Rainbow connection number and the number of blocks*

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Abstract

An edge-colored graph G is *rainbow connected* if every pair of vertices of G are connected by a path whose edges have distinct colors. The *rainbow connection number* $rc(G)$ of G is defined to be the minimum integer t such that there exists an edge-coloring of G with t colors that makes G rainbow connected. For a graph G without any cut vertex, i.e., a 2-connected graph, of order n, it was proved that $rc(G) \leq \lceil \frac{n}{2} \rceil$ and the bound is tight. In this paper, we prove that for a connected graph G of order n with at least one cut vertex, $rc(G) \leq \frac{n+r-1}{2}$, where r is the number of blocks of G with even orders, and the upper bound is tight. Moreover, we also obtain a tight upper bound $\lfloor (2n-2)/3 \rfloor$ for the rainbow connection number of a bridgeless (2-edge-connected) graph of order n.

Keywords: rainbow edge-coloring, rainbow connection number, cut vertex, block decomposition.

AMS subject classification 2010: 05C40, 05C15.

1 Introduction

All graphs considered in this paper are simple, finite and undirected. For notation and terminology not defined here, we refer to [2]. In an edge-colored graph G, a path is called a *rainbow path* if the colors of its edges are distinct. The graph G is called *rainbow connected* if every pair of vertices are connected by at least one rainbow path in G. An edge-coloring of a connected graph G that makes G rainbow connected is called *a rainbow edge-coloring (rainbow coloring* for short) of G. The minimum number of colors required

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to rainbow color G is called the *rainbow connection number* of G, denoted by $rc(G)$. It is obvious that $rc(G) \leq d(G)$ for any connected graph G, where $d(G)$ denotes the diameter of G. If a graph G has an edge-coloring c and G' is a subgraph of G, $c(G')$ denotes the set of colors appearing in G'. An edge-coloring using k colors is addressed as a k-edge-coloring. If P is a path, the length of P, which is the number of edges in P, is denoted by $\ell(P)$.

Let G' be a subgraph of a graph G. An *ear* of G' in G is a nontrivial path in G whose end vertices lie in G' but whose internal vertices are not. An *ear decomposition* of a 2-connected graph G is a sequence of subgraphs $G_0, G_1, \cdots, G_k$ of G satisfying that (1) G_0 is a cycle of G; (2) $G_i = G_{i-1} \bigcup P_i$ $(1 \leq i \leq k)$, where P_i is an ear of G_{i-1} in G; (3) $G_{i-1}(1 \leq i \leq k)$ is a proper subgraph of G_i; (4) $G_k = G$. If $\ell(P_1) \geq \ell(P_2) \geq \cdots \geq \ell(P_k)$, we say that the ear decomposition is *nonincreasing*. From the above definition, every graph G_i in an ear decomposition is 2-connected.

A *block* of a graph G is a maximal connected subgraph of G that does not have any cut vertex. So every block of a nontrivial connected graph is either a K_2 or a 2-connected subgraph. All the blocks of a graph G form a *block decomposition* of G. A block B is called an *even (odd) block* if the order of B is even (odd).

Let c be a rainbow k-edge-coloring of a connected graph G. If a rainbow path P in G has length k, we call P *a complete rainbow path*; otherwise, it is *an incomplete rainbow path*. A rainbow edge-coloring c of G is *incomplete* if for any vertex $u \in V(G)$, G has at most one vertex v such that all the rainbow paths between u and v are complete; otherwise, it is *complete*.

The concept of rainbow coloring was introduced by Chartrand et al. in [5]. For more knowledge, we refer to [10, 11]. In [6], it was proved that computing the rainbow connection number of a graph is NP-hard. Hence, tight upper bounds of the rainbow connection number for a connected graph have been a subject of investigation. The authors of [4] proved that $rc(G) \leq 3n/(\delta + 1) + 3$, where δ is the minimum degree of the connected graph G. The authors of [1] obtained an upper bound of the rainbow connection number in term of radius: For every bridgeless graph G with radius r, $rc(G) \leq r(r+2)$. Moreover, for every integer $r \geq 1$, there exists a bridgeless graph with radius r and $rc(G) = r(r+2)$. Later, the authors of [7] generalized the bound to graphs with bridges, which is a little bit complicated to restate and therefore omitted.

For 2-connected graphs, there exist the following results.

Lemma 1.1. *[9] Let G be a Hamiltonian graph of order n $(n \geq 3)$. Then G has an incomplete $\lceil \frac{n}{2} \rceil$-rainbow coloring, i.e., $rc(G) \leq \lceil \frac{n}{2} \rceil$.*

Lemma 1.2. *[9] Let G be a 2-connected non-Hamiltonian graph of order n $(n \geq 4)$. If G has at most one ear with length 2 in a nonincreasing ear decomposition, then G has a incomplete $\lceil \frac{n}{2} \rceil$-rainbow coloring, i.e., $rc(G) \leq \lceil \frac{n}{2} \rceil$.*

Theorem 1.1. *[9, 8] Let G be a 2-connected graph of order n $(n \geq 3)$. Then $rc(G) \leq \lceil \frac{n}{2} \rceil$,* *and the upper bound is tight for $n \geq 4$.*

Proposition 1.1. *[3] If G is a connected bridgeless (2-edge-connected) graph with n vertices, then $rc(G) \leq 4n/5 - 1$.*

In this paper, we will study the rainbow connection number of a connected graph with at least one cut vertex and obtain a tight upper bound. Besides, a tight upper bound for a 2-edge-connected (bridgeless) graph is also obtained.

2 Main results

We first show that every 2-connected graph G with odd number of vertices has a rainbow edge-coloring with a nice property.

Lemma 2.1. *Let G be a 2-connected graph of order n ($n \geq 3$) and v_0 be any vertex of G. If n is odd, then G has a rainbow $\lceil \frac{n}{2} \rceil$-edge-coloring c such that there exists a color x of the edge-coloring satisfying that every vertex of G can be connected by a rainbow path P to v_0 with $x \notin c(P)$.*

Proof. Since G is 2-connected, G has a nonincreasing ear decomposition $G_0, G_1, \cdots, G_q(= G)$ $(q \geq 0)$ satisfying that (1) G_0 is a cycle with $v_0 \in V(G_0)$; (2) $G_i = G_{i-1} \bigcup P_i$, where P_i $(1 \leq i \leq q)$ is one of the longest ears of G_{i-1} in G; (3) $\ell(P_1) \geq \ell(P_2) \geq \cdots \geq \ell(P_q)$. In the sequel, every nonincreasing ear decomposition of a 2-connected graph G satisfies the above tree conditions. We consider the following two cases.

Case 1. No ear of $P_1, \cdots, P_q$ has an even length.

In this case, since G has an odd order, G_0 must be an odd cycle. Assume that $G_0 = v_0v_1 \cdots v_{2k}v_{2k+1}(= v_0)$ with $k \geq 1$. Define a $(k+1)$-edge-coloring c_0 of G_0 by $c_0(v_{i-1}v_i) = x_i$ for i with $1 \leq i \leq k+1$ and $c_0(v_{i-1}v_i) = x_{i-k-1}$ for i with $k+2 \leq i \leq 2k+1$. It can be checked that c_0 is a rainbow $\lceil \frac{|V(G_0)|}{2} \rceil$-edge-coloring of G_0 such that every vertex of G_0 can be connected by a rainbow path P in G_0 to v_0 with $x_{k+1} \notin c_0(P)$. If $G_0 = G$, the conclusion holds.

Now assume that $G_0 \neq G$ and $P_1 = v'_0v'_1 \cdots v'_{2s}v'_{2s+1}(s \geq 0)$ with $V(G_0) \bigcap V(P_1) = \{v'_0, v'_{2s+1}\}$. Define an edge-coloring c_1 of $G_1 = G_0 \bigcup P_1$ by $c_1(e) = c_0(e)$ for $e \in E(G_0)$, $c_1(v'_{i-1}v'_i) = y_i$ for i with $1 \leq i \leq s$, $c_1(v'_sv'_{s+1}) = x'$ and $c_1(v'_{i-1}v'_i) = y_{i-s-1}$ for i with $s+2 \leq i \leq 2s+1$, where $y_1, \cdots, y_s$ are new colors and x' is a color that already appeared in G_0. Here, if $\ell(P_1) = 1$, i.e., $s = 0$, we just color the only edge $v'_0v'_1$ of P_1 by a color that appeared in G_0. It can be checked that c_1 is a rainbow $\lceil \frac{|V(G_1)|}{2} \rceil$-edge-coloring of G_1. From the definition of c_1, every vertex of G_0 can be connected by a rainbow path P in G_0 to v_0 with $x_{k+1} \notin c_1(P)$. Let P' and P'' be the rainbow paths, respectively, from v'_0 and v'_{2s+1} to v_0 in G_0 such that $x_{k+1} \notin c_1(P')$ and $x_{k+1} \notin c_1(P'')$. For any vertex v'_j $(1 \leq j \leq s)$, $v'_jP_1v'_0P'v_0$ is a rainbow path in G_1 from v'_j to v_0 such that $x_{k+1} \notin c_1(v'_jP_1v'_0P'v_0)$. For any vertex v'_j $(s+1 \leq j \leq 2s)$, we can choose $v'_jP_1v'_{2s+1}P''v_0$ as a rainbow path in G_1 from v'_j to v_0 such that $x_{k+1} \notin c_1(v'_jP_1v'_{2s+1}P''v_0)$. Hence, c_1 is a required rainbow edge-coloring of G_1.

If $G_1 = G$, the conclusion holds. Otherwise, repeating the above process of defining c_1 from c_0, we can obtain a rainbow $\lceil\frac{|V(G_i)|}{2}\rceil$-edge-coloring of G_i $(2 \leq i \leq q)$ such that every vertex of G_i can be connected by a rainbow path P in G_i to v_0 with $x_{k+1} \notin c_i(P)$. Therefore, c_q is a required rainbow $\lceil\frac{n}{2}\rceil$-edge-coloring of G.

Case 2. At least one of $P_1, \cdots, P_q$ has an even length.

Suppose that P_t $(1 \leq t \leq q)$ is the last added ear with an even length. So $P_{t+1}, \cdots, P_s$ have odd lengths. Once we show that G_t has a required rainbow $\lceil\frac{n_t}{2}\rceil$-edge-coloring by arguments similar to those used in the proof of Case 1, we can show that G has the required rainbow $\lceil\frac{n}{2}\rceil$-edge-coloring. We will consider the following two cases:

Subcase 2.1. At most one of the ears $P_1, \cdots, P_{t-1}$ has length 2.

Assume that $P_t = v'_0v'_1 \cdots v'_{2s-1}v'_{2s}$ such that $V(P_t) \bigcap V(G_{t-1}) = \{v'_0, v'_{2s}\}$. It is obvious that $G_0, G_1, \cdots, G_{t-1}$ is a nonincreasing ear decomposition of G_{t-1} with at most one ear with length 2. Note that G_{t-1} has at least 4 vertices. From Lemmas 1.1 and 1.2, G_{t-1} has an incomplete rainbow $\lceil\frac{|V(G_{t-1})|}{2}\rceil$-edge-coloring c_{t-1}. In G_{t-1}, there exists an incomplete rainbow path P' from v_0 to one of v'_0 and v'_{2s} (say v'_{2s}). Assume that x' is a color of the coloring c_{t-1} with $x' \notin c_{t-1}(P')$. Define an edge-coloring c_t of $G_t = G_{t-1} \bigcup P_t$ by $c_t(e) = c_{t-1}(e)$ for $e \in E(G_{t-1})$, $c_t(v'_{i-1}v'_i) = x_i$ for i with $1 \leq i \leq s$, $c_t(v'_sv'_{s+1}) = x'$ and $c_t(v'_{i-1}v'_i) = x_{i-s-1}$ for i with $s+2 \leq i \leq 2s$, where $x_1, \cdots, x_s$ are new colors. It can be checked that c_t is a rainbow $\lceil\frac{|V(G_t)|}{2}\rceil$-edge-coloring of G_t. From the definition of coloring c_t, every vertex of G_{t-1} has a rainbow path P in G_{t-1} to v_0 with $x_s \notin c_t(P)$. Let P'' be a rainbow path in G_{t-1} from v'_0 to v_0. For any vertex v'_j $(1 \leq j \leq s-1)$, $v'_jP_tv'_0P''v_0$ is a rainbow path in G_t from v'_j to v_0 such that $x_s \notin c_t(v'_jP_tv'_0P''v_0)$. For any vertex v'_j $(s \leq j \leq 2s-1)$, we have $v'_jP_tv'_{2s}P'v_0$ is a rainbow path in G_t from v'_j to v_0 such that $x_s \notin c_t(v'_jP_tv'_{2s}P'v_0)$. So every vertex of G_t has a rainbow path P in G_t to v_0 with $x_s \notin c_t(P)$. Hence, c_t is a required rainbow edge-coloring of G_t.

Subcase 2.2. At least two ears of $P_1, \cdots, P_{t-1}$ have length 2.

In this case, it is obvious that $\ell(P_t) = 2$ and $\ell(P_{t+1}) = \cdots = \ell(P_q) = 1$. Assume that $\ell(P_1) \geq \cdots \geq \ell(P_h) \geq 3$ and $\ell(P_{h+1}) = \cdots = \ell(P_t) = 2$. Note that the endvertices of every P_j with $h+1 \leq j \leq t$ belong to $V(G_t)$. Here at least three ears have length 2, i.e., $t-h \geq 3$. From Theorem 1.1, G_h has a rainbow $\lceil\frac{|V(G_h)|}{2}\rceil$-edge-coloring c_h. Assume that $P_j = a_jv_jb_j$ $(h+1 \leq j \leq t)$ such that $V(P_j) \bigcap V(G_h) = \{a_j, b_j\}$. Define an edge-coloring c_t of G_t by $c_t(e) = c_h(e)$ for $e \in E(G_h)$, $c_t(a_jv_j) = x_1$ for j with $h+1 \leq j \leq t$ and $c_t(v_jb_j) = x_2$ for j with $h+1 \leq j \leq t$, where x_1, x_2 are new colors. When there are exactly 3 ears in the nonincreasing ear decomposition, i.e., $t-h = 3$, then $|V(G_h)|$ is even. So c_t uses exactly $\lceil\frac{|V(G_t)|}{2}\rceil$ colors. If $t-h \geq 4$, c_t is a rainbow edge-coloring of G_t with at most $\lceil\frac{|V(G_t)|}{2}\rceil$ colors. It is easy to check that every vertex of G_t has a rainbow path P to

v_0 with $x_2 \notin c_t(P)$. Therefore, G_t has a required rainbow $\lceil\frac{|V(G_t)|}{2}\rceil$-edge-coloring. $\square$

Theorem 2.1. *Let G be a connected graph of order n $(n \geq 3)$ and G has a block decomposition $B_1, \cdots, B_q$ $(q \geq 2)$, where r blocks are even blocks. Then $rc(G) \leq \frac{n+r-1}{2}$ and the upper bound is tight.*

Proof. Let G be a connected graph of order n with q $(q \geq 2)$ blocks in its block decomposition. If G has at least one even block, we choose $G_1 = B_1$ being an even block of G; otherwise, $G_1 = B_1$ being an odd block of G. Since $q \geq 2$ and G is connected, G has a block B_2 such that $V(G_1) \bigcap V(B_2) = \{v_1\}$. Let $G_2 = G_1 \bigcup B_2$. So G_2 is a connected graph which consists of two blocks B_1, B_2. Repeating the process of adding B_2 to G_1, we obtain a sequence of subgraphs $G_1, G_2, \cdots, G_q$ such that G_i $(1 \leq i \leq q)$ is a connected graph and $G_i = B_1 \bigcup B_2 \bigcup \cdots \bigcup B_i$ $(2 \leq i \leq q)$ with $V(G_{i-1}) \bigcap V(B_i) = \{v_{i-1}\}$ for i with $2 \leq i \leq q$. Denote the order of B_i $(1 \leq i \leq q)$ by n_i. From Theorem 1.1 and $rc(K_2) = 1$, every block B has a rainbow $\lceil\frac{|V(B)|}{2}\rceil$-edge-coloring. We will consider the following two cases.

Case 1. $r \geq 1$.

From the definition of G_1, $G_1 = B_1$ is an even block and G_1 has a rainbow $\lfloor\frac{n_1}{2}\rfloor$-edge-coloring c_1. If B_2 is an even block, color the edges of B_2 with $\lfloor\frac{n_2}{2}\rfloor$ new colors such that B_2 is rainbow connected. It is obvious that G_2 is rainbow connected and the obtained edge-coloring c_2 of G_2 uses $\lfloor\frac{n_1}{2}\rfloor + \lfloor\frac{n_2}{2}\rfloor$ colors. Consider the case that B_2 is an odd block. From Lemma 2.1, B_2 has a rainbow edge-coloring c'_2 with $\lceil\frac{n_2}{2}\rceil$ new colors such that there exists a color x' of c'_2 satisfying that every vertex of B_2 has a rainbow path P in B_2 to v_1 with $x' \notin c'_2(P)$. Replacing the color x' of c'_2 by a color x that already appeared in G_1, we obtain an edge-coloring c_2 of G_2 with $\lfloor\frac{n_1}{2}\rfloor + \lfloor\frac{n_2}{2}\rfloor$ colors. It is obvious that G_1 and B_2 are rainbow connected, respectively. Consider two vertices $v' \in V(G_1)$ and $v'' \in V(B_2)$. From the definition of c_2, there are two rainbow paths P' in G_1 from v' to v_1 and P'' in B_2 from v_1 to v'' such that $x \notin c_2(P'')$. So $v'P'v_1P''v''$ is a rainbow path from v' to v'' in G_2. Hence, c_2 is a rainbow edge-coloring of G_2 with $\lfloor\frac{n_1}{2}\rfloor + \lfloor\frac{n_2}{2}\rfloor$ colors.

If $q \geq 3$, we can repeat the process of defining c_2 from c_1 to obtain a rainbow edge-coloring c_q of $G_q(= G)$ with $\lfloor\frac{n_1}{2}\rfloor + \lfloor\frac{n_2}{2}\rfloor + \cdots + \lfloor\frac{n_q}{2}\rfloor$ colors.

Case 2. $r = 0$.

In this case, $G_2 = B_1 \bigcup B_2$ consists of two odd blocks. From Lemma 2.1, B_i $(i = 1, 2)$ has a rainbow $\lceil\frac{n_i}{2}\rceil$-edge-coloring c'_i such that x'_i is a color of c'_i satisfying that every vertex of B_i $(i = 1, 2)$ has a rainbow path P in B_i to v_1 with $x'_i \notin c'_i(P)$. Note that $c'_1(B_1) \bigcap c'_2(B_2) = \emptyset$. Assume that x_i $(i = 1, 2)$ is a color of c'_i such that $x_i \neq x'_i$. Replacing x'_1 by x_2 in B_1 and x'_2 by x_1 in B_2, we obtain an edge-coloring c_2 of G_2 with $\lfloor\frac{n_1}{2}\rfloor + \lfloor\frac{n_2}{2}\rfloor$ colors. It is obvious that B_i $(i = 1, 2)$ is rainbow connected. Consider two vertices $v' \in V(B_1)$ and $v'' \in V(B_2)$. From the definition of c_2, there exist two rainbow paths P' in B_1 from v' to v_1 and P'' in B_2 from v_1 to v'' such that $x_2 \notin c_2(P')$ and $x_1 \notin c_2(P'')$. So $v'P'v_1P''v''$ is a rainbow path in G_2 from v' to v''. Hence, c_2 is a rainbow edge-coloring of G_2 with $\lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor$ colors. If $q \geq 3$, we can color the blocks $B_3, \cdots, B_q$ similar to Case 1 to obtain a rainbow edge-coloring of G with $\lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor + \cdots + \lfloor \frac{n_q}{2} \rfloor$ colors.

Therefore, in any case we have that $rc(G) \leq \lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor + \cdots + \lfloor \frac{n_q}{2} \rfloor = \frac{n+r-1}{2}$.

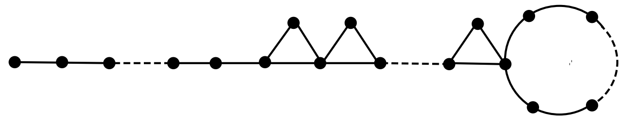

Figure 1. A graph of order n with rK_2, $(q-r-1)K_3$ and one odd cycle $C_{n-2q+r+2}$.

In order to prove that the upper bound is tight, we will show that for any integers n, r, q, if there exist graphs of order n with r even blocks and $q-r$ odd blocks, then one of such graphs has a rainbow connection number $\frac{n+r-1}{2}$.

In fact, if there exists a connected graph of order n with r even blocks, then $n+r$ must be an odd number. The graph G of order n in Figure 1 consists of r even blocks K_2, $q-r-1$ odd cycles K_3 and one odd cycle $C_{n-2q+r+2}$. Since $d(G) = \frac{n+r-1}{2}$ and $d(G) \leq rc(G) \leq \frac{n+r-1}{2}$, we have $rc(G) = \frac{n+r-1}{2}$. □

We know that for any connected graph G of order n, $rc(G) \leq n-1$ with equality if and only if G is a tree. Since the number of even blocks in any connected graph G of order n is at most $n-1$ (when G is a tree), from the bound of Theorem 2.1 we have $rc(G) \leq (n+r-1)/2 \leq (n+n-1-1)/2 = n-1$. Hence, the upper bound in the Theorem 2.1 generalizes the bound $n-1$.

In the following, we give a tight upper bound of the rainbow connection number for a 2-edge-connected graph which improves the result of Proposition 1.1.

Theorem 2.2. *Let G be a 2-edge-connected graph of order n $(n \geq 3)$. Then we have $rc(G) \leq \lfloor (2n-2)/3 \rfloor$ and the upper bound is tight.*

Proof. Suppose that G has the block decomposition $B_1, B_2, \cdots, B_q$. Since G is 2-edge-connected, we have $|B_i| \geq 3, 1 \leq i \leq q$. And if B_i is an even block, then $|B_i| \geq 4$. If G has r even blocks, then $3r+1 \leq n$, i.e., $r \leq \frac{n-1}{3}$. From Theorem 2.1, $rc(G) \leq \frac{n+r-1}{2} \leq \frac{2n-2}{3}$. Since $rc(G)$ is an integer, we have $rc(G) \leq \lfloor (2n-2)/3 \rfloor$.

The three graphs G_1, G_2, G_3 in Figure 2 are 2-edge-connected. The order of G_i $(i = 1, 2, 3)$ is $n_i = 3k+i$, and $d(G_1) = d(G_2) = 2k$ and $d(G_3) = 2k+1$. From the above result and $d(G) \leq rc(G)$, we have that $rc(G_1) = rc(G_2) = 2k$ and $rc(G_3) = 2k+1$, i.e., $rc(G_i) = \lfloor (2n_i - 2)/3 \rfloor$ for $i = 1, 2, 3$. Hence, the upper bound is tight. □

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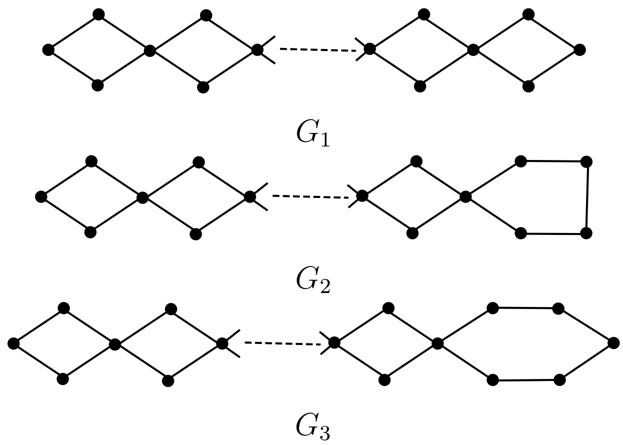

Figure 2. Graphs for the tightness of Theorem 2.2.

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