Further hardness results on the rainbow vertex-connection number of graphs*

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Abstract

A vertex-colored graph G is *rainbow vertex-connected* if any pair of vertices in G are connected by a path whose internal vertices have distinct colors, which was introduced by Krivelevich and Yuster. The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. In a previous paper we showed that it is NP-Complete to decide whether a given graph G has $rvc(G) = 2$. In this paper we show that for every integer $k \geq 2$, deciding whether $rvc(G) \leq k$ is NP-Hard. We also show that for any fixed integer $k \geq 2$, this problem belongs to NP-class, and so it becomes NP-Complete.

Keywords: vertex-colored graph, rainbow vertex-connection number, NP-Hard, NP-Complete.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. Undefined terminology and notation can be found in [2].

Let G be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \cdots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path P of G is a *rainbow path* if no two edges of P are colored the same. The graph G is called *rainbow-connected* if for any pair of vertices u and v of G, there is a rainbow $u - v$ path. The minimum number of colors for which there is an edge-coloring of G such that G is rainbow connected is called the *rainbow connection number*, denoted by $rc(G)$. Clearly, if a graph is rainbow connected, then it is also connected. Conversely, any connected graph has a trivial edge-coloring that makes it rainbow connected, just assign each edge a distinct color. An easy

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observation is that if G has n vertices then $rc(G) \leq n-1$, since one may color the edges of a spanning tree with distinct colors, and color the remaining edges with one of the colors already used. It is easy to see that if H is a connected spanning subgraph of G, then $rc(G) \leq rc(H)$. We note the trivial fact that $rc(G) = 1$ if and only if G is a clique, the fact that $rc(G) = n-1$ if and only if G is a tree, and the easy observation that a cycle with $k \geq 4$ vertices has a rainbow connection number $\lceil k/2 \rceil$. Also notice that $rc(G) \geq diam(G)$, where $diam(G)$ is the diameter of G.

Similar to the concept of rainbow connection number, Krivelevich and Yuster [7] proposed the concept of rainbow vertex-connection. Let G be a nontrivial connected graph with a vertex-coloring $c : V(G) \rightarrow \{1, 2, \cdots, k\}, k \in \mathbb{N}$. A path P of G is *rainbow vertex-connected* if its internal vertices have distinct colors. The graph G is *rainbow vertex-connected* if any pair of vertices are connected by a rainbow vertex-connected path. In particular, if k colors are used, then G is rainbow k-vertex-connected. The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. An easy observation is that if G is of order n then $rvc(G) \leq n-2$, $rvc(G) = 0$ if and only if G is a complete graph, and $rvc(G) = 1$ if and only if $diam(G) = 2$. Notice that $rvc(G) \geq diam(G) - 1$ with equality if the diameter is 1 or 2. For the rainbow connection number and the rainbow vertex-connection number, some examples were given to show that there is no upper bound for one of parameters in terms of the other in [7]. Krivelevich and Yuster [7] proved that if G is a graph with n vertices and minimum degree δ, then $rvc(G) < 11n/\delta$. Li and Shi used a similar proof technique and greatly improved this bound, see [9].

The computational complexity of rainbow connection number has been studied extensively. In [3], Caro et al. conjectured that computing $rc(G)$ is an NP-Hard problem, and that even deciding whether a graph has $rc(G) = 2$ is NP-Complete. Later, Chakraborty et al. confirmed this conjecture in [4]. They also conjectured that for every integer $k \geq 2$, to decide whether $rc(G) \leq k$ is NP-Hard. Recently, Ananth and Nasre confirmed the conjecture in [1]. Li and Li [8] showed that for any fixed integer $k \geq 2$, to decide whether $rc(G) \leq k$ is actually NP-Complete. For the rainbow vertex-connection number we got a similar complexity result in [6].

Theorem 1 *[6] Given a graph G, deciding whether $rvc(G) = 2$ is NP-Complete. Thus, computing $rvc(G)$ is NP-Hard.*

As a generalization of the above result, in this paper we will show the following result:

Theorem 2 *For every integer $k \geq 2$, to decide whether $rvc(G) \leq k$ is NP-Hard. Moreover, for any fixed integer $k \geq 2$, the problem belongs to NP-class, and therefore it is NP-Complete.*

In order to prove this theorem, we first show that an intermediate problem called the k-subset rainbow vertex-connection problem is NP-Hard by giving a reduction from the vertex-coloring problem. We then establish the polynomial-time equivalence of the k-subset rainbow vertex-connection problem and the problem of deciding whether $rvc(G) \leq k$ for a graph G.

2 Proof of Theorem 2

We first describe the problem of k-subset rainbow vertex-connection: given a graph G and a set of pairs $P \subseteq V(G) \times V(G)$, decide whether there is a vertex-coloring of G with k colors such that every pair of vertices $(u, v) \in P$ is rainbow vertex-connected. Recall that the k-vertex-coloring problem is as follows: given a graph G and an integer k, whether there exists an assignment of at most k colors to the vertices of G such that no pair of adjacent vertices are colored the same. It is known that this k-vertex-coloring problem is NP-Hard for $k \geq 3$. Now we reduce the k-vertex-coloring problem to the k-subset rainbow vertex-connection problem, which shows that the problem of k-subset rainbow vertex-connection is NP-Hard.

Lemma 1 *The problem of k-vertex-coloring is polynomially reducible to the problem of k-subset rainbow vertex-connection.*

Proof. Let $G = (V, E)$ be an instance of the k-vertex-coloring problem, we construct a graph $\langle G' = (V', E'), P \rangle$ as follows:

For every vertex $v \in V$ we introduce a new vertex x_v. We set

$$V' = V \cup \{x_v : \ v \in V\} \textit{ and } E' = E \cup \{(v, x_v) : \ v \in V\}.$$

Now we define the set P as follows:

$$P = \{(x_u, x_v) : (u, v) \in E\}.$$

It remains to verify that G is vertex-colorable using $k(\geq 3)$ colors if and only if there is a vertex-coloring of G' with k colors such that every pair of vertices $(x_u, x_v) \in P$ is rainbow vertex-connected.

Let c be the proper k-vertex-coloring of G. We define the vertex-coloring c' of G' by $c'(x_v) = c'(v) = c(v)$. If $(x_u, x_v) \in P$, then $(u, v) \in E$, $c(u) \neq c(v)$, and so $c'(u) \neq c'(v)$, $x_u u v x_v$ is a rainbow vertex-connected path between x_u and x_v.

In the other direction, assume that c' is a k-vertex-coloring of G' such that every pair of vertices $(x_u, x_v) \in P$ is rainbow vertex-connected. We define the vertex-coloring c of G by $c(v) = c'(v)$. For every $(u, v) \in E$, $(x_u, x_v) \in P$, since the rainbow vertex-connected path between x_u and x_v must go through u and v, $c'(u) \neq c'(v)$, and so $c(u) \neq c(v)$, thus c is the proper k-vertex-coloring of G. ∎

In the following, we prove that the problem of deciding whether a graph is k-subset rainbow vertex-connection is polynomial-time equivalent to the problem of deciding whether $rvc(G) \leq k$ for a graph G.

Lemma 2 *The following problems are polynomial-time equivalent:*
1. Given a graph G, decide whether $rvc(G) \leq k$.
2. Given a graph G and a set $P \subseteq V(G) \times V(G)$ of pairs of vertices, decide whether there is a vertex-coloring of G with k colors such that every pair of vertices $(u, v) \in P$ is rainbow vertex-connected.

Proof. It is sufficient to demonstrate a reduction from Problem 2 to Problem 1. Let $\langle G = (V, E), P\rangle$ be any instance of Problem 2. We construct a graph $G_k = (V_k, E_k)$ such that G is a subgraph of G_k and $rvc(G_k) \leq k$ if and only if G is k-subset rainbow vertex-connected. We prove the correctness of the reduction by induction on k. For $k = 2$ and $k = 3$, we give explicit constructions and show that the reduction is valid. Then we show our inductive step to get G_k and prove the correctness of the reduction.

Construction of G_2: Let $G_2 = (V_2, E_2)$ where the vertex set V_2 is defined as follows:

$$\begin{aligned} V_2 &= \{u\} \cup V_2^{(0)} \cup V_2^{(2)} \\ V_2^{(0)} &= \{v_{i,0}^{(1)}, v_{i,0}^{(2)} : i \in \{1, 2, \cdots, n\}\} \cup \{w_{i,j}^{(1)}, w_{i,j}^{(2)} : (v_i, v_j) \in (V \times V)\backslash P\} \\ V_2^{(2)} &= \{v_{i,2} : i \in \{1, 2, \cdots n\}\} \end{aligned}$$

and the edge set E_2 is defined as:

$$\begin{aligned} E_2 &= E_2^{(1)} \cup E_2^{(2)} \cup E_2^{(3)} \cup E_2^{(4)} \cup E_2^{(5)} \cup E_2^{(6)} \\ E_2^{(1)} &= \{(u, x) : x \in V_2^{(0)}\} \\ E_2^{(2)} &= \{(v_{i,0}^{(1)}, v_{i,0}^{(2)}) : i \in \{1, 2, \cdots, n\}\} \\ E_2^{(3)} &= \{(w_{i,j}^{(1)}, w_{i,j}^{(2)}) : (v_i, v_j) \in (V \times V)\backslash P\} \\ E_2^{(4)} &= \{(v_{i,2}, v_{i,0}^{(1)}), (v_{i,2}, v_{i,0}^{(2)}) : i \in \{1, 2, \cdots, n\}\} \\ E_2^{(5)} &= \{(v_{i,2}, w_{i,j}^{(1)}), (v_{j,2}, w_{i,j}^{(2)}) : (v_i, v_j) \in (V \times V)\backslash P\} \\ E_2^{(6)} &= \{(v_{i,2}, v_{j,2}) : (v_i, v_j) \in E(G)\} \end{aligned}$$

Denote $H_2 = G_2[\{v_{i,2} : i \in \{1, 2, \ldots, n\}\}]$. Let $P_2 = \{(v_{i,2}, v_{j,2}) : (v_i, v_j) \in P\}$. The graph G_2 satisfies the property that for all $(v_{i,2}, v_{j,2}) \in P_2$ there is no path of length ≤ 3 between $v_{i,2}$ and $v_{j,2}$ in $G_2 \backslash E(H_2)$ and also for all $(v_{i,2}, v_{j,2}) \notin P_2$ the length of the shortest path between $v_{i,2}$ and $v_{j,2}$ in $G_2 \setminus E(H_2)$ is 3.

Let $c : V \to \{1, 2\}$ be a 2-vertex-coloring of G such that every pair of vertices in P is rainbow vertex-connected . Define the vertex-coloring c_2 of G_2 as follows:

- $c_2(u) = 1$.
- $c_2(v_{i,0}^{(1)}) = 1$ and $c_2(v_{i,0}^{(2)}) = 2$ for $i \in \{1, 2, \cdots, n\}$.

  $c_2(w_{i,j}^{(1)}) = 1$ and $c_2(w_{i,j}^{(2)}) = 2$, for all $w_{i,j}^{(\alpha)} \in V_2^{(0)}, \alpha \in \{1, 2\}$.
- $c_2(v_{i,2}) = c(v_i)$, for $i \in \{1, 2, \cdots, n\}$.

It can be easily verified that $rvc(G_2) \leq 2$ if and only if G is 2-subset rainbow vertex-connected.

Construction of G_3: Let $G_3 = (V_3, E_3)$ where the vertex set V_3 is defined as follows:

$$\begin{aligned}
V_3 &= V_3^{(0)} \cup V_3^{(1)} \cup V_3^{(3)} \\
V_3^{(0)} &= \{v_{i,0}^{(1)}, v_{i,0}^{(2)} : i \in \{1, 2, \cdots, n\}\} \cup \{u_{i,j}^{(1)}, u_{i,j}^{(2)} : (v_i, v_j) \in (V \times V) \backslash P\} \\
V_3^{(1)} &= \{v_{i,1}^{(1)}, v_{i,1}^{(2)} : i \in \{1, 2, \cdots, n\}\} \cup \{w_{i,j}^{(1)}, w_{i,j}^{(2)} : (v_i, v_j) \in (V \times V) \backslash P\} \\
V_3^{(3)} &= \{v_{i,3} : i \in \{1, 2, \cdots, n\}\}
\end{aligned}$$

and the edge set E_3 is defined as:

$$\begin{aligned}
E_3 &= E_3^{(1)} \cup E_3^{(2)} \cup E_3^{(3)} \cup E_3^{(4)} \cup E_3^{(5)} \cup E_3^{(6)} \cup E_3^{(7)} \\
E_3^{(1)} &= \{(x, y) : x, y \in V_3^{(0)}\} \\
E_3^{(2)} &= \{(v_{i,0}^{(\alpha)}, v_{i,1}^{(\beta)}) : i \in \{1, 2, \cdots, n\},\ \alpha, \beta \in \{1, 2\}\} \\
E_3^{(3)} &= \{(u_{i,j}^{(\alpha)}, w_{i,j}^{(\beta)}) : (v_i, v_j) \in (V \times V) \backslash P,\ \alpha, \beta \in \{1, 2\}\} \\
E_3^{(4)} &= \{(v_{i,1}^{(1)}, v_{i,1}^{(2)}) : i \in \{1, 2, \cdots, n\}\} \\
E_3^{(5)} &= \{(v_{i,3}, v_{i,1}^{(1)}), (v_{i,3}, v_{i,1}^{(2)}) : i \in \{1, 2, \cdots, n\}\} \\
E_3^{(6)} &= \{(v_{i,3}, w_{i,j}^{(1)}), (v_{j,3}, w_{i,j}^{(2)}) : (v_i, v_j) \in (V \times V) \backslash P\} \\
E_3^{(7)} &= \{(v_{i,3}, v_{j,3}) : (v_i, v_j) \in E(G)\}
\end{aligned}$$

Denote $H_3 = G_3[\{v_{i,3} : i \in \{1, 2, \ldots, n\}\}]$. Let $P_3 = \{(v_{i,3}, v_{j,3}) : (v_i, v_j) \in P\}$. The graph G_3 satisfies the property that for all $(v_{i,3}, v_{j,3}) \in P_3$ there is no path of length ≤ 4 between $v_{i,3}$ and $v_{j,3}$ in $G_3 \backslash E(H_3)$ and also for all $(v_{i,3}, v_{j,3}) \notin P_3$ the length of the shortest path between $v_{i,3}$ and $v_{j,3}$ in $G_3 \setminus E(H_3)$ is 4.

Let $c : V \to \{1, 2, 3\}$ be a 3-vertex-coloring of G such that every pair of vertices in P is rainbow vertex-connected. Define the vertex-coloring c_3 of G_3 as follows:

- $c_3(v_{i,0}^{(1)}) = 1$ and $c_3(v_{i,0}^{(2)}) = 2$, for $i \in \{1, 2, \cdots, n\}$,

  $c_3(u_{i,j}^{(1)}) = 1$ and $c_3(u_{i,j}^{(2)}) = 2$, for $u_{i,j}^{(1)}, u_{i,j}^{(2)} \in V_3^{(0)}$.
- $c_3(v_{i,1}^{(1)}) = 2$ and $c_3(v_{i,1}^{(2)}) = 3$, for $i \in \{1, 2, \cdots, n\}$,

  $c_3(w_{i,j}^{(1)}) = 2$ and $c_3(w_{i,j}^{(2)}) = 3$, for $w_{i,j}^{(1)}, w_{i,j}^{(2)} \in V_3^{(1)}$.

- $c_3(v_{i,3}) = c(v_i)$, for $i \in \{1, 2, \cdots, n\}$.

It can be easily verified that $rvc(G_3) \leq 3$ if and only if G is 3-subset rainbow vertex-connected.

Inductive construction of G_k: Assuming that we have constructed $G_{k-2} = (V_{k-2}, E_{k-2})$, the graph $G_k = (V_k, E_k)$ is then constructed as follows: Each base vertex $v_{i,k-2}$ in V_{k-2} is split into the vertices $v^{(1)}_{i,k-2}, v^{(2)}_{i,k-2}$ and edges are added between them. Any edge of the form $(x, v_{i,k-2})$ is replaced by $(x, v^{(1)}_{i,k-2})$, $(x, v^{(2)}_{i,k-2})$. After doing this, we add the vertices $v_{i,k}$ and edges $(v_{i,k}, v^{(1)}_{i,k-2})$, $(v_{i,k}, v^{(2)}_{i,k-2})$ for $i \in \{1, 2, \cdots, n\}$. Formally the graph G_k is defined as follows:

When k is even: $V_k = \{u\} \cup V^{(0)}_k \cup V^{(2)}_k \cup \cdots \cup V^{(k)}_k$, where

$$\begin{aligned} V^{(i)}_k =& V^{(i)}_{k-2}, \quad for \quad i = 0, 2, \cdots, k-4; \\ V^{(k-2)}_k =& \{v^{(1)}_{i,k-2}, v^{(2)}_{i,k-2} : i \in \{1, 2, \cdots, n\}\}; \\ V^{(k)}_k =& \{v_{i,k} : i \in \{1, 2, \cdots, n\}\}. \end{aligned}$$

When k is odd: $V_k = V^{(0)}_k \cup V^{(1)}_k \cup V^{(3)}_k \cup \cdots \cup V^{(k)}_k$, where

$$\begin{aligned} V^{(i)}_k =& V^{(i)}_{k-2}, \quad for \quad i = 0, 1, 3, \cdots, k-4; \\ V^{(k-2)}_k =& \{v^{(1)}_{i,k-2}, v^{(2)}_{i,k-2} : i \in \{1, 2, \cdots, n\}\}; \\ V^{(k)}_k =& \{v_{i,k} : i \in \{1, 2, \cdots, n\}\}. \end{aligned}$$

For all $k \geq 4$, E_k is defined as follows:

$$\begin{aligned} E_k =& E_{k-2} \setminus (E_{k-2}(V^{(k-4)}_{k-2}, V^{(k-2)}_{k-2}) \cup E(H_{k-2})) \\ & \cup \{(v^{(\alpha)}_{i,k-2}, x) : (v_{i,k-2}, x) \in E_{k-2}(V^{(k-4)}_{k-2}, V^{(k-2)}_{k-2}), i \in \{1, 2, \cdots n\}, \alpha \in \{1, 2\}\} \\ & \cup \{(v^{(1)}_{i,k-2}, v^{(2)}_{i,k-2}) : i \in \{1, 2, \cdots, n\}\} \\ & \cup \{(v_{i,k}, v^{(\alpha)}_{i,k-2}) : i \in \{1, 2, \cdots, n\}, \alpha \in \{1, 2\}\} \cup E(H_k) \end{aligned}$$

where $E(H_l) = \{(v_{i,l}, v_{j,l}) : (v_i, v_j) \in E(G)\}$ and $E_{k-2}(V^{(k-4)}_{k-2}, V^{(k-2)}_{k-2}) = \{(u, v) : u \in V^{(k-4)}_{k-2}, v \in V^{(k-2)}_{k-2}\}$.

Let $P_k = \{(v_{i,k}, v_{j,k}) : (v_i, v_j) \in P\}$. Then we show that the graph G_k satisfies the following properties as claims:

Claim 1 *For any $(v_{i,k}, v_{j,k}) \in P_k$, there is no path of length less than $k+2$ between $v_{i,k}$ and $v_{j,k}$ in $G_k \setminus E(H_k)$.*

Proof. It has been shown that the assertion is true for G_2 and G_3. Assume that the assertion is true for G_{k-2}. Let $(v_i, v_j) \in P$, then $(v_{i,k-2}, v_{j,k-2}) \in P_{k-2}$, and hence by induction, there is no path of length less than k between $v_{i,k-2}$ and $v_{j,k-2}$ in $G_{k-2} \backslash E(H_{k-2})$. By the construction of G_k, we do not shorten the paths between any two vertices, so the paths from $v_{i,k-2}^{(\alpha)}$ to $v_{j,k-2}^{(\beta)}$ will still be of length at least k for $\alpha, \beta \in \{1, 2\}$. Consider the graph $G_k \setminus E(H_k)$. Since the neighbors of the vertex $v_{i,k}$ are only $v_{i,k}^{(1)}, v_{i,k}^{(2)}$, the path between $v_{i,k}$ and $v_{j,k}$ must be $v_{i,k} v_{i,k-2}^{(\alpha)} \dots v_{j,k-2}^{(\beta)} v_{j,k}$ for $\alpha = 1$ *or* $2, \beta = 1$ *or* 2, thus their lengths are at least $k+2$. ∎

Claim 2 *For any $(v_{i,k}, v_{j,k}) \notin P_k$, the shortest path between $v_{i,k}$ and $v_{j,k}$ is of length $k+1$ in $G_k \setminus E(H_k)$.*

Proof. It has been shown that the assertion is true for G_2 and G_3. Suppose that the assertion is true for G_{k-2}. Let $(v_i, v_j) \notin P$, then $(v_{i,k-2}, v_{j,k-2}) \notin P$, and hence by induction, the shortest path between $v_{i,k-2}$ and $v_{j,k-2}$ is of length $k-1$ in $G_{k-2} \setminus E(H_{k-2})$. By the construction of G_k, we do not shorten the paths between any two vertices, so the shortest path between $v_{i,k-2}^{(\alpha)}$ and $v_{j,k-2}^{(\beta)}$ will still be of length $k-1$ for $\alpha, \beta \in \{1, 2\}$. Consider the graph $G_k \setminus E(H_k)$. Since the neighbors of the vertex $v_{i,k}$ are only $v_{i,k}^{(1)}, v_{i,k}^{(2)}$, the shortest path between $v_{i,k}$ and $v_{i,k}$ must be $v_{i,k} v_{i,k-2}^{(\alpha)} \cdots v_{j,k-2}^{(\beta)} v_{j,k-2}$ for $\alpha = 1$ *or* $2, \beta = 1$ *or* 2, thus the length of the path is $k+1$. ∎

Claim 3 *G is k-subset rainbow vertex-connected if and only if G_k is k-rainbow vertex-connected.*

Proof. Denote $H_k = G_k[\{v_{i,k} : i \in \{1, 2, \cdots, n\}\}]$. It can be seen that H_k is isomorphic to G.

If G_k is k-rainbow vertex-connected, let $c_k : V(G_k) \to \{1, 2, \cdots, k\}$ be a vertex-coloring of G_k with k colors such that every pair of vertices in G_k is rainbow vertex-connected. We define the vertex-coloring c of G as follows: $c(v_i) = c_k(v_{i,k})$ for $i \in \{1, 2, \cdots, n\}$. If $(v_i, v_j) \in P$, then $(v_{i,k}, v_{j,k}) \in P_k$. By Claim 1, there is no path between $v_{i,k}$ and $v_{j,k}$ with length less than $k+2$ in $G_k \setminus E(H_k)$. Hence the entire rainbow vertex-connected path between $v_{i,k}$ and $v_{j,k}$ must lie in H_k itself. Correspondingly, there is a rainbow vertex-connected path between v_i and v_j in G. Thus, G is k-subset rainbow vertex-connected.

In the other direction, if G is k-subset rainbow vertex-connected, let $c : V(G) \to \{1, 2, \cdots, k\}$ be a vertex-coloring of G with k colors such that every pair of vertices in P is rainbow vertex-connected. We define the vertex-coloring c_k of G_k by induction. We have given the vertex-colorings c_2, c_3 of G_2, G_3. Assume that $c_{k-2} : V(G_{k-2}) \to \{1, 2, \cdots, k-2\}$ is a vertex-coloring of G_{k-2} such that G_{k-2} is rainbow vertex-connected. We define the vertex-coloring c_k of G_k as follows:

When k is even:

- $c_k(u) = k - 1$.

- $c_k(v) = c_{k-2}(v)$, for $v \in V_k^{(0)} \cup V_k^{(2)} \cup \cdots \cup V_k^{(k-4)}$.
- $c_k(v_{i,k-2}^{(1)}) = k-1, c_k(v_{i,k-2}^{(2)}) = k$, for $i \in \{1, 2, \cdots, n\}$.
- $c_k(v_{i,k}) = c(v_i)$, for $i \in \{1, 2, \cdots, n\}$.

When k is odd:

- $c_k(v_{i,0}^{(1)}) = c_{k-2}(v_{i,0}^{(1)}), c_k(v_{i,0}^{(2)}) = k-1$, for $i \in \{1, 2, \cdots, n\}$.
  $c_k(u_{i,j}^{(1)}) = c_{k-2}(u_{i,j}^{(1)}), c_k(u_{i,j}^{(2)}) = k-1$ for $u_{i,j}^{(1)}, u_{i,j}^{(2)} \in V_k^{(0)}$.
- $c_k(v) = c_{k-2}(v)$, for $v \in V_k^{(1)} \cup V_k^{(3)} \cup \cdots \cup V_k^{(k-4)}$.
- $c_k(v_{i,k-2}^{(1)}) = k-1, c_k(v_{i,k-2}^{(2)}) = k$, for $i \in \{1, 2, \cdots, n\}$.
- $c_k(v_{i,k}) = c(v_i)$, for $i \in \{1, 2, \cdots, n\}$.

Proposition 1 *The vertex-coloring c_k of G_k defined above makes G_k rainbow vertex-connected.*

Proof. Let $v, w \in V_k$, we now show that v, w are rainbow vertex-connected in G_k.

Case 1. k is even.

By the vertex-coloring c_k, we have $c_k(v_{i,j}^{(1)}) = j+1$, $c_k(v_{i,j}^{(2)}) = j+2$, $c_k(u) = k-1$ and $c_k(v_{i,k}) = c(v_i)$ for $i \in \{1, 2, \cdots, n\}, j \in \{0, 2, \cdots, k-2\}$.

Subcase 1.1. $v \in V_k^{(p)}$, $w \in V_k^{(q)}$, where $p, q \in \{0, 2, \cdots, k-2\}$.

If $v = v_{i,p}^{(\alpha)}$, $w = v_{j,q}^{(\beta)}$ for $\alpha, \beta \in \{1, 2\}$, then $vv_{i,p-2}^{(1)}v_{i,p-4}^{(1)} \cdots v_{i,0}^{(1)}uv_{j,0}^{(2)} \cdots v_{j,q-2}^{(2)}w$ is the rainbow vertex-connected path between v and w.

If $v = v_{i_1,p}^{(\alpha)}$, $w = w_{i,j}^{(\beta)}$ for $\alpha, \beta \in \{1, 2\}$, then $vv_{i_1,p-2}^{(1)}v_{i_1,p-4}^{(1)} \cdots v_{i_1,0}^{(1)}uw$ is the rainbow vertex-connected path between v and w.

If $v = w_{i_1,j_1}^{(\alpha)}$, $w = w_{i_2,j_2}^{(\beta)}$ for $\alpha, \beta \in \{1, 2\}$, then vuw is the rainbow vertex-connected path between v and w.

Subcase 1.2. $v = v_{i,k}$, $w \in V_k^{(q)}$, where $q \in \{0, 2, \cdots, k-2\}$.

If $w = v_{j,q}^{(\alpha)}$ for $\alpha \in \{1, 2\}$, then $vv_{i,k-2}^{(2)}v_{i,k-4}^{(1)} \cdots v_{i,0}^{(1)}uv_{j,0}^{(2)} \cdots v_{j,q-2}^{(2)}w$ is the rainbow vertex-connected path between v and w.

If $w = w_{i,j}^{(\alpha)}$ for $\alpha \in \{1, 2\}$, then $vv_{i,k-2}^{(2)}v_{i,k-4}^{(1)} \cdots v_{i,0}^{(1)}uw$ is the rainbow vertex-connected path between v and w.

Subcase 1.3. $v = v_{i,k}$, $w = v_{j,k}$.

If $(v_{i,k}, v_{j,k}) \in P_k$, then $(v_i, v_j) \in P$. By the vertex-coloring c of G, there is a rainbow vertex-connected path between v_i and v_j in G. Correspondingly, since $c_k(v_{i,k}) = c(v_i)$, there is a rainbow vertex-connected path between $v_{i,k}$ and $v_{j,k}$ in G_k.

If $(v_{i,k}, v_{j,k}) \notin P_k$, then $v_{i,k}v^{(1)}_{i,k-2}v^{(1)}_{i,k-4}\cdots v^{(1)}_{i,2}w^{(1)}_{i,j}w^{(2)}_{i,j}v^{(2)}_{j,2}\cdots v^{(2)}_{j,k-2}v_{j,k}$ is the rainbow vertex-connected path between $v_{i,k}$ and $v_{j,k}$.

Case 2. k is odd.

By the vertex-coloring c_k, we have

$c_k(v^{(1)}_{i,j}) = j + 1$, $c_k(v^{(2)}_{i,j}) = j + 2$, for $j \in \{1, 3, \cdots, k - 2\}$,

$c_k(v^{(1)}_{i,0}) = 1$, $c_k(v^{(2)}_{i,0}) = k - 1$, for $i \in \{1, 2, \cdots, n\}$,

$c_k(u^{(1)}_{i,j}) = 1$, $c_k(u^{(2)}_{i,j}) = k - 1$, for $u^{(1)}_{i,j}, u^{(2)}_{i,j} \in V^{(0)}_k$,

$c_k(w^{(1)}_{i,j}) = 2$, $c_k(w^{(2)}_{i,j}) = 3$, for $w^{(1)}_{i,j}, w^{(2)}_{i,j} \in V^{(1)}_k$,

$c_k(v_{i,k}) = c(v_i)$, for $i \in \{1, 2, \cdots, n\}$.

Subcase 2.1. $v \in V^{(p)}_k$, $w \in V^{(q)}_k$, where $p, q \in \{1, 3, \cdots, k - 2\}$.

If $v = v^{(\alpha)}_{i,p}$, $w = v^{(\beta)}_{j,q}$ for $\alpha, \beta \in \{1, 2\}$, then $vv^{(1)}_{i,p-2}v^{(1)}_{i,p-4}\cdots v^{(1)}_{i,0}v^{(2)}_{j,0}v^{(2)}_{j,1}\cdots v^{(2)}_{j,q-2}w$ is the rainbow vertex-connected path between v and w.

If $v = v^{(\alpha)}_{i,p}$, $w = w^{(\beta)}_{i,j}$ for $\alpha, \beta \in \{1, 2\}$, then $vv^{(1)}_{i,p-2}v^{(1)}_{i,p-4}\cdots v^{(1)}_{i,0}u^{(2)}_{i,j}w$ is the rainbow vertex-connected path between v and w.

If $v = w^{(\alpha)}_{i_1,j_1}$, $w = w^{(\beta)}_{i_2,j_2}$ for $\alpha, \beta \in \{1, 2\}$, then $vu^{(1)}_{i_1,j_1}u^{(2)}_{i_2,j_2}w$ is the rainbow vertex-connected path between v and w.

Subcase 2.2. $v = v_{i,k}$, $w \in V^{(q)}_k$, where $q \in \{1, 3, \cdots, k - 2\}$.

If $w = v^{(\alpha)}_{j,q}$ for $\alpha \in \{1, 2\}$, then $vv^{(2)}_{i,k-2}v^{(1)}_{i,k-4}\cdots v^{(1)}_{i,1}v^{(1)}_{i,0}v^{(2)}_{j,0}v^{(2)}_{j,2}\cdots v^{(2)}_{j,q-2}w$ is the rainbow vertex-connected path between v and w.

If $w = w^{(\alpha)}_{i,j}$ for $\alpha \in \{1, 2\}$, then $vv^{(2)}_{i,k-2}v^{(1)}_{i,k-4}\cdots v^{(1)}_{i,1}v^{(1)}_{i,0}u^{(2)}_{i,j}w$ is the rainbow vertex-connected path between v and w.

Subcase 2.3. $v = v_{i,k}$, $w = v_{j,k}$.

If $(v_{i,k}, v_{j,k}) \in P_k$, then $(v_i, v_j) \in P$. By the vertex-coloring c of G, there is a rainbow vertex-connected path between v_i and v_j in G. Correspondingly, since $c_k(v_{i,k}) = c(v_i)$, there is a rainbow vertex-connected path between $v_{i,k}$ and $v_{j,k}$ in G_k.

If $(v_{i,k}, v_{j,k}) \notin P_k$, then $v_{i,k}v^{(1)}_{i,k-2}v^{(1)}_{i,k-4}\cdots v^{(1)}_{i,3}w^{(1)}_{i,j}u^{(1)}_{i,j}w^{(2)}_{i,j}v^{(2)}_{j,3}\cdots v^{(2)}_{j,k-2}v_{j,k}$ is the rainbow vertex-connected path between $v_{i,k}$ and $v_{j,k}$. ∎

Proof of Theorem 2: From the above Lemmas 1 and 2, the first part of Theorem 2, the NP-Hardness, follows immediately.

In the following we will prove the second part of Theorem 2. Recall that a problem belongs to NP-class if given any instance of the problem whose answer is "yes", there is a certificate validating this fact which can be checked in polynomial time. For any fixed integer k, to prove the problem of deciding whether $rvc(G) \leq k$ is in NP-class, we can choose a rainbow k-vertex-coloring of G as a certificate. For checking a rainbow k-vertex-coloring, we only need to check that k colors are used and for any two vertices u and v of G, there exists a rainbow vertex-connected path between u and v. Notice that for any two vertices u and v of G, there are at most $n^{\ell-1}$ $u-v$ paths of length ℓ, since if we let $P = uv_1v_2\cdots v_{\ell-1}v$, then there are less than n choices for each v_i ($i \in \{1, 2, \ldots, \ell-1\}$). Therefore, G contains at most $\sum_{\ell=1}^{k+1} n^{\ell-1} = \frac{n^{k+1}-1}{n} \leq n^k$ $u-v$ paths of length at most $k+1$. Then, check these paths in turn until one finds one path whose internal vertices have distinct colors. It follows that the time used for checking is at most $O(n^k \cdot n^2 \cdot n^2) = O(n^{k+4})$. Since k is a fixed integer, we conclude that the certificate can be checked in polynomial time, which implies that the problem of deciding whether $rvc(G) \leq k$ belongs to NP-class, and therefore it is NP-Complete. ∎

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