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On Spieß's conjecture on harmonic numbers

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ABSTRACT

Let H_n be the n-th harmonic number and let $H_n^{(2)}$ be the n-th generalized harmonic number of order two. Spieß proved that for a nonnegative integer m and for $t = 1, 2,$ and 3, the sum $R(m, t) = \sum_{k=0}^{n} k^m H_k^t$ can be represented as a polynomial in H_n with polynomial coefficients in n plus $H_n^{(2)}$ multiplied by a polynomial in n. For $t = 3$, we show that the coefficient of $H_n^{(2)}$ in Spieß's formula equals $B_m/2$, where B_m is the m-th Bernoulli number. Spieß further conjectured for $t \geq 4$ such a summation takes the same form as for $t \leq 3$. We find a counterexample for $t = 4$. However, we prove that the structure theorem of Spieß holds for the sum $\sum_{k=0}^{n} p(k) H_k^4$ when the polynomial $p(k)$ satisfies a certain condition. We also give a structure theorem for the sum $\sum_{k=0}^{n} k^m H_k H_k^{(2)}$. Our proofs rely on Abel's lemma on summation by parts.

1. Introduction

For a positive integer n and an integer r, the n-th generalized harmonic number of order r is defined by

$$H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}.$$

For convenience, we set $H_n^{(r)} = 0$ for $n \leq 0$. For $r = 1$, $H_n = H_n^{(1)}$ is the n-th harmonic number. Identities involving harmonic numbers and the generalized harmonic numbers have been extensively studied in the literature; see, for example, [1–4].

Spieß [3] considered the following summation

$$R(m, t) = \sum_{k=0}^{n} k^m H_k^t,$$

where m and t are nonnegative integers. He obtained the following structure theorem.

Theorem 1.1 *([3, Theorem 30]). Let $p(k)$ be a polynomial in k of degree m. Then for $t = 1, 2,$ or 3, there exist polynomials $q_0(n), \ldots, q_t(n)$ and $C(n)$ of degree at most $m + 1$ such that*

$$\sum_{k=0}^{n} p(k) H_k^t = \sum_{i=0}^{t} q_i(n) H_n^i + C(n) H_n^{(2)}, \quad (1.1)$$

for all nonnegative integers n. Moreover, $C(n) = 0$ when $t = 1, 2$.

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Using the above structure theorem, we can establish identities by interpolation with the method of undetermined coefficients; see [3, Section 5] for more details of this method. For example, we have

$$\sum_{k=0}^{n} H_k^2 = (n+1)H_n^2 - (2n+1)H_n + 2n,$$

$$\sum_{k=0}^{n} kH_k^2 = \frac{n(n+1)}{2}H_n^2 - \frac{n^2-n-1}{2}H_n + \frac{n(n-3)}{4}.$$

For $t = 3$, we determine the polynomial $C(n)$ in Theorem 1.2 in terms of the Bernoulli numbers.

Theorem 1.2. *For nonnegative integers n and m, there exist polynomials $q_1(n)$, $q_2(n)$, and $q_3(n)$ of degree at most $m+1$, such that*

$$\sum_{k=0}^{n} k^m H_k^3 = s_m(n)H_n^3 + q_1(n)H_n^2 + q_2(n)H_n + q_3(n) + \frac{B_m}{2}H_n^{(2)}, \tag{1.2}$$

where $s_m(n) = \sum_{k=0}^{n} k^m$ and B_m is the m-th Bernoulli number.

Spieß [3] conjectured that Theorem 1.1 holds for any positive integer $t \geq 4$. We find a counterexample for $R(0, t)$. More precisely, by interpolation with the method of undetermined coefficients, we can show that the sum

$$R(0,4) = \sum_{k=0}^{n} H_k^4$$

cannot be represented by the form as conjectured by Spieß. However, the following theorem shows that the conjecture holds for $t = 4$ when the polynomial $p(k)$ satisfies a certain condition.

Theorem 1.3. *Let $p(k) = \sum_{i=0}^{m} a_i k^i$ be a polynomial in k of degree m such that the coefficients a_i satisfy $\sum_{i=0}^{m} a_i B_i = 0$. Then there exist polynomials $S_m(n)$, $q_1(n)$, $q_2(n)$, $q_3(n)$ and $q_4(n)$ of degree at most $m+1$, together with a constant C such that*

$$\sum_{k=0}^{n} p(k)H_k^4 = S_m(n)H_n^4 + q_1(n)H_n^3 + q_2(n)H_n^2 + q_3(n)H_n + q_4(n) + CH_n^{(2)}. \tag{1.3}$$

In particular, we have $S_m(n) = \sum_{k=0}^{n} p(k)$ and

$$C = -\sum_{i=0}^{m} a_i \left(\frac{2}{i+1} \sum_{k=0}^{i} \binom{i+1}{k} B_k \sum_{j=0}^{i-k} \binom{i-k}{j} B_j - \frac{i}{2} B_{i-1} \right).$$

Moreover, we consider the summation $\sum_{k=0}^{n} k^m H_k H_k^{(2)}$ and obtain a similar structure theorem.

Theorem 1.4. *Let n and m be nonnegative integers. There exist polynomials $q_1(n)$, $q_2(n)$ and $q_3(n)$ of degree at most $m+1$, m and m, respectively, such that*

$$\sum_{k=0}^{n} k^m H_k H_k^{(2)} = s_m(n)H_n H_n^{(2)} + q_1(n)H_n^{(2)} + q_2(n)H_n + q_3(n) - \frac{B_m}{2}H_n^2, \tag{1.4}$$

where $s_m(n) = \sum_{k=0}^{n} k^m$.

Our proofs of Theorems 1.2–1.4 rely on Abel's lemma on summation by parts. In Section 2, we give a proof of Theorem 1.2. The proofs of Theorems 1.3 and 1.4 are similar to that of Theorem 1.2 and hence are omitted. We conclude this paper with several examples.

2. The proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2 by using Abel's lemma on summation by parts, which is stated as follows.

Lemma 2.1 (*Abel's Lemma*). *Let $\{a_k\}$ and $\{b_k\}$ be two sequences. Then we have*

$$\sum_{k=m}^{n-1} (a_{k+1} - a_k)b_k = \sum_{k=m}^{n-1} a_{k+1}(b_k - b_{k+1}) + a_n b_n - a_m b_m. \tag{2.1}$$

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Abel's lemma can be used to prove identities on harmonic numbers; see, for example, Graham, Knuth and Patashnik [1]. For example, by taking $a_k = H_k^{(q)}$ and $b_k = H_k^{(p)}$ in (2.1), we are led to the following well-known formula

$$\sum_{k=1}^{n} \frac{H_k^{(p)}}{k^q} + \sum_{k=1}^{n} \frac{H_k^{(q)}}{k^p} = H_n^{(p)} H_n^{(q)} + H_n^{(p+q)}.$$

In particular, we have

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{H_n^2 + H_n^{(2)}}{2}. \tag{2.2}$$

Proof of Theorem 1.2. Let $s_m(n) = 0^m + 1^m + \cdots + n^m$, where $0^m = \delta_{0,m}$. It is well-known that

$$s_m(n) = \frac{1}{m+1} \sum_{i=0}^{m} \binom{m+1}{i} B_i (n+1)^{m-i+1}. \tag{2.3}$$

Note that $s_m(k)$ can be written as

$$\sum_{i=1}^{m+1} a_i (k+1)^i$$

with $a_1 = B_m$ and $a_2 = mB_{m-1}/2$.

By Abel's lemma, we find

$$\begin{aligned} \sum_{k=0}^{n} k^m H_k^3 &= \sum_{k=0}^{n} (s_m(k) - s_m(k-1)) H_k^3 \\ &= -\sum_{k=0}^{n} s_m(k)(H_{k+1}^3 - H_k^3) + s_m(n) H_{n+1}^3 \\ &= -\sum_{k=0}^{n-1} s_m(k) \left(\frac{3H_k^2}{k+1} + \frac{3H_k}{(k+1)^2} + \frac{1}{(k+1)^3} \right) + s_m(n) H_n^3. \end{aligned} \tag{2.4}$$

We proceed to compute the three sums on the right hand side of (2.4). First, we see that

$$\sum_{k=0}^{n-1} s_m(k) \frac{3H_k^2}{k+1}$$

can be represented in the form as in (1.1) since

$$\sum_{k=0}^{n-1} p(k) H_k^t = \sum_{k=0}^{n} p(k) H_k^t - p(n) H_n^t.$$

Second, it is easily checked that

$$\sum_{k=0}^{n-1} s_m(k) \frac{3H_k}{(k+1)^2} = \sum_{k=0}^{n-1} 3 \left(\sum_{i=0}^{m-1} a_i (k+1)^i \right) H_k + 3B_m \sum_{k=0}^{n-1} \frac{H_k}{k+1}. \tag{2.5}$$

Hence the first sum of the right hand side can also be represented in the form as in Theorem 1.1. According to (2.2), the second sum on the right hand of (2.5) can be evaluated as follows

$$3B_m \sum_{k=0}^{n-1} \frac{H_k}{k+1} = \frac{3B_m}{2} (H_n^2 - H_n^{(2)}).$$

Finally, for the third sum on the right hand side of (2.4), we have

$$\sum_{k=0}^{n-1} \frac{s_m(k)}{(k+1)^3} = \sum_{k=0}^{n-2} \left(\sum_{i=0}^{m-2} a_i (k+1)^i \right) + a_2 H_n + B_m H_n^{(2)}.$$

Summing up the three summations on the right hand side of (2.4), we find that the coefficient of $H_n^{(2)}$ equals $B_m/2$. This completes the proof. □

3. Examples

Applying interpolation with the method of undetermined coefficients, we conclude this paper by giving the following examples from Theorems 1.2–1.4.

$$\sum_{k=0}^{n} H_k^3 = (n+1)H_n^3 - \frac{3}{2}(2n+1)H_n^2 + 3(2n+1)H_n - 6n + \frac{1}{2}H_n^{(2)},$$

$$\sum_{k=0}^{n} kH_k^3 = \frac{n(n+1)}{2}H_n^3 - \frac{3(n^2-n-1)}{4}H_n^2 + \frac{3n^2-9n-5}{4}H_n - \frac{3n(n-7)}{8} - \frac{1}{4}H_n^{(2)},$$

$$\sum_{k=0}^{n} k^2H_k^3 = \frac{n(n+1)(2n+1)}{6}H_n^3 - \frac{4n^3-3n^2-n+3}{12}H_n^2 + \frac{8n^3-15n^2+25n+15}{36}H_n - \frac{n(16n^2-57n+203)}{216} + \frac{1}{12}H_n^{(2)},$$

$$\sum_{k=0}^{n} (2k+1)H_k^4 = (n+1)^2H_n^4 - 2n(n+1)H_n^3 + (3n^2+3n+1)H_n^2 - (3n^2+3n+1)H_n + \frac{3n(n+1)}{2},$$

$$\sum_{k=0}^{n} (3k^2+k)H_k^4 = n(n+1)^2H_n^4 - \frac{4}{3}n(n-1)(n+1)H_n^3 + \frac{1}{3}n(4n+1)(n-1)H_n^2 - \frac{1}{9}(8n^3-15n^2+7n+6)H_n + \frac{1}{54}n(16n^2-57n+95) - \frac{1}{3}H_n^{(2)},$$

$$\sum_{k=0}^{n} H_kH_k^{(2)} = (n+1)H_nH_n^{(2)} - \frac{2n+1}{2}H_n^{(2)} + H_n - \frac{1}{2}H_n^2,$$

$$\sum_{k=0}^{n} kH_kH_k^{(2)} = \frac{n(n+1)}{2}H_nH_n^{(2)} + \frac{1+n-n^2}{4}H_n^{(2)} - \frac{2n+3}{4}H_n + \frac{3}{4}n + \frac{1}{4}H_n^2.$$

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