

The generalized 3-connectivity of random graphs*

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Abstract

The generalized connectivity of a graph G was introduced by Chartrand et al. Let S be a nonempty set of vertices of G , and $\kappa(S)$ denote the largest number of internally disjoint trees connecting S in G . Then for an integer r with $2 \leq r \leq n$, the *generalized r -connectivity* $\kappa_r(G)$ of G is the minimum $\kappa(S)$ where S runs over all the r -subsets of the vertex set of G . Obviously, $\kappa_2(G) = \kappa(G)$, is the vertex connectivity of G , and hence the generalized connectivity is a natural generalization of the vertex connectivity. In this paper, we study the generalized 3-connectivity of random graphs and prove that for every fixed integer $k \geq 1$,

$$p = \frac{\log n + (k + 1) \log \log n - \log \log \log n}{n}$$

is a sharp threshold function for the property $\kappa_3(G(n, p)) \geq k$, which could be seen as a counterpart of Bollobás and Thomason's result for vertex connectivity.

Keywords: connectivity; internally disjoint trees; generalized connectivity; random graph; threshold function

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1 Introduction

All graphs considered here are finite, undirected, and have no loops or multiple edges. For standard graph-theoretic notation and terminology the reader is referred to [1]. In particular, denote by $e[S]$ the number of edges in the induced subgraph by a

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set S of vertices. The generalized connectivity of a graph G , introduced by Chartrand et al. in [3, 4], is a generalization of the concept of the vertex connectivity. Let G be a nontrivial connected graph of order n and r an integer with $2 \leq r \leq n$. For a set S of r vertices of G , a collection T_1, T_2, \dots, T_k of trees in G is said to be *internally disjoint* ones connecting S if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for any pair of distinct integers i and j with $1 \leq i, j \leq k$. Let $\kappa(S)$ denote the largest number of internally disjoint trees connecting S in G . The *generalized r -connectivity*, denoted by $\kappa_r(G)$, of G is then defined by $\kappa_r(G) = \min\{\kappa(S) \mid S \subseteq V(G) \text{ and } |S| = r\}$. Thus, $\kappa_2(G) = \kappa(G)$, where $\kappa(G)$ denotes the vertex connectivity of G . Set $\kappa_r(G) = 0$ when G is disconnected. There have been many results on the generalized connectivity, see [5, 6, 7, 8, 9, 10, 11]. For total survey, we refer to [12].

The generalized connectivity can be motivated by its interesting interpretation in practice. For example, suppose that G represents a network. If one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree for connecting a set of vertices is usually called a Steiner tree, and popularly used in the physical design of VLSI, see [14]. Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose.

In this paper, we study the generalized 3-connectivity of random graphs. The two most frequently occurring probability models of random graphs are $G(n, M)$ and $G(n, p)$. The first one consists of all graphs with n vertices having M edges, in which each graph have the same probability. The model $G(n, p)$ consists of all graphs with n vertices in which the edges are chosen independently and with a same probability p . We say that an event \mathcal{A} happens *almost surely* if its happening probability approaches 1 as $n \rightarrow \infty$, i.e., $Pr[\mathcal{A}] = 1 - o_n(1)$. Sometimes, it is addressed as *a.s.* for short. We will always assume that n is the variable that tends to infinity.

For a graph property P , a function $p(n)$ is called a *threshold function* of P if

- for every $r(n) = O(p(n))$, $G(n, r(n))$ almost surely satisfies P ; and
- for every $r'(n) = o(p(n))$, $G(n, r'(n))$ almost surely does not satisfy P .

Furthermore, $p(n)$ is called a *sharp threshold function* of P if there exist two positive constants c and C such that

- for every $r(n) \geq C \cdot p(n)$, $G(n, r(n))$ almost surely satisfies P ; and
- for every $r'(n) \leq c \cdot p(n)$, $G(n, r'(n))$ almost surely does not satisfy P .

As well-known, for the vertex connectivity Bollobás and Thomason obtained the following result.

Theorem 1 [2] *If $k \in \mathbb{N}$ and $y \in \mathbb{R}$ are fixed, and $M = \frac{n}{2}(\log n + k \log \log n + y + o(1)) \in \mathbb{N}$, then*

$$\Pr [\kappa(G(n, M)) = k] \rightarrow 1 - e^{-e^{-y/k!}}$$

and

$$\Pr [\kappa(G(n, M)) = k + 1] \rightarrow e^{-e^{-y/k!}}.$$

Our main result for the generalized 3-connectivity of random graphs is as follows, which could be seen as a counterpart of Theorem 1.

Theorem 2 *Let $k \geq 1$ be a fixed integer. Then*

$$p = \frac{\log n + (k + 1) \log \log n - \log \log \log n}{n}$$

is a sharp threshold function for the property $\kappa_3(G(n, p)) \geq k$.

2 Main results

Throughout the paper \log always denotes the natural logarithm, and we assume that $k \geq 1$ is a fixed integer. To establish a sharp threshold function for a graph property the proof should be two-fold. We first show one easy direction. The following result is given by Li et al. in [11], which will be used later.

Lemma 1 [11] *For any connected graph G , $\kappa_3(G) \leq \kappa(G)$. Moreover, the upper bound is sharp.*

We first prove the following result.

Theorem 3 $\kappa_3(G(n, \frac{1}{2} \frac{\log n + (k+1) \log \log n - \log \log \log n}{n})) \leq k - 1$ almost surely holds.

We need the following lemma. We call a property Q *convex* if $F \subset G \subset H$ and F satisfies Q , then H satisfies Q imply that G satisfies Q , where F, G, H are some graphs. Set $N = \frac{1}{2}n(n - 1)$.

Lemma 2 [1] *If Q is a convex property and $p(1 - p)N \rightarrow \infty$, then $G(n, p)$ almost surely satisfies Q if and only if for every fixed x , $G(n, M)$ almost surely satisfies Q , where $M = \lfloor pN + x(p(1 - p)N)^{1/2} \rfloor$.*

Proof of Theorem 3: Let $p = \frac{\log n + (k+1) \log \log n - \log \log \log n}{n}$ and $M' = \lfloor \frac{1}{2}pN + x\{\frac{1}{2}p(1 - \frac{1}{2}p)N\}^{1/2} \rfloor$ for any $x \in \mathbb{R}$, i.e., $M' = \frac{n}{4}(\log n + (k+1) \log \log n - \log \log \log n + o(1))$. It is easy to check that $\frac{1}{2}p(1 - \frac{1}{2}p)N \rightarrow \infty$.

Let $M_1 = \frac{n}{2}(\log n + (k-1) \log \log n + y + o(1)) \in \mathbb{N}$. By Theorem 1, we have

$$\Pr[\kappa(G(n, M_1)) = k-1] \rightarrow 1 - e^{-e^{-y/(k-1)!}}.$$

Hence, for any $\varepsilon > 0$, there exists an $N' \in \mathbb{N}$ and a $Y \in \mathbb{R}^+$, such that for any $y < -Y$,

$$1 - e^{-e^{-y/(k-1)!}} - \Pr[\kappa(G(n, M_1)) = k-1] < \frac{\varepsilon}{2} \quad \text{and} \quad e^{-e^{-y/(k-1)!}} < \frac{\varepsilon}{2}.$$

On the other hand, there exists an integer $N_1 \in \mathbb{N}$, such that for any $n > N_1$, $M' < M_1$. We have

$$\begin{aligned} \Pr[\kappa(G(n, M')) \leq k-1] &= \sum_{i=0}^{\infty} \Pr[\kappa(G(n, M')) \leq k-1 | \kappa(G(n, M_1)) = i] \\ &\quad \cdot \Pr[\kappa(G(n, M_1)) = i] \\ &\geq \Pr[\kappa(G(n, M')) \leq k-1 | \kappa(G(n, M_1)) = k-1] \\ &\quad \cdot \Pr[\kappa(G(n, M_1)) = k-1] \\ &= \Pr[\kappa(G(n, M')) \leq k-1, \kappa(G(n, M_1)) = k-1] \\ &= \Pr[\kappa(G(n, M_1)) = k-1] \end{aligned}$$

Hence, we have for any $n > \max\{N', N_1\}$,

$$\begin{aligned} 1 - \Pr[\kappa(G(n, M')) \leq k-1] &\leq 1 - \Pr[\kappa(G(n, M_1)) = k-1] \\ &< e^{-e^{-y/(k-1)!}} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Thus, $\kappa(G(n, M')) \leq k-1$ almost surely holds. Obviously, the property that the connectivity of a given graph is at most $k-1$, is a convex property. By Lemmas 1 and 2, $\kappa_3(G(n, \frac{1}{2} \frac{\log n + (k+1) \log \log n - \log \log \log n}{n})) \leq k-1$ almost surely holds. \blacksquare

We leave with the other direction stated below.

Theorem 4 $\kappa_3(G(n, \frac{\log n + (k+1) \log \log n - \log \log \log n}{n})) \geq k$ almost surely holds.

From now on, p is always $\frac{\log n + (k+1) \log \log n - \log \log \log n}{n}$.

Lemma 3 $k \leq \kappa(G(n, p)) \leq k+1$ almost surely holds.

Proof. We prove this lemma by a similar method used in the proof of Theorem 3. Let $M = \lfloor pN + x\{p(1-p)N\}^{1/2} \rfloor$ for any $x \in \mathbb{R}$, i.e., $M = \frac{n}{2}(\log n + (k+1)\log \log n - \log \log \log n + o(1))$. It is easy to check that $p(1-p)N \rightarrow \infty$. Let $M_0 = \frac{n}{2}(\log n + (k+1)\log \log n + y + o(1)) \in \mathbb{N}$, $M_2 = \frac{n}{2}(\log n + k\log \log n + y + o(1)) \in \mathbb{N}$. By Theorem 1,

$$\Pr[\kappa(G(n, M_2)) = k] \rightarrow 1 - e^{-e^{-y/k!}}.$$

Hence, for any $\varepsilon > 0$, there exists an $N'' \in \mathbb{N}$ and a $Y \in \mathbb{R}^+$, such that for any $y < -Y$,

$$1 - e^{-e^{-y/k!}} - \Pr[\kappa(G(n, M_2)) = k] < \frac{\varepsilon}{2} \quad \text{and} \quad e^{-e^{-y/k!}} < \frac{\varepsilon}{2}.$$

On the other hand, there exists an integer $N_2 \in \mathbb{N}$, such that for any $n > N_2$, $M > M_2$. We have

$$\begin{aligned} \Pr[\kappa(G(n, M)) \geq k] &= \sum_{i=0}^{\infty} \Pr[\kappa(G(n, M)) \geq k | \kappa(G(n, M_2)) = i] \\ &\quad \cdot \Pr[\kappa(G(n, M_2)) = i] \\ &\geq \Pr[\kappa(G(n, M)) \geq k | \kappa(G(n, M_2)) = k] \\ &\quad \cdot \Pr[\kappa(G(n, M_2)) = k] \\ &= \Pr[\kappa(G(n, M)) \geq k, \kappa(G(n, M_2)) = k] \\ &= \Pr[\kappa(G(n, M_2)) = k] \end{aligned}$$

Hence, we get for any $n > \max\{N'', N_2\}$,

$$\begin{aligned} 1 - \Pr[\kappa(G(n, M)) \geq k] &\leq 1 - \Pr[\kappa(G(n, M_2)) = k] \\ &< e^{-e^{-y/k!}} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Thus, $\kappa(G(n, M)) \geq k$ almost surely holds. By Lemma 2, $\kappa(G(n, p)) \geq k$ almost surely holds.

Similarly, we can prove that $\kappa(G(n, p)) \leq k+1$. By Theorem 1,

$$\Pr[\kappa(G(n, M_0)) = k+1] \rightarrow 1 - e^{-e^{-y/(k+1)!}}.$$

Hence, for any $\varepsilon > 0$, there exists an $N^* \in \mathbb{N}$ and a $Y \in \mathbb{R}^+$, such that for any $y < -Y$,

$$1 - e^{-e^{-y/(k+1)!}} - \Pr[\kappa(G(n, M_0)) = k+1] < \frac{\varepsilon}{2} \quad \text{and} \quad e^{-e^{-y/(k+1)!}} < \frac{\varepsilon}{2}.$$

On the other hand, there exists an integer $N_3 \in \mathbb{N}$, such that for any $n > N_3$, $-\log \log \log n < -Y$. Namely, $M < M_0$, and then

$$\begin{aligned}
\Pr[\kappa(G(n, M)) \leq k+1] &= \sum_{i=0}^{\infty} \Pr[\kappa(G(n, M)) \leq k+1 | \kappa(G(n, M_0)) = i] \\
&\quad \cdot \Pr[\kappa(G(n, M_0)) = i] \\
&\geq \Pr[\kappa(G(n, M)) \leq k+1 | \kappa(G(n, M_0)) = k+1] \\
&\quad \cdot \Pr[\kappa(G(n, M_0)) = k+1] \\
&= \Pr[\kappa(G(n, M)) \leq k+1, \kappa(G(n, M_0)) = k+1] \\
&= \Pr[\kappa(G(n, M_0)) = k+1].
\end{aligned}$$

Hence, we have for any $n > \max\{N^*, N_3\}$,

$$\begin{aligned}
1 - \Pr[\kappa(G(n, M)) \leq k+1] &\leq 1 - \Pr[\kappa(G(n, M_0)) = k+1] \\
&< e^{-e^{-x/(k+1)!}} + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Thus, $\kappa(G(n, M)) \leq k+1$ almost surely holds. By Lemma 2, $\kappa(G(n, p)) \leq k+1$ almost surely holds. \blacksquare

From Lemma 3, we know that the minimum degree of $G(n, p)$ is a.s. at least k .

Let $G = G(n, p)$ and $D = \frac{\log n}{\log \log n}$. Almost surely, the diameter of G is asymptotically equal to D , see for example [1]. We call a vertex v *large* if it is of degree $d_G(v) \geq \frac{\log n}{100}$, and *small* otherwise.

Lemma 4 *Almost surely, there does not exist two small vertices within distance at most $\frac{3}{4}D$ in G .*

Proof. Denote by $\text{dist}_G(x, y)$ the distance between x and y in G . We have

$$\begin{aligned}
&\Pr\left[\exists x, y \in V(G) : d_G(x), d_G(y) < \frac{\log n}{100}, \text{dist}_G(x, y) \leq \frac{3D}{4}\right] \\
&\leq \binom{n}{2} \sum_{j=1}^{\frac{3D}{4}} \binom{n-2}{j-1} p^j \left(\sum_{i=0}^{\frac{\log n}{100}} \binom{n-(j+1)}{i} p^i (1-p)^{n-(i+1)} \right)^2 \\
&\leq \sum_{j=1}^{\frac{3D}{4}} n(2 \log n)^j \left\{ \sum_{i=0}^{\frac{\log n}{100}} \binom{n-1}{\frac{\log n}{100}} p^{\frac{\log n}{100}} (1-p)^{n-1-\frac{\log n}{100}} \right\}^2 \\
&\leq \sum_{j=1}^{\frac{3D}{4}} n(2 \log n)^j \left\{ \frac{\log n}{100} \left(\frac{ne}{\log n} \right)^{\frac{\log n}{100}} \left(\frac{\log n}{100} \right)^{\frac{\log n}{100}} e^{-\frac{\log n}{n} \left(n-1-\frac{\log n}{100} \right)} \right\}^2
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^{\frac{3D}{4}} n(2 \log n)^j \left(\frac{\log n}{100} (100e^{1+o(1)})^{\frac{\log n}{100}} \cdot \frac{1}{n} \right)^2 \\
&\leq \frac{3 \log n}{4 \log \log n} (2 \log n)^{\frac{3 \log n}{4 \log \log n}} \left(\frac{\log n}{100} \right)^2 \frac{1}{n} n^{\frac{8}{50}} \leq n^{-\frac{9}{100}}.
\end{aligned}$$

The proof is thus complete. ■

Lemma 5 *For a fixed $t \in \mathbb{N}$ and $0 < \alpha < 1$, almost surely, there does not exist a subset $S \subset V(G)$, such that $|S| \leq \alpha t D$ and $e[S] \geq |S| + t$.*

Proof. For convenience, let $s = |S|$. Then we have

$$\begin{aligned}
&\Pr[\exists S : s \leq \alpha t D, e[S] \geq s + t] \\
&\leq \sum_{s \leq \alpha t D} \binom{n}{s} \binom{\binom{s}{2}}{s+t} p^{s+t} \\
&\leq \sum_{s \leq \alpha t D} \left(\frac{ne}{s} \right)^s \left(\frac{\frac{1}{2}s(s-1)e}{s+t} \right)^{s+t} p^{s+t} \\
&\leq \sum_{s \leq \alpha t D} \left(\frac{e^2 s}{2(s+t)} \log n \right)^s \left(\frac{se \log n}{n} \right)^t \\
&\leq \alpha t \frac{\log n}{\log \log n} (e^{2+o(1)} \log n)^{\alpha t \frac{\log n}{\log \log n}} \left(\frac{e \alpha t \frac{\log^2 n}{\log \log n}}{n} \right)^t \\
&< \frac{1}{n^{(1-\alpha-o(1))t}}.
\end{aligned}$$

The proof is thus complete. ■

Remark 1 *Let T be a rooted tree of depth at most $3D/4$ and let v be a vertex not in T , but with b neighbors in T . Let S consist of v , the neighbors of v in T and the ancestors of these neighbors. Then $|S| \leq 3bD/4 + 1 + 1 \leq 4bD/5$ and $e[S] = |S| + b - 2$. It follows from the proof of Lemma 5 with $\alpha = 4/5$, $t = 16$, that we must have $b \leq 18$, with probability $1 - o(n^{-(1/5-o(1))t}) \geq 1 - o(n^{-3})$.*

Remark 2 *Let \mathcal{P} be a set of at most k vertex disjoint paths and trees, each containing at most $5D/2$ edges, and let v be a vertex not in \mathcal{P} , but with c neighbors in \mathcal{P} . Let $S = \{v\} \cup V(\mathcal{P})$, $|S| \leq 5kD/2 + k + 1 \leq 11kD/4$ and $e[S] = |S| + c - k - 1$. By Lemma 5 with $\alpha = 1/4$, $t = 11k$, we deduce that with probability at least $1 - o(n^{-3})$, $c \leq 12k + 1$.*

We first deal with large vertices. The following lemma points out that for every pair of large vertices in $V(G)$, there exists a special subgraph containing them, which can be used to find trees connecting given vertices. Recall that a t -ary tree with a designated root is a tree whose non-leaf vertices all have exactly t children.

Lemma 6 *Let $\varepsilon = \varepsilon(n) = \frac{1}{\log \log n}$. Then, almost surely, for any pair of large vertices u and $v \in V(G)$, there exists a subgraph $G_{u,v}$ of G that consists of two vertex disjoint $\frac{\log n}{101}$ -ary trees T_u and T_v rooted at u and v , respectively, each having depth $(\frac{3}{4} - \varepsilon)D$.*

Proof. We will show that for any pair of large vertices u and v , the two trees described in Lemma 6 exist with probability $1 - o(n^{-3})$.

Firstly, we grow a tree from u using BFS until it reaches depth $(\frac{3}{4} - \varepsilon)D$. Then we grow a tree starting from v again using BFS until it reaches depth $(\frac{3}{4} - \varepsilon)D$.

We use the notation $S_i^{(x)}$ for the number of vertices at depth i of the BFS tree rooted at x .

As growing T_u , when we grow the tree from a vertex x at depth i to depth $i + 1$, there may exist some *bad edges* which connect x to vertices already in T_u .

Remark 1 implies that with probability $1 - o(n^{-3})$, there exist at most 18 bad edges from x .

For small vertices, from Lemma 4 we can easily get that in the first $3D/8$ levels, there exists at most one small vertex at each level a.s.. Furthermore, once a small vertex appears in the BFS tree, there will be no small vertex in the subtree rooted at that small vertex. Though there may be more than one small vertex in depth $3D/8 + 1$, the number of them will not exceed the number of branches at root u , since one branch contains at most one small vertex in depth $3D/8 + 1$, a.s.. Then in depth $3D/8 + 2$, the number of small vertices of that level will be no more than the number of vertices in the depth $3D/8 + 1$ contained in the branches which have no small vertex in the previous levels. For the remaining levels of that BFS tree, we can conclude the similar result. And note that there will exist no small vertex in the following levels of branches which contain small vertices in depth at least $3D/8 + 1$ of that BFS tree, a.s.. Hence the number of small vertices contained in each level is much smaller than the increase of the number of vertices in each level. Denote by $t_i^{(u)}$ the number of small vertices of depth i . Thus we get the following recursion:

$$S_{i+1}^{(u)} \geq \left(\frac{\log n}{100} - 18 \right) \left(S_i^{(u)} - t_i^{(u)} \right) \geq \frac{\log n}{101} S_i^{(u)}$$

We call the operation of deleting some vertices from a tree as *prune a tree*. It is clear that we can make the current BFS tree a $\frac{\log n}{101}$ -ary tree by pruning.

Then we grow T_v , similarly. The only difference is that now we also say that an edge is *bad* if the other endpoint is in T_u .

Hence,

$$S_{i+1}^{(v)} \geq \left(\frac{\log n}{100} - 36 \right) \left(S_i^{(v)} - t_i^{(v)} \right) \geq \frac{\log n}{101} S_i^{(v)}$$

After pruning, we can obtain the required subgraph $G_{u,v}$. \blacksquare

Proof of Theorem 4: In order to prove Theorem 4, we will show that for any three vertices, we can find at least k internally disjoint trees connecting them in G .

Given three vertices u, v and w , we first assume that they are all large vertices. With the aid of Lemma 6, construct two vertex disjoint $\frac{\log n}{101}$ -ary trees T_u and T_v rooted at u and v , respectively, each having depth $(\frac{3}{4} - \varepsilon)D$.

For every tree T , denote the set of leaves of T by $L(T)$. Let $u_1, \dots, u_{\frac{\log n}{101}}$ ($v_1, \dots, v_{\frac{\log n}{101}}$) be the vertices in the first depth of T_u (T_v respectively). For each u_i (v_i), denote by T_{u_i} (T_{v_i}) the subtree of T_u (T_v) of depth $(\frac{3}{4} - \varepsilon)D - 1$ rooted at u_i (v_i), $i = 1, \dots, \frac{\log n}{101}$. Call these T_{u_i} (T_{v_i}) *vice trees*.

For a fixed T_{u_i} , let the random variable A_i denote the number of edges between $L(T_{u_i})$ and $L(T_v)$. Then A_i follows the binomial distribution, i.e., $A_i \sim \text{Bin} \left(\left(\frac{\log n}{101} \right)^{(\frac{3}{4} - \varepsilon)D - 1} \cdot \left(\frac{\log n}{101} \right)^{(\frac{3}{4} - \varepsilon)D}, p \right)$. The expectation value of A_i

$$\mathbb{E}[A_i] = p \left(\frac{\log n}{101} \right)^{2(\frac{3}{4} - \varepsilon)D - 1} \geq \frac{101}{n} \left(\frac{\log n}{101} \right)^{2(\frac{3}{4} - \varepsilon)D} \geq 101n^{\frac{1}{2} - 2\varepsilon - \frac{9.2(\frac{3}{4} - \varepsilon)}{\log \log n}}.$$

By Chernoff Bounds [13],

$$\begin{aligned} \Pr \left[A_i < \frac{100}{101} \mathbb{E}[A_i] \right] &\leq e^{-\frac{1}{2} \times \frac{1}{101^2} \mathbb{E}[A_i]} \leq e^{-\frac{1}{2} \times \frac{1}{101^2} \times 101n^{\frac{1}{2} - 2\varepsilon - \frac{9.2(\frac{3}{4} - \varepsilon)}{\log \log n}}} \\ &= e^{-\log n \left(\frac{1}{2} \times \frac{1}{101} \frac{1}{\log n} n^{\frac{1}{2} - 2\varepsilon - \frac{9.2(\frac{3}{4} - \varepsilon)}{\log \log n}} \right)} \\ &= n^{-\frac{1}{2} \times \frac{1}{101} \frac{1}{\log n} n^{\frac{1}{2} - 2\varepsilon - \frac{9.2(\frac{3}{4} - \varepsilon)}{\log \log n}}} \leq n^{-n^{\frac{1}{2} - o(1)}}. \end{aligned}$$

Now, for a fixed T_{v_j} , let A_{ij} denote the number of edges between $L(T_{u_i})$ and $L(T_{v_j})$. Then $A_{ij} \sim \text{Bin} \left(\left(\frac{\log n}{101} \right)^{(\frac{3}{4} - \varepsilon)D - 1} \cdot \left(\frac{\log n}{101} \right)^{(\frac{3}{4} - \varepsilon)D - 1}, p \right)$. We have

$$\begin{aligned} \mathbb{E}[A_{ij}] &= \left(\frac{\log n}{101} \right)^{2(\frac{3}{4} - \varepsilon)D - 2} \cdot p \leq \frac{101^2}{\log^2 n} \left(\frac{\log n}{101} \right)^{2(\frac{3}{4} - \varepsilon)D} \frac{2 \log n}{n} \\ &= \frac{101^2 \times 2}{\log n} \cdot n^{\frac{1}{2} - 2\varepsilon - \frac{9.2(\frac{3}{4} - \varepsilon)}{\log \log n}}. \end{aligned}$$

Also, we can deduce that $\mathbb{E}[A_{ij}] \geq \frac{101^2}{\log n} \cdot n^{\frac{1}{2}-2\varepsilon-\frac{9.2(\frac{3}{4}-\varepsilon)}{\log \log n}}$. By applying Chernoff Bounds,

$$\begin{aligned}
\Pr[A_{ij} > 8\mathbb{E}[A_{ij}]] &\leq \frac{e^{7\mathbb{E}[A_{ij}]}}{8^8\mathbb{E}[A_{ij]}^8} \leq \frac{e^{7\frac{101^2 \times 2}{\log n} \cdot n^{\frac{1}{2}-2\varepsilon-\frac{9.2(\frac{3}{4}-\varepsilon)}{\log \log n}}}}{8^8 \frac{101^2}{\log n} \cdot n^{\frac{1}{2}-2\varepsilon-\frac{9.2(\frac{3}{4}-\varepsilon)}{\log \log n}}} \\
&= \frac{e^{\frac{101^2 \times 14}{\log n} \cdot n^{\frac{1}{2}-2\varepsilon-\frac{9.2(\frac{3}{4}-\varepsilon)}{\log \log n}}}}{e^{\frac{101^2 \times 8 \times \log 8}{\log n} \cdot n^{\frac{1}{2}-2\varepsilon-\frac{9.2(\frac{3}{4}-\varepsilon)}{\log \log n}}}} \\
&= e^{(14-8 \log 8) \frac{101^2}{\log n} \cdot n^{\frac{1}{2}-2\varepsilon-\frac{9.2(\frac{3}{4}-\varepsilon)}{\log \log n}}} \\
&= n^{-\frac{2.64 \times 101^2}{\log^2 n} \cdot n^{\frac{1}{2}-2\varepsilon-\frac{9.2(\frac{3}{4}-\varepsilon)}{\log \log n}}} \leq n^{-n^{\frac{1}{2}-o(1)}}.
\end{aligned}$$

By the Union Bound, with probability at least $1 - \frac{\log n}{101} o(n^{-n^{\frac{1}{2}-o(1)}}) \geq 1 - o(n^{-n^{2/5}})$, we have that for every T_{v_j} , the number of edges between $L(T_{u_i})$ and $L(T_{v_j})$ is at most $8\mathbb{E}[A_{ij}] = 8p \left(\frac{\log n}{101}\right)^{2(\frac{3}{4}-\varepsilon)D-2}$.

Therefore, with probability at least $1 - o(n^{-n^{\frac{1}{2}-o(1)}}) - o(n^{-n^{2/5}}) = 1 - o(n^{-n^{2/5}})$ there are at least $\frac{\frac{100}{101} \left(\frac{\log n}{101}\right)^{2(\frac{3}{4}-\varepsilon)D-1} p}{8 \left(\frac{\log n}{101}\right)^{2(\frac{3}{4}-\varepsilon)D-2} p} = \frac{100}{101^2 \times 8} \log n$ vice trees T_{v_j} , such that vertices in $L(T_{u_i})$ and $L(T_{v_j})$ can be connected with edges. Moreover, using the Union Bounds, with probability at least $1 - \frac{\log n}{101} o(n^{-n^{2/5}}) \geq 1 - o(n^{-n^{1/5}})$, each vice tree of T_u can be connected to $\frac{100}{101^2 \times 8} \log n$ vice trees of T_v with edges. Hence there are at least $\frac{100}{101^2 \times 8} \log n$ pairs $\{T_{u_i}, T_{v_j}\}$ such that vertices of $L(T_{u_i})$ and $L(T_{v_j})$ can be connected by edges.

For convenience, let $a \log n = \frac{100}{101^2 \times 8} \log n$. Without loss of generality, assume these $a \log n$ pairs be T_{u_ℓ} and T_{v_ℓ} , $\ell = 1, 2, \dots, a \log n$. Now we show that, for the remaining large vertex w , we can find at least k internally disjoint trees connecting u , v and w .

Note that we can assume that w is not in T_u and T_v , since otherwise we can prune the tree by deleting the subtree rooted at w (just like the way to deal with small vertices), and we can still obtain $\frac{\log n}{101}$ -ary trees rooted at u and v , respectively.

With the similar argument in the proof of Lemma 6, we can construct a $\frac{\log n}{101}$ -ary tree T_w of depth $(\frac{1}{4} + 2\varepsilon)D$ rooted at w , and T_u, T_v, T_w are vertex disjoint. Note that at this time the number of small vertices in each level is at most one.

Let $w_1, \dots, w_{\frac{\log n}{101}}$ be the vertices of the first depth of T_w . Let $Q_i = T_{u_i} \cup T_{v_i}$, $i = 1, 2, \dots, a \log n$. Then $|Q_i| > 2 \left(\frac{\log n}{101}\right)^{\left(\frac{3}{4}-\varepsilon\right)D-1}$. For any fixed Q_j , let q_j denote the

probability that there exists at least one edge between T_{w_j} and Q_j . Then

$$\begin{aligned}
q_j &= 1 - \Pr[\text{there is no edge between } T_{w_j} \text{ and } Q_j] \\
&= 1 - (1 - p)^{|Q_j| \cdot |T_{w_j}|} \\
&\geq 1 - (1 - p)^{2 \left(\frac{\log n}{101}\right)^{\left(\frac{3}{4} - \varepsilon\right)D-1} \cdot \left(\frac{\log n}{101}\right)^{\left(\frac{1}{4} + 2\varepsilon\right)D-1}} \\
&\geq 1 - e^{-2 \frac{\log n}{n} \left(\frac{\log n}{101}\right)^{(1+\varepsilon)D-2}} \\
&= 1 - e^{-2 \log n \cdot \frac{101^2}{\log^2 n} \cdot n^{\varepsilon - \frac{4.6(1+\varepsilon)}{\log \log n}}} \\
&= 1 - e^{-2 \frac{101^2}{\log n} \cdot n^{\varepsilon - \frac{4.6(1+\varepsilon)}{\log \log n}}}.
\end{aligned}$$

Since for any $i \neq j$, $q_i = q_j$, let $q = q_i = q_j$, and let \mathcal{A} be the event that there are at most $k - 1$ pairs of $\{T_{w_\ell}, Q_\ell\}$, such that there exist edges between T_{w_ℓ} and Q_ℓ , where $\ell = 1, \dots, a \log n$.

Consider the upper bound of the probability that \mathcal{A} happens, we can deduce that

$$\begin{aligned}
\Pr[\mathcal{A}] &\leq \sum_{i=0}^{k-1} \binom{a \log n}{i} q^i (1 - q)^{a \log n - i} \\
&\leq k \left(\frac{a \log n \cdot e}{k - 1}\right)^{k-1} (1 - q)^{\frac{a}{2} \log n} \\
&< k \left(\frac{a \log n \cdot e}{k - 1}\right)^{k-1} e^{-2 \frac{101^2}{\log n} \cdot n^{\varepsilon - \frac{4.6(1+\varepsilon)}{\log \log n}} \cdot \frac{a}{2} \log n} \\
&= k \left(\frac{a \log n \cdot e}{k - 1}\right)^{k-1} n^{-\frac{100}{8} \frac{\varepsilon - \frac{4.6(1+\varepsilon)}{\log \log n}}{\log n}} < n^{-10}.
\end{aligned}$$

This indicates there are at least k pairs of $\{T_{w_\ell}, Q_\ell\}$, such that T_{w_ℓ} and Q_ℓ can be connected by edges, a.s..

Without loss of generality, assume that there exists edges connecting T_{w_ℓ} and Q_ℓ , where $\ell = 1, \dots, k$. Now we will construct k internally disjoint trees connecting u , v and w . For each i with $i = 1, \dots, k$, suppose that $w' \in V(T_{w_i})$ is adjacent to $x \in V(T_{u_i})$, edge yz connects $L(T_{u_i})$ and $L(T_{v_i})$, where $y \in L(T_{u_i})$, $z \in L(T_{v_i})$. Let $P_{T_{u_i}}(x, y)$ denote the path connecting x and y in T_{u_i} .

- If u_i is not in $P_{T_{u_i}}(x, y)$, we construct the tree $\{uu_i\} \cup P_{T_{u_i}}(u_i, x) \cup P_{T_{u_i}}(x, y) \cup \{yz\} \cup P_{T_{v_i}}(z, v_i) \cup \{v_i v\} \cup P_{T_{w_i}}(w', w_i) \cup \{w_i w\}$.

- If u_i is contained in $P_{T_{u_i}}(x, y)$, we construct the tree $\{uu_i\} \cup P_{T_{u_i}}(x, y) \cup \{yz\} \cup P_{T_{v_i}}(z, v_i) \cup \{v_i v\} \cup P_{T_{w_i}}(w', w_i) \cup \{w_i w\}$.

For the case that $x \in V(T_{v_i})$, the tree connecting u , v and w can be constructed similarly.

Thus we construct k trees connecting large vertices u , v and w , and it is easy to get that all these trees are internally disjoint.

Now we deal with small vertices. From the previous argument, if the three given vertices are all large, we can find at least k internally disjoint trees connecting them. So we assume that there are at least one small vertex of the given vertices u , v and w . By Lemma 4, it is easy to obtain the following three facts.

1. The neighbors of a small vertex are large vertices, a.s..
2. A large vertex can have at most one neighbor that is small, a.s..
3. Any two small vertices have no common neighbors, a.s..

Combining the three facts above, we can take k large neighbors of u , v , w , denoted by u_1, \dots, u_k , v_1, \dots, v_k and w_1, \dots, w_k , respectively, and all these $3k$ vertices are different.

Firstly, using the method described before, we can find a tree T_1^* connecting u_1 , v_1 and w_1 . Note that the number of edges in T_1^* is at most $3\left(\frac{3}{4} - \varepsilon\right)D + \left(\frac{1}{4} + 2\varepsilon\right)D = \left(\frac{5}{2} - \varepsilon\right)D$.

Then we find a tree T_2^* connecting u_2 , v_2 and w_2 . In order to make T_1^* and T_2^* internally disjoint, when we construct BFS tree rooted at u_2 , v_2 and w_2 , we treat the edges with one endpoint in $V(T_1^*)$ as bad edges, too. By Remark 2 and the similar argument as we deal with BFS tree rooted at a large vertex, we can find a tree T_2^* connecting u_2 , v_2 and w_2 .

Continue that process, until we find trees T_j^* connecting u_j , v_j and w_j for all $j = 1, \dots, k$.

Let $T_i = T_i^* \cup \{uu_i, vv_i, ww_i\}$ for $i = 1, \dots, k$. Apparently, these are k internally disjoint connecting u , v and w . Thus, for any fixed three vertices u , v , w , with probability at least $1 - o(n^{-3}) - o(n^{-n^{1/5}}) - o(n^{-10}) = 1 - o(n^{-3})$, we can find k vertex disjoint connecting them.

Consequently, for all possible three vertices u , v , w , by the Union Bound, we can find k internally disjoint connecting them with probability at least $1 - n^3 \cdot o(n^{-3}) = 1 - o(1)$. ■

Proof of Theorem 2: It follows immediately from Theorems 3 and 4. ■

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