Solutions to conjectures on the (k, ℓ)-rainbow index of complete graphs*

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Abstract

The (k, ℓ)-rainbow index $rx_{k,\ell}(G)$ of a graph G was introduced by Chartrand et al. For the complete graph K_n of order $n \geq 6$, they showed that $rx_{3,\ell}(K_n) = 3$ for $\ell = 1, 2$. Furthermore, they conjectured that for every positive integer ℓ, there exists a positive integer N such that $rx_{3,\ell}(K_n) = 3$ for every integer $n \geq N$. More generally, they conjectured that for every pair of positive integers k and ℓ with $k \geq 3$, there exists a positive integer N such that $rx_{k,\ell}(K_n) = k$ for every integer $n \geq N$. This paper provides solutions to these conjectures.

Keywords: rainbow connectivity; rainbow tree; rainbow index.

1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [2] for graph theoretical notation and terminology not described here. Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \to \{1, 2, \ldots, t\}, t \in \mathbb{N}$, where adjacent edges may be colored the same. A path is said to be a *rainbow path* if no two edges on the path have the same color. An edge-colored graph G is called *rainbow connected* if for every pair of distinct vertices of G there exists a rainbow path connecting them. The *rainbow connection number* of a graph G, denoted by $rc(G)$, is defined as the minimum number of

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colors that are needed in order to make G rainbow connected. The *rainbow k-connectivity* of G, denoted $rc_k(G)$, is defined as the minimum number of colors in an edge-coloring of G such that every two distinct vertices of G are connected by k internally disjoint rainbow paths. These concepts were introduced by Chartrand et al. [3]. Recently, several results have been published on rainbow connectivity. We refer the readers to [5, 6] for details.

Similarly, a tree T in G is called a *rainbow tree* if no two edges of T have the same color. For $S \subseteq V(G)$, a *rainbow S-tree* is a rainbow tree connecting the vertices of S. Suppose that $\{T_1, T_2, \dots, T_\ell\}$ is a set of rainbow S-trees. They are called *internally disjoint* if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \bigcap V(T_j) = S$ for every pair of distinct integers i, j with $1 \leq i, j \leq \ell$ (note that the trees are vertex-disjoint in $G \setminus S$). Given two positive integers k, ℓ with $k \geq 2$, the *(k,ℓ)-rainbow index* $rx_{k,\ell}(G)$ of G is the minimum number of colors needed in an edge-coloring of G such that for any set S of k vertices of G, there exist ℓ internally disjoint rainbow S-trees. In particular, for $\ell = 1$, we often write $rx_k(G)$ rather than $rx_{k,1}(G)$ and call it the *k-rainbow index.* It is easy to see that $rx_{2,\ell}(G) = rc_\ell(G)$. So the (k,ℓ)-rainbow index can be viewed as a generalization of rainbow connectivity. In the sequel, we always assume $k \geq 3$. The concept of (k,ℓ)-rainbow index was also introduced by Chartrand et al. [4]. They determined the k-rainbow index of all unicyclic graphs and the $(3,\ell)$-rainbow index of complete graphs for $\ell = 1, 2$. At the end of [4], they proposed the following two conjectures:

Conjecture 1. *For every positive integer ℓ, there exists a positive integer N such that $rx_{3,\ell}(K_n) = 3$ for every integer $n \geq N$.*

Conjecture 2. *For every pair of positive integers k, ℓ with $k \geq 3$, there exists a positive integer N such that $rx_{k,\ell}(K_n) = k$ for every integer $n \geq N$.*

In this paper, we will use the probabilistic method [1] to establish the above two conjectures.

2 Solution to the conjectures

It is easy to see that the second conjecture implies the first one. So, if the second conjecture is proved, the first one follows then. In this section, we will prove Conjecture 2. Firstly, let us start with a lemma.

Lemma 1. *For every pair of positive integers k, ℓ with $k \geq 3$, there exists a positive integer $N_1 = 4\lceil (\frac{k+\ell-1}{\ln(1-k!/k^k)})^2 \rceil$ such that $rx_{k,\ell}(K_n) \leq k$ for every integer $n \geq N_1$.*

Proof. Let $C = \{1, 2, \ldots, k\}$ be a set of k different colors. We color the edges of K_n with the colors from C randomly and independently. For $S \subseteq V(K_n)$ with $|S| = k$, define A_S as the event that there exist at least ℓ internally disjoint rainbow S-trees. If $\Pr[\bigcap_S A_S] > 0$, then there exists a suitable k-edge-coloring, which implies that $rx_{k,\ell}(K_n) \leq k$.

Let $S \subseteq V(K_n)$ with $|S| = k$. Without loss of generality, suppose $S = \{v_1, v_2, \ldots, v_k\}$. For any vertex $u \in V(K_n) \setminus S$, let $T(u)$ denote a star with u as its center and $E(T(u)) = \{uv_1, uv_2, \ldots, uv_k\}$. Clearly, $T(u)$ is an S-tree. Moreover, for $u_1, u_2 \in V(K_n) \setminus S$ and $u_1 \neq u_2$, $T(u_1)$ and $T(u_2)$ are two internally disjoint S-trees. Let $\mathcal{T}^* = \{T(u) | u \in V(K_n) \setminus S\}$. Then $\mathcal{T}^*$ is a set of $n-k$ internally disjoint S-trees. It is easy to see that $p := \Pr[\, T \in \mathcal{T}^*$ is a rainbow S-tree $] = k!/k^k$. (Throughout this paper, $\mathcal{T}^*$ and p are always defined in this way.) Denote by B_S the event that there exist at most $\ell - 1$ internally disjoint rainbow S-trees in $\mathcal{T}^*$. Here we assume that $n \geq k + \ell \geq 4$. Then $n - k > \ell - 1$ and

$$\Pr[\,\overline{A_S}\,] \leq \Pr[\,B_S\,] \leq \tbinom{n-k}{\ell-1}(1-p)^{n-k-(\ell-1)} < n^{\ell-1}(1-p)^{n-k-\ell+1}.$$

As an immediate consequence, we obtain

$$\begin{aligned} Pr[\bigcap_S A_S\,] &= 1 - Pr[\bigcup_S \overline{A_S}\,] \\ &\geq 1 - \sum_S Pr[\,\overline{A_S}\,] \\ &> 1 - \sum_S n^{\ell-1}(1-p)^{n-k-\ell+1} \\ &= 1 - \binom{n}{k} n^{\ell-1}(1-p)^{n-(k+\ell-1)} \\ &> 1 - n^{k+\ell-1}(1-p)^{n-(k+\ell-1)}. \end{aligned}$$

Now we are in the position to estimate the value of N_1 according to the inequality $n^{k+\ell-1}(1-p)^{n-(k+\ell-1)} \leq 1$, which leads to $\Pr[\bigcap_S A_S\,] > 0$. This inequality is equivalent to

$$(\tfrac{n}{1-p})^{k+\ell-1} \leq (\tfrac{1}{1-p})^n.$$

Taking the natural logarithm, we obtain

$$(k+\ell-1)\ln\tfrac{n}{1-p} \leq n \ln\tfrac{1}{1-p}.$$

That is,

$$\tfrac{k+\ell-1}{\ln(1/(1-p))} \leq \tfrac{n}{\ln n + \ln(1/(1-p))}.$$

Let $f(k) = \frac{1}{1-p} = \frac{1}{1-k!/k^k}$. Obviously, $f(k)$ is monotonically decreasing in $[3, +\infty)$. So, $f(k) \leq f(3) \approx 1.286$. Since $n \geq 4 > \frac{1}{1-p}$, $\ln n > \ln \frac{1}{1-p}$, then $\frac{n}{\ln n + \ln(1/(1-p))} > \frac{n}{2\ln n}$. Note that $\ln x < \sqrt{x}$ holds for $x \geq 4$. Thus, when $n \geq k + \ell \geq 4$, we have $\frac{n}{\ln n+\ln(1/(1-p))} > \frac{\sqrt{n}}{2}$. Setting $\frac{k+\ell-1}{\ln(1/(1-p))} \leq \frac{\sqrt{n}}{2}$, we get $n \geq 4(\frac{k+\ell-1}{\ln(1/(1-p))})^2$. Then, the inequality $\frac{k+\ell-1}{\ln(1/(1-p))} < \frac{n}{\ln n+\ln(1/(1-p))}$ holds for $n \geq \max\{k+\ell, 4(\frac{k+\ell-1}{\ln(1/(1-p))})^2\} = 4(\frac{k+l-1}{\ln(1/(1-p))})^2$. In other words, if $n \geq N_1 = 4\lceil(\frac{k+\ell-1}{\ln(1-k!/k^k)})^2\rceil$, then $Pr[\bigcap_S A_S] > 0$, as desired. □

To solve Conjecture 2 completely, we have to determine an integer N_2 such that for every integer $n \geq N_2$, $rx_{k,\ell}(K_n) \geq k$. First we recall the concept of Ramsey number, which will be used in our proof. The Ramsey number $R(t, s)$ is the smallest integer n such that every 2-edge-coloring of K_n contains either a complete subgraph on t vertices, all of whose edges are assigned color 1, or a complete subgraph on s vertices, all of whose edges are assigned color 2. For positive integers t_i with $1 \leq i \leq r$, the multicolor Ramsey number $R(t_1, t_2, \ldots, t_r)$ is defined as the smallest integer n such that for every r-edge-coloring of K_n, there exists an $i \in \{1, 2, \ldots, r\}$ such that K_n contains a complete subgraph on t_i vertices, all of whose edges are assigned color i. When $t_1 = t_2 = \cdots = t_r = t$, $R(t_1, t_2, \ldots, t_r)$ is abbreviated to $R_r(t)$. The existence of such a positive integer is guaranteed by the Ramsey's classical result [8]. A survey on the Ramsey number of graphs can be found in [7]. A typical upper bound for the multicolor Ramsey number is as follows, which can be found in any related textbooks, see [2] for example. For all positive integers t_i with $1 \leq i \leq r$,

$$R(t_1 + 1, t_2 + 1, \ldots, t_r + 1) \leq \frac{(t_1+t_2+\cdots+t_r)!}{t_1!t_2!\cdots t_r!}. \quad (1)$$

One may find more refined upper bounds in the existing literature; see [7] for example.

For $S \subseteq V(G)$ with $|S| = k$, let $\mathcal{T}$ be a maximum set of internally disjoint rainbow S-trees in G. Let $\mathcal{T}_1$ be the set of rainbow S-trees in $\mathcal{T}$, all of whose edges belong to $E(G[S])$, and let $\mathcal{T}_2$ be the set of rainbow S-trees in $\mathcal{T}$ containing at least one edge from $E_G[S, \overline{S}]$. Clearly, $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. (Throughout this paper, $\mathcal{T}$, $\mathcal{T}_1$, $\mathcal{T}_2$ are always defined in this way.)

Lemma 2. *For $S \subseteq V(G)$ with $|S| = k$, let T be a rainbow S-tree. If $T \in \mathcal{T}_1$, then T uses exactly $k - 1$ different colors; if $T \in \mathcal{T}_2$, then T uses at least k different colors.*

Proof. It is easy to see that for each rainbow S-tree $T \in \mathcal{T}_1$, T has exactly $k - 1$ edges. Then, exactly $k - 1$ different colors are used. For each rainbow S-tree $T \in \mathcal{T}_2$, T contains at least one vertex in $V(G) \setminus S$. Then, T has at least $k + 1$ vertices. So the number of edges of T is at least k, which implies that T uses at least k different colors. □

We proceed with the following lemma.

Lemma 3. *For every pair of positive integers k, ℓ with $k \geq 3$,*

(i) if $\ell > \lfloor \frac{k}{2} \rfloor$, then $rx_{k,\ell}(K_n) \geq k$ for every integer $n \geq N_2 = k$;

(ii) if $\ell \leq \lfloor \frac{k}{2} \rfloor$, then $rx_{k,\ell}(K_n) \geq k$ for every integer $n \geq N_2 = R_{k-1}(k)$.

Proof. We distinguish two cases.

Case 1. $\ell > \lfloor \frac{k}{2} \rfloor$.

For any set S of k vertices in K_n, the subgraph induced by S, denoted by $G[S]$, is a complete graph of order k. So, by Theorem 3.1 of [4] we know that $G[S]$ contains at most $\lfloor \frac{k}{2} \rfloor$ edge-disjoint spanning trees. From $\ell > \lfloor \frac{k}{2} \rfloor$, we can derive that there must exist one rainbow S-tree in $\mathcal{T}_2$, which uses at least k different colors by Lemma 2. Thus $rx_{k,\ell}(K_n) \geq k$ for every integer $n \geq k$.

Case 2. $\ell \leq \lfloor \frac{k}{2} \rfloor$.

From Ramsey's theorem, we know that if $k \geq 3$ and $n \geq R_{k-1}(k)$, then in any $(k-1)$-edge-coloring of K_n, one will find a monochromatic subgraph K_k. Now, take S as the set of k vertices of the monochromatic subgraph K_k. Then, $\mathcal{T}_1 = \emptyset$. In other words, all the rainbow S-trees belong to $\mathcal{T}_2$. Similar to *Case 1*, we have $rx_{k,\ell}(K_n) \geq k$ for every integer $n \geq R_{k-1}(k)$. $\square$

Combining Lemmas 1 and 3, we come to the following conclusion, which solves Conjecture 2.

Theorem 1. *For every pair of positive integers k, ℓ with $k \geq 3$,*

(i) if $\ell > \lfloor \frac{k}{2} \rfloor$, then there exists a positive integer $N = 4\lceil (\frac{k+\ell-1}{\ln(1-k!/k^k)})^2 \rceil$ such that $rx_{k,\ell}(K_n) = k$ for every integer $n \geq N$.

(ii) if $\ell \leq \lfloor \frac{k}{2} \rfloor$, there exists a positive integer $N = \max\{4\lceil (\frac{k+\ell-1}{\ln(1-k!/k^k)})^2 \rceil, R_{k-1}(k)\}$ such that $rx_{k,\ell}(K_n) = k$ for every integer $n \geq N$.

Note that although this gives a lower bound N for the order n of a complete graph with $rx_{k,\ell}(K_n) = k$, the bound is far from the best. Also, note that from (1) we can get a rough upper bound for the Ramsey number $R_{k-1}(k) \leq \frac{((k-1)^2)!}{((k-1)!)^{k-1}}$. The next section will use this bound to investigate an exact asymptotic value of N for the $(3, \ell)$-rainbow index $rx_{3,\ell}(K_n)$.

3 Exact asymptotic solution for Conjecture 1

In this section, we focus on the exact asymptotic solution of N for the $(3,\ell)$-rainbow index of K_n. To start with, we present a result derived from Theorem 1.

Lemma 4. *For every positive integers ℓ, there exists an integer $N = 4\lceil(\frac{\ell+2}{\ln 9/7})^2\rceil$ such that $rx_{3,\ell}(K_n) = 3$ for every integer $n \geq N$.*

Proof. From Lemma 1, we know that $rx_{3,\ell}(K_n) \leq 3$ for every integer $n \geq 4\lceil(\frac{\ell+2}{\ln 9/7})^2\rceil$. On the other hand, it follows from Lemma 3 that $rx_{3,\ell}(K_n) \geq 3$ for every integer $n \geq 6$. Since $4\lceil(\frac{\ell+2}{\ln 9/7})^2\rceil > 6$ holds for all integers $\ell \geq 1$, $rx_{3,\ell}(K_n) = 3$ for every integer $n \geq 4\lceil(\frac{\ell+2}{\ln 9/7})^2\rceil$. $\square$

One can see that the value of N in Lemma 4 is $O(\ell^2)$, which is far from the best. As the next step, we will improve N to $\frac{9}{2}\ell + o(\ell)$ in a certain range for ℓ, and show that it is asymptotically the best possible. To see this, we start with a general lemma for all integers $k \geq 3$.

Lemma 5. *Let ε be a constant with $0 < \varepsilon < 1$, k, ℓ be two integers with $k \geq 3$ and $\ell \geq \frac{k!}{k^k}(\theta - k)(1-\varepsilon) + 1$, where $\theta = \theta(\varepsilon, k)$ is the largest solution of $x^k e^{-\frac{k!}{2k^k}\varepsilon^2(x-k)} = 1$. Then, $rx_{k,\ell}(K_n) \leq k$ for every integer $n \geq \lceil \frac{k^k(\ell-1)}{k!(1-\varepsilon)} + k\rceil$.*

Proof. Here we follow the notations $C, S, A_S, T(u), p, \mathcal{T}^*$ in the proof of Lemma 1. Color the edges of K_n with the colors from C randomly and independently. Just as in Lemma 1, our aim is to obtain $\Pr[\bigcap_S A_S\,] > 0$. We assume $n > k$.

Let X be the number of rainbow S-trees in $\mathcal{T}^*$. Clearly, $X \sim Bi(n-k, p)$ (the Binomial distribution) and $EX = (n-k)p$. Using the Chernoff Bound [1], we get

$$Pr[\,\overline{A_S}\,] \leq Pr[\,X \leq \ell{-}1\,] = Pr[\,X \leq (n{-}k)p(1{-}\frac{(n-k)p-\ell+1}{(n-k)p})\,] \leq e^{-\frac{1}{2}[\frac{(n-k)p-\ell+1}{(n-k)p}]^2 p(n-k)}.$$

Note that the condition $n \geq \frac{\ell-1}{p(1-\varepsilon)} + k$ ensures $(n-k)p > \ell - 1$. So we can apply the Chernoff Bound to scaling the above inequalities. Also since $n \geq \frac{\ell-1}{p(1-\varepsilon)} + k$, then $\frac{(n-k)p-\ell+1}{(n-k)p} \geq \varepsilon$, and thus $Pr[\,\overline{A_S}\,] \leq e^{-\frac{1}{2}\varepsilon^2 p(n-k)}$. So,

$$\begin{aligned}
Pr[\bigcap_S A_S\,] &= 1 - Pr[\bigcup_S \overline{A_S}\,] \\
&\geq 1 - \sum_S Pr[\,\overline{A_S}\,] \\
&\geq 1 - \sum_S e^{-\frac{1}{2}\varepsilon^2 p(n-k)}
\end{aligned}$$

$$
\begin{aligned}
&= 1-\binom{n}{k}e^{-\frac{1}{2}\varepsilon^2 p(n-k)}\\
&> 1-n^k e^{-\frac{1}{2}\varepsilon^2 p(n-k)}.
\end{aligned}
$$

Obviously, the function $f(x)=x^k e^{-\frac{1}{2}\varepsilon^2 p(x-k)}$ eventually decreases and tends to 0 as $x\to+\infty$. Let $\theta=\theta(\varepsilon,k)$ be the largest solution of $x^k e^{-\frac{1}{2}\varepsilon^2 p(x-k)}=1$. Then, if $n\geq\theta$, then $n^k e^{-\frac{1}{2}\varepsilon^2 p(n-k)}\leq 1$, and consequently, $Pr[\bigcap\limits_S A_S\,]>0$, as desired. On the other hand, since $\ell\geq p(\theta-k)(1-\varepsilon)+1$, then $n\geq\frac{(\ell-1)}{p(1-\varepsilon)}+k\geq\theta$, which completes our proof. $\square$

Let $k=3$. From Lemma 5 we know that if $0<\varepsilon<1$, ℓ is an integer with $\ell\geq\frac{2}{9}(\theta-3)(1-\varepsilon)+1$ where $\theta=\theta(\varepsilon)$ is the largest solution of $x^3e^{-\frac{1}{9}\varepsilon^2(x-3)}=1$, then $rx_{3,\ell}(K_n)\leq 3$ for every integer $n\geq\lceil\frac{9(\ell-1)}{2(1-\varepsilon)}+3\rceil$. On the other hand, it follows from Lemma 3 that $rx_{3,\ell}(K_n)\geq 3$ for all integers $n\geq 6$. Thus we get the following theorem.

Theorem 2. *Let ε be a constant with $0<\varepsilon<1$, and let ℓ be an integer with $\ell\geq\frac{2}{9}(\theta-3)(1-\varepsilon)+1$ where $\theta=\theta(\varepsilon)$ is the largest solution of $x^3e^{-\frac{1}{9}\varepsilon^2(x-3)}=1$. Then, there exists an integer $N=\max\{6,\lceil\frac{9(\ell-1)}{2(1-\varepsilon)}+3\rceil\}$ such that $rx_{3,\ell}(K_n)=3$ for every integer $n\geq N$.*

For example, if we set $\varepsilon=\frac{1}{2}$, then $\theta\approx 712.415$. The result shows that for $\ell\geq 80$, $rx_{3,\ell}(K_n)=3$ holds for every integer $n\geq 9\ell-6$. If we set $\varepsilon=\frac{2}{3}$, then $\theta\approx 360.699$. The result shows that for $\ell\geq 28$, $rx_{3,\ell}(K_n)=3$ holds for every integer $n\geq\frac{3}{2}(9\ell-7)$.

Now we have improved N from $O(\ell^2)$ to $\frac{9}{2}\ell+o(\ell)$. A natural question is how small the integer N can be. The next lemma will show that $\frac{9}{2}\ell+o(\ell)$ is asymptotically the best possible.

Lemma 6. *For any 3-edge-coloring of K_n, there exists a set $S\subseteq V(K_n)$ with $|S|=3$ such that the number of internally disjoint rainbow S-trees is at most $\frac{2(n-1)^2}{9(n-2)}+3$.*

Proof. Let C be an arbitrary 3-edge-coloring of K_n. For every set $S\subseteq V(K_n)$ with $|S|=3$, we define the following three variables:

- $X(S)$ is the number of internally disjoint rainbow S-trees;
- $X_1(S)$ is the number of internally disjoint rainbow S-trees that contains at least one edge in $E(G[S])$;
- $X_2(S)$ is the number of internally disjoint rainbow S-trees in $\mathcal{T}^*=\{T(u)|u\in V(K_n)\setminus S\}$.

In fact, $X(S) = X_1(S) + X_2(S)$. Moreover, $X_1(S) \leq 3$ since there are exactly three edges in $E(G[S])$.

For any vertex $v \in V(K_n)$, we define Y_v as the number of distinct rainbow stars with 3 edges and with v as its center. Denote by $d_i(v)$ $(1 \leq i \leq 3)$ the number of edges of color i incident with v. Apparently, $d_1(v) + d_2(v) + d_3(v) = d(v) = n - 1$. Counting the distinct rainbow stars in two ways, we have $\sum\limits_S X_2(S) = \sum\limits_v Y_v$. Then

$$\begin{aligned}
EX &= \frac{1}{\binom{n}{3}} \sum_S X(S) \\
&= \frac{1}{\binom{n}{3}} (\sum_S X_1(S) + \sum_S X_2(S)) \\
&\leq \frac{1}{\binom{n}{3}} (\sum_S 3 + \sum_v Y_v) \\
&= 3 + \frac{1}{\binom{n}{3}} \sum_v d_1(v) d_2(v) d_3(v) \\
&\leq 3 + \frac{1}{\binom{n}{3}} \sum_v (\frac{d_1(v) + d_2(v) + d_3(v)}{3})^3 \\
&= 3 + \frac{1}{\binom{n}{3}} \sum_v (\frac{n-1}{3})^3 \\
&= 3 + \frac{n}{\binom{n}{3}} (\frac{n-1}{3})^3 \\
&= 3 + \frac{2(n-1)^2}{9(n-2)}.
\end{aligned}$$

Therefore, there exists a set S of three vertices such that the number of internally disjoint rainbow S-trees is at most $\frac{2(n-1)^2}{9(n-2)} + 3$. $\square$

It follows from the above lemma that $\ell \leq \frac{2(n-1)^2}{9(n-2)} + 3$, which is approximately equivalent to $n \geq \frac{9}{2}\ell + o(\ell)$. Therefore, $\frac{9}{2}\ell + o(\ell)$ is asymptotically the best possible for the lower bound on N.

4 Concluding remarks

In this paper, we solve the two conjectures posed in [4]. First we prove that for every pair of positive integers k, ℓ with $k \geq 3$, if $n \geq 4\lceil (\frac{k+\ell-1}{\ln(1-k!/k^k)})^2 \rceil$, then $rx_{k,\ell}(K_n) \leq k$. Recall that the Ramsey number $R_{k-1}(k)$ is the smallest number n such that any $(k-1)$-edge-coloring of K_n yields a monochromatic subgraph K_k. So, if $n \geq R_{k-1}(k)$, then

$rx_{k,\ell}(K_n) \geq k$ (note that $R_{k-1}(k) \leq \frac{((k-1)^2)!}{((k-1)!)^{k-1}}$). Thus, we get that $rx_{k,\ell}(K_n) = k$ for every integer $n \geq N = \max\{4\lceil(\frac{k+\ell-1}{\ln(1-k!/k^k)})^2\rceil, R_{k-1}(k)\}$, which solves Conjecture 2. Then, we try to get a more exact asymptotic expression of N for the special case $k = 3$. Using the Chernoff Bound, we obtain that if $n \geq N = \max\{6, \lceil\frac{9(\ell-1)}{2(1-\varepsilon)} + 3\rceil\}$, where $0 < \varepsilon < 1$, then $rx_{3,\ell}(K_n) = 3$; moreover the bound $\frac{9}{2}\ell + o(\ell)$ is asymptotically the best possible for N in Conjecture 1.

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