A NOTE ON CONNECTED CUBIC CAYLEY GRAPHS

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Abstract. Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected cubic Cayley graph, and let $X = \mathsf{Aut}\Gamma$ be the automorphism group of Γ. It is proved in this note that either G has non-trivial core $\cap_{x\in X} G^x$ in X, or the stabilizer X_u of a vertex u of Γ is non-abelian.

1. Introduction

All graphs are assumed to be finite, simple and undirected.

Let Γ be a graph. We use $V\Gamma$ and $\mathsf{Aut}\Gamma$ to denote its vertex set and automorphism group, respectively. An *arc* in Γ is an ordered pair of adjacent vertices. The graph Γ is said to be *arc-transitive* if $\mathsf{Aut}\Gamma$ is transitive on the set of arcs of Γ.

Let G be a finite group and let S be a subset of G such that $S = S^{-1} := \{x^{-1} \mid x \in S\}$ and S does not contain the identity of G. The *Cayley graph* of G with respect to S, denoted by $\mathrm{Cay}(G, S)$, is the graph with vertex set G such that $g, h \in G$ are adjacent if and only if $hg^{-1} \in S$. Then $\mathrm{Cay}(G, S)$ is a regular graph of valency $|S|$ and $\mathrm{Cay}(G, S)$ is connected if and only if $G = \langle S\rangle$, that is, S is a generating set of G.

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph. The underlying group G can be viewed as a regular subgroup of $\mathsf{Aut}\Gamma$, which acts on G by right multiplication. Clearly, $\mathsf{Aut}\Gamma$ contains the normal subgroup $\mathsf{Core}_{\mathsf{Aut}\Gamma}(G) := \cap_{x\in\mathsf{Aut}\Gamma} G^x$, which is called the *core* of G in $\mathsf{Aut}\Gamma$. If $\mathsf{Core}_{\mathsf{Aut}\Gamma}(G) = 1$ then Γ is said to be *core-free* with respect to G.

In [4] a classification was given for the core-free arc-transitive cubic Cayley graphs. Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected arc-transitive cubic Cayley graph which is core-free with respect to G. Employing a well-known result of Tutte (refer to [2, 18f]), it was proved that $\mathsf{Aut}\Gamma$ is isomorphic to one of 14 non-abelian subgroups of the symmetric group S_{48}, and that Γ is isomorphic to one of 15 cubic Cayley graphs. This motivates us to make an attempt towards classifying or characterizing the core-free cubic Cayley graphs which are not arc-transitive.

The goal of this note is to point out the following fact about the vertex-stabilizers of connected cubic Cayley graphs.

Theorem 1. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected cubic Cayley graph and let $G \le X \le \mathsf{Aut}\Gamma$. Let H be the stabilizer in X of the vertex of Γ corresponding to the identity of G. If $\mathsf{Core}_X(G) = 1$, then H is non-abelian.*

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2. A technical lemma

For a group X and a subgroup $H \le X$, we denote by $\mathbf{C}_X(H)$ and $\mathbf{N}_X(H)$ the centralizer and normalizer of H in X, respectively; for two groups N and H, we denote by $N{:}H$ a semi-direct product of N by H.

For a nonempty set Δ, we denote by $\mathrm{Sym}(\Delta)$ the symmetric group on Δ. Let X be a subgroup of $\mathrm{Sym}(\Delta)$. A subset Δ_1 of Δ is X-invariant if $\Delta_1^x = \Delta_1$ for all $x \in X$. For $x \in X$ and an X-invariant subset Δ_1 of Δ, we denote by x^{Δ_1} the restriction of x to Δ_1. Write $X^{\Delta_1} := \{x^{\Delta_1} \mid x \in X\}$. Then X^{Δ_1} is a permutation group on Δ_1.

The following lemma plays an important part in the proof of Theorem 1.

Lemma 2. *Let Δ be a set of size 2^n for an integer $n \ge 2$. Suppose that H is a regular subgroup of $\mathrm{Sym}(\Delta)$ such that $H \cong \mathbb{Z}_2^n$. Let P be a subgroup of index 2 in H. Then H is normal in $\mathbf{N}_{\mathrm{Sym}(\Delta)}(P)$ and $\mathbf{N}_{\mathrm{Sym}(\Delta)}(P) = H{:}A$ for a subgroup A of $\mathrm{Sym}(\Delta)$ with $A \cong \mathbb{Z}_2^{n-1}{:}\mathsf{Aut}(\mathbb{Z}_2^{n-1})$.*

Proof. By the assumption, $P \cong \mathbb{Z}_2^{n-1}$ and P is semiregular on Δ. Then P has two orbits on Δ, say Δ_1 and Δ_2. We fix an element $h \in H \setminus P$. Then $H = \langle h\rangle \times P$ and $\Delta_1^h = \Delta_2$. Taking $\delta_1 \in \Delta_1$ and setting $\delta_2 = \delta_1^h$, we have that

$$\Delta_i = \delta_i^P := \{\delta_i^y \mid y \in P\},\ i = 1, 2.$$

Let $H_i = P^{\Delta_i}$ for $i = 1, 2$. Then $H_i \cong P \cong \mathbb{Z}_2^{n-1}$, and H_i is a regular subgroup of $\mathrm{Sym}(\Delta_i)$. Let A_i be the point-stabilizer of δ_i in $\mathbf{N}_{\mathrm{Sym}(\Delta_i)}(H_i)$. Then

$$\mathbf{N}_{\mathrm{Sym}(\Delta_i)}(H_i) = H_i{:}A_i \text{ and } A_i \cong \mathsf{Aut}(H_i) \cong \mathsf{Aut}(\mathbb{Z}_2^{n-1}),$$

refer to [3, Corollary 4.2B].

For convenience, if $x \in \mathrm{Sym}(\Delta_i)$ then we use the notation $\widetilde{x}$ to denote the element of $\mathrm{Sym}(\Delta)$ acting in the same way as x on Δ_i and acting trivially on $\Delta \setminus \Delta_i$. Set $\widetilde{H_i} = \{\widetilde{y_i} \mid y_i \in H_i\}$ and $\widetilde{A_i} = \{\widetilde{a_i} \mid a_i \in A_i\}$, where $i = 1, 2$. Then $\widetilde{H_i} \cong H_i$ and $\widetilde{A_i} \cong A_i$. It is easily shown that $\widetilde{H_1}^h = \widetilde{H_2}$ and $\widetilde{A_1}^h = \widetilde{A_2}$.

For $y \in P$, let $y_1 = y^{\Delta_1}$. Then $y = \widetilde{y_1}\widetilde{y_1}^h = \widetilde{y_1}^h\widetilde{y_1}$. Since $\widetilde{H_1} \cong H_1 \cong P \cong \mathbb{Z}_2^{n-1}$, it follows that $P = \{\widetilde{y_1}\widetilde{y_1}^h \mid y_1 \in H_1\}$. Let $D = \langle \widetilde{a}\widetilde{a}^h \mid a \in A_1\rangle$. Then $D \cong A_1 \cong \mathsf{Aut}(\mathbb{Z}_2^{n-1})$. If $a \in A_1$ and $y_1 \in H_1$, then $y_1^a \in H_1$ and $\widetilde{y_1}^{\widetilde{a}} = \widetilde{y_1^a}$, hence

$$(\widetilde{y_1}\widetilde{y_1}^h)^{\widetilde{a}\widetilde{a}^h} = \widetilde{y_1}^{\widetilde{a}\widetilde{a}^h}\widetilde{y_1}^{h\widetilde{a}\widetilde{a}^h} = \widetilde{y_1}^{\widetilde{a}}\widetilde{y_1}^{\widetilde{a}h} = \widetilde{y_1^a}\widetilde{y_1^a}^h \in P. \tag{1}$$

It follows that D lies in $\mathbf{N}_{\mathrm{Sym}(\Delta)}(P)$.

Let $\widetilde{a}\widetilde{a}^h \in D \cap \mathbf{C}_{\mathrm{Sym}(\Delta)}(P)$, where $a \in A_1$. For every $y_1 \in H_1$, by (1), we have that

$$\widetilde{y_1}\widetilde{y_1}^h = (\widetilde{y_1}\widetilde{y_1}^h)^{\widetilde{a}\widetilde{a}^h} = \widetilde{y_1^a}\widetilde{y_1^a}^h,$$

yielding $y_1 = y_1^a$. It follows that $a \in A_1 \cap \mathbf{C}_{\mathrm{Sym}(\Delta_1)}(H_1)$. Since H_1 is a regular abelian subgroup of $\mathrm{Sym}(\Delta_1)$, we have that $\mathbf{C}_{\mathrm{Sym}(\Delta_1)}(H_1) = H_1$ by [3, Theorem 4.2A]. Thus $a \in A_1 \cap H_1 = 1$, hence $\widetilde{a}\widetilde{a}^h = 1$. It follows that $D \cap \mathbf{C}_{\mathrm{Sym}(\Delta)}(P) = 1$. Thus, since $\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)$ is normal in $\mathbf{N}_{\mathrm{Sym}(\Delta)}(P)$, we have that

$$\mathbf{N}_{\mathrm{Sym}(\Delta)}(P) \ge \langle D, \mathbf{C}_{\mathrm{Sym}(\Delta)}(P)\rangle = \mathbf{C}_{\mathrm{Sym}(\Delta)}(P){:}D.$$

In particular, $|\mathbf{N}_{\mathrm{Sym}(\Delta)}(P)|$ is divisible by $|\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)||D|$.

Since the quotient group $\mathbf{N}_{\mathrm{Sym}(\Delta)}(P)/\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)$ is isomorphic to a subgroup of $\mathsf{Aut}(P)$, we know that $|\mathbf{N}_{\mathrm{Sym}(\Delta)}(P):\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)|$ divides $|\mathsf{Aut}(P)|$. Since $D\cong\mathsf{Aut}(\mathbb{Z}_2^{n-1})$ and $P\cong\mathbb{Z}_2^{n-1}$, the order of $\mathbf{N}_{\mathrm{Sym}(\Delta)}(P)$ divides $|\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)||D|$. Therefore, $\mathbf{N}_{\mathrm{Sym}(\Delta)}(P)=\mathbf{C}_{\mathrm{Sym}(\Delta)}(P){:}D$.

We now determine $\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)$. Clearly, $H\le\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)$. Since P is a normal subgroup of $\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)$, the P-orbits Δ_1 and Δ_2 give a $\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)$-invariant partition of Δ. Let K be the set-wise stabilizer of Δ_1 in $\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)$. Since $h\notin K$ we thus have that $|\mathbf{C}_{\mathrm{Sym}(\Delta)}(P):K|=2$, and so $\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)=K{:}\langle h\rangle$. Recall that $H_1=P^{\Delta_1}$. It is easily shown that $\widetilde{H_1}\le\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)$, hence $\widetilde{H_1}\le K$, and so $\widetilde{H_1}^h\le K^h=K$. Thus $\widetilde{H_1}\times\widetilde{H_1}^h\le K$.

For $x\in K$, let $x_i=x^{\Delta_i}$, where $i=1,2$. Then $x=\widetilde{x_1}\widetilde{x_2}$ and $x_i\in\mathbf{C}_{\mathrm{Sym}(\Delta_i)}(P^{\Delta_i})$. Note that $H_i=P^{\Delta_i}$ is a regular abelian subgroup of $\mathrm{Sym}(\Delta_i)$. By [3, Theorem 4.2A], $\mathbf{C}_{\mathrm{Sym}(\Delta_i)}(H_i)=H_i$. It follows that $x_i\in H_i$, and hence $x=\widetilde{x_1}\widetilde{x_2}\in\widetilde{H_1}\times\widetilde{H_2}=\widetilde{H_1}\times\widetilde{H_1}^h$. Therefore, $K\le\widetilde{H_1}\times\widetilde{H_1}^h$, and so $K=\widetilde{H_1}\times\widetilde{H_1}^h$. Recalling that $P=\{\widetilde{y}\widetilde{y}^h\mid y\in H_1\}$, it follows that $K=P\times\widetilde{H_1}$, and hence

$$\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)=K{:}\langle h\rangle=(P\times\widetilde{H_1}){:}\langle h\rangle=\langle P,h,\widetilde{H_1}\rangle=\langle H,\widetilde{H_1}\rangle.$$

For each $y_1\in H_1$, we have that $\widetilde{y_1}^2=1$ and $\widetilde{y_1}^h\widetilde{y_1}=\widetilde{y_1}\widetilde{y_1}^h\in P$, hence

$$h^{\widetilde{y_1}}=\widetilde{y_1}h\widetilde{y_1}=hh^{-1}\widetilde{y_1}h\widetilde{y_1}=h\widetilde{y_1}^h\widetilde{y_1}=h\widetilde{y_1}\widetilde{y_1}^h\in H.$$

It follows that $\widetilde{H_1}$ normalizes $H=\langle P,h\rangle$, and hence that H is normal in $\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)=\langle H,\widetilde{H_1}\rangle$, yielding $\mathbf{C}_{\mathrm{Sym}(\Delta)}(P)=H{:}\widetilde{H_1}$. Thus $\mathbf{N}_{\mathrm{Sym}(\Delta)}(P)=(H{:}\widetilde{H_1}){:}D$.

For each $a\in A_1$, since $\widetilde{a}\widetilde{a}^h=\widetilde{a}^h\widetilde{a}$, we have that $(\widetilde{a}^{-1}(\widetilde{a}^{-1})^h)(\widetilde{a}\widetilde{a}^h)=1$. Hence

$$h^{\widetilde{a}\widetilde{a}^h}=h\widetilde{a}^{-1}h\widetilde{a}^{-1}h\widetilde{a}h\widetilde{a}h=h(\widetilde{a}^{-1}(\widetilde{a}^{-1})^h)(\widetilde{a}\widetilde{a}^h)=h.$$

Since $D=\langle\widetilde{a}\widetilde{a}^h\mid a\in A_1\rangle$, we know that D centralizes h. Then D normalizes $H=\langle P,h\rangle$ as D normalizes P. Since $\widetilde{H_1}$ normalizes H, we conclude that H is normal in $\mathbf{N}_{\mathrm{Sym}(\Delta)}(P)=(H{:}\widetilde{H_1}){:}D$. It is easily shown that D normalizes $\widetilde{H_1}$. Then

$$\mathbf{N}_{\mathrm{Sym}(\Delta)}(P)=(H{:}\widetilde{H_1}){:}D=H{:}(\widetilde{H_1}{:}D)$$

and $\widetilde{H_1}{:}D\cong\mathbb{Z}_2^{n-1}{:}\mathsf{Aut}(\mathbb{Z}_2^{n-1})$. Thus the lemma follows. □

3. Proof of Theorem 1

Let $\mathit{\Gamma}=\mathrm{Cay}(G,S)$ be a connected cubic Cayley graph. Assume that $G\le X\le\mathsf{Aut}\mathit{\Gamma}$. Since $\mathit{\Gamma}$ is cubic, $|G|\ge4$. Assume further that $\mathsf{Core}_X(G)=1$. Then $X\ne G$.

Let u be the vertex of $\mathit{\Gamma}$ corresponding to the identity element of G and let $H=X_u$ be the stabilizer of u in X. Then $X=GH$, $G\cap H=1$ and $H\ne1$.

If X acts transitively on the arc set of Γ then, by [4], $X = \mathsf{Aut}\Gamma$ and H is non-abelian, and so the result follows.

Assume in the following that X is not transitive on the arc set of Γ. Then H is a non-trivial 2-group, and H has exactly two orbits, say $\{v_1\}$ and $\{v_2, v_3\}$, on the neighborhood $\Gamma(u)$ of u in Γ. Since X is transitive on $V\Gamma$, there exist $z_i \in X$ with $u^{z_i} = v_i$, where $i = 1$, 2. It is easily shown that

$$\{y \in X \mid u^y \in \Gamma(u)\} = H\{z_1, z_2\}H \subset \langle z_1, z_2, H\rangle.$$

Recall that Γ is connected. For each $x \in X$ denote by $p(x)$ the distance between u and u^x in Γ. Now we show that $x \in \langle z_1, z_2, H\rangle$ by induction on $p(x)$. If $p(x) = 0$ then $u^x = u$, yielding $x \in H$. Suppose that $x \in \langle z_1, z_2, H\rangle$ for $x \in X$ with $p(x) = l - 1$, where $l \geq 1$. Let $x \in X$ with $p(x) = l$. Take a vertex $u^y \in \Gamma(u)$ which is at distance $l - 1$ from u^x. Then $y \in H\{z_1, z_2\}H \subset \langle z_1, z_2, H\rangle$, and $p(xy^{-1}) = l - 1$. By the induction hypothesis, $xy^{-1} \in \langle z_1, z_2, H\rangle$. It follows that $x \in \langle z_1, z_2, H\rangle$. Therefore,

$$X = \langle z_1, z_2, H\rangle. \tag{2}$$

Set $P = X_u \cap X_{v_2}$. Then $|H : P| = 2$. We claim that

$$z_1 \in \mathbf{N}_X(H) \text{ and } z_2 \in \mathbf{N}_X(P). \tag{3}$$

Since H fixes v_1, we have that

$$H = X_u = X_{v_1} = X_{u^{z_1}} = X_u^{z_1} = H^{z_1},$$

hence $z_1 \in \mathbf{N}_X(H)$. Noting that $\Gamma(u) = \{v_1, v_2, v_3\}$, we have that

$$u \in \Gamma(v_2) = \Gamma(u^{z_2}) = \Gamma(u)^{z_2} = \{v_1^{z_2}, v_2^{z_2}, v_3^{z_2}\}.$$

If $u = v_1^{z_2}$, then

$$H = X_{v_1^{z_2}} = X_{v_1}^{z_2} = X_u^{z_2} = X_{u^{z_2}} = X_{v_2},$$

hence H fixes v_2, a contradiction. Thus $u = v_j^{z_2}$ for $j = 2$ or 3. Since H is transitive on $\{v_2, v_3\}$, there is some $h \in H$ with $v_j = v_2^h$. Then

$$u = v_j^{z_2} = v_2^{hz_2} \text{ and } v_2 = u^{z_2} = u^{hz_2},$$

that is, hz_2 interchanges u and v_2. This yields that hz_2 normalizes $P = X_u \cap X_{v_2}$. Since P is normal in H as $|H : P| = 2$, we have that $H \leq \mathbf{N}_X(P)$. Thus $z_2 \in \mathbf{N}_X(P)$.

Let N be a characteristic subgroup of both H and P. By the above (3), we conclude that $N^{z_1} = N$ and $N^{z_2} = N$. It follows that N is normal in $\langle z_1, z_2, H\rangle = X$, yielding $N = 1$ as N fixes u. This easy observation will be useful to our further argument.

Recall that H is a non-trivial 2-group. Suppose, by contradiction, that H is abelian.

Let K be the subgroup generated by all involutions in H. Then $K \neq 1$ and K is a characteristic subgroup of H. Since H and P cannot have a non-trivial characteristic subgroup in common, $K \not\leq P$. This means that H contains an element h of order 2 with $h \notin P$. Since $|H : P| = 2$ and H is abelian, $H = \langle h, P\rangle = \langle h\rangle \times P$. Recall that the Frattini subgroup of H is the smallest normal subgroup $\Phi(H)$ with $H/\Phi(H)$ being elementary abelian (refer to [1, (23.2)]). It is easily shown that $\Phi(P) = \Phi(H)$. Thus $\Phi(H)$ is a common characteristic subgroup of H and P, and hence $\Phi(H) = 1$. It follows that $H \cong \mathbb{Z}_2^n$ for some positive integer n.

Let $\Delta := \{Gh \mid h \in H\}$. Recalling that $X = GH$, the set Δ consists of all right cosets of G in X. Consider the action of X on Δ by right multiplication. Since $X = GH = HG$ and $G \cap H = 1$, H acts regularly on Δ, and $X_\delta = G$ for some $\delta \in \Delta$. Moreover, the kernel of this action is $\mathsf{Core}_X(G)$. Since $\mathsf{Core}_X(G) = 1$, this action is faithful. Thus X can be viewed as a subgroup of $\mathrm{Sym}(\Delta)$, while H is a regular subgroup of X (acting on Δ). Recall that $|X_\delta| = |G| \geq 4$. Then X, as a permutation group on Δ, has a point-stabilizer of order at least 4. It follows that $|\Delta| \geq 4$. Thus $2^n = |H| = |\Delta| \geq 4$, hence $n \geq 2$.

By Lemma 2, $\mathbf{N}_X(P) \leq \mathbf{N}_{\mathrm{Sym}(\Delta)}(P) \leq \mathbf{N}_{\mathrm{Sym}(\Delta)}(H)$. Then, by (3), both z_1 and z_2 normalize H, and hence H is normal in $\langle z_1, z_2, H\rangle$. Thus H is normal in X by (2). Since H fixes the vertex $u \in V\mathit{\Gamma}$, it follows that $H = 1$, which contradicts $|H| \geq 4$.

Therefore, H is non-abelian. This completes the proof of Theorem 1.

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