjoint S-trees, where $\{w_1, w_2, \cdots, w_{n-4}\} = V(G) \setminus \{x, y, z, u_2\}$.

From the above arguments, we conclude that $\kappa(S) \geq n-3$ for $S \subseteq V(G)$. Thus $\kappa_3(G) \geq n-3$. From this together with Theorem 4, $\kappa_3(G) = n-3$.

Case 3. $\overline{G} = P_4 \cup (n-4)K_1$.

This case can be proved by an argument similar to Cases 1 and 2. □

Remark 2. In this paper, we characterize graphs with $\kappa_3(G) = n-2, n-3$. There exists an interesting problem: To characterize graphs with $\kappa_3(G) = 1$.

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