LEFT CELLS CONTAINING A FULLY COMMUTATIVE ELEMENT

Jian-yi Shi

Department of Mathematics, East China Normal
University, Shanghai, 200062, P.R.China
and
Center for Combinatorics, Nankai
University, Tianjin, 300071, P.R.China

Abstract. Let W be a finite or an affine Coxeter group and W_{c} the set of all the fully commutative elements in W. For any left cell L of W containing some fully commutative element, our main result of the paper is to prove that there exists a unique element (say w^L) in $L \cap W_{\text{c}}$ such that any $z \in L$ has the form $z = xw^L$ with $\ell(z) = \ell(x) + \ell(w^L)$ for some $x \in W$. This implies that L is left connected, verifying a conjecture of Lusztig in our case.

Introduction.

Let $W = (W, S)$ be a Coxeter group with S the distinguished generator set. The fully commutative elements of W were defined by Stembridge: $w \in W$ is fully commutative, if any two reduced expressions of w can be transformed from each other by only applying the relations $st = ts$ with $s, t \in S$ and $o(st) = 2$ ($o(st)$ being the order of st), or equivalently, w has no reduced expression of the form $w = x(sts...)y$, where $sts...$ is a string of length $o(st) > 2$ for some $s \neq t$ in S. The fully commutative elements were studied extensively by a number of people (see [3], [6], [7], [15], [16], [17], [18]). Let W_{c} be the set of all the fully commutative elements in W.

Let W be a finite or an affine Coxeter group. The aim of this paper is to prove a structural property for any left cell of W containing some element of W_{c}: if $z \in W$ satisfies

Supported by Nankai Univ., the NSF of China, the SF of the Univ. Doctoral Program of ME of China, the Shanghai Priority Academic Discipline, and the CST of Shanghai

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

$z \underset{L}{\sim} w$ for some $w \in F'_{\rm c}$ (see 1.8) then z is a left extension of w (see 1.1 and Theorem 2.1). So any $w \in F'_{\rm c}$ is the unique shortest element in the left cell of W containing it and that $F'_{\rm c}$ forms a representative set for all the left cells L of W with $L \cap W_{\rm c} \neq \emptyset$.

A subset K of W is *left connected*, if for any $x, y \in K$, there exists a sequence of elements $x_0 = x, x_1, ..., x_r = y$ in K with some $r \geqslant 0$ such that $x_{i-1}x_i^{-1} \in S$ for every $1 \leqslant i \leqslant r$. Lusztig conjectured in [2] that if W is an affine Weyl group then any left cell L of W is left connected. The conjecture is supported by all the existing data. Then Theorem 2.1 verifies the conjecture in the case where L contains some element of $W_{\rm c}$.

Since the generalized Coxeter elements are fully commutative, this paper generalizes a result in my previous paper [14, Theorem 4.5]; the latter described any left cell of W containing some generalized Coxeter element.

Note that the conclusion of Theorem 2.1 was proved in [17] for the case where W is a Weyl or an affine Weyl group, using the knowledge of distinguished involutions of W in $W_{\rm c}$. The proof given in the present paper is independent of that in [17], without assuming the knowledge of distinguished involutions in $W_{\rm c}$, and is applicable to a more general case: W is a finite or an affine Coxeter group.

The contents of the paper are organized as follows. We collect some notations, terms and known results concerning cells and fully commutative elements of a Coxeter group W in Section 1. Then the main result of the paper is proved in Section 2.

§1. Some results on fully commutative elements.

Let (W, S) be a Coxeter system. In the Introduction we defined the set $W_{\rm c}$ of all the fully commutative elements of W. In this section, we collect some notations, terms and known results for later use.

1.1. Let $\leqslant$ be the Bruhat–Chevalley order and $\ell(w)$ the length function on W. Given $J \subseteq S$, let w_J be the longest element in the subgroup W_J of W generated by J. Call J *fully commutative* if the element w_J is so.

For $w, x, y \in W$, we use the notation $w = x \cdot y$ to mean $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$. In this case, we say that w is a *left* (resp., *right*) *extension* of y (resp., x), and say that y

(resp., x) is a *left* (resp., *right*) *retraction* of w. More generally, we say z is a *retraction* of w (or w is an *extension* of z), if $w = x \cdot z \cdot y$ for some $x, y \in W$.

We have the following results on the elements in W_{c}:

Lemma. (see [17, Lemma 1.1]) *For $w \in W_{\mathrm{c}}$, let $w = s_1 s_2 ... s_r$ be a reduced expression of w with $s_i \in S$.*

(1) The multi-set $\{s_1, s_2, ..., s_r\}$ only depends on w but not on the choice of a reduced expression.

(2) For any $s \in S$ with $sw \in W_{\mathrm{c}}$, the equation $sw = ws$ holds if and only if $ss_i = s_i s$ for any $1 \leqslant i \leqslant r$.

(3) If $s, t \in S$ satisfy $sw = wt \in W_{\mathrm{c}}$, then $s = t$.

(4) If $w \in W_{\mathrm{c}}$ then any retraction of w is also in W_{c}. In particular, if $w \in W_{\mathrm{c}}$ has an expression $w = x \cdot w_J \cdot y$ with $x, y \in W$ and $J \subseteq S$, then J is fully commutative.

1.2. Let $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) be the preorder on W defined as in [8], and let $\underset{L}{\sim}$ (resp., $\underset{R}{\sim}$, $\underset{LR}{\sim}$) be the equivalence relation on W determined by $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$). The corresponding equivalence classes are called *left* (resp., *right*, *two-sided*) *cells* of W.

1.3. For any $w \in W$, let $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$.

Assume $m = o(st) > 2$ for some $s, t \in S$. A sequence of elements

$$\underbrace{ys,\ yst,\ ysts,\ \ldots}_{m-1 \text{ terms}}$$

is called a *right* $\{s, t\}$*-string* (or just a *right string*) if $y \in W$ satisfies $\mathcal{R}(y) \cap \{s, t\} = \emptyset$.

We say that z is obtained from w by a *right* $\{s, t\}$*-star operation* (or a *right star operation* for brevity), if z, w are two neighboring terms in a right $\{s, t\}$-string. Clearly, a resulting element z of a right $\{s, t\}$-star operation on w, when it exists, need not be unique unless w is a terminal term of the right $\{s, t\}$-string containing it.

Similarly, we can define a *left* $\{s, t\}$*-string* and a *left* $\{s, t\}$*-star operation* on an element.

Lemma. *(1) If $x, y \in W$ can be obtained from each other by successively applying left (resp., right) star operations, then $x \underset{L}{\sim} y$ (resp., $x \underset{R}{\sim} y$).*

(2) The set W_c is invariant under star operations.

Proof. (1) follows easily from the definition of the relations $\underset{L}{\sim}$ and $\underset{R}{\sim}$ on W. (2) is just [17, Proposition 2.10]. $\square$

From now on, we always assume that W is a finite or an affine Coxeter group unless otherwise specified.

1.4. In [10], [11], Lusztig defined a function $a : W \longrightarrow \mathbb{N} \cup \{\infty\}$ and proved the following results (we further assume that W is a Weyl or an affine Weyl group when the results involve the function a).

(a) $a(w_J) = \ell(w_J)$ for $J \subseteq S$ with W_J finite (see [10, Proposition 2.4] and [11, Proposition 1.2]). In particular, when J is fully commutative, we have $a(w_J) = |J|$, the cardinality of the set J.

(b) If $x \underset{LR}{\leqslant} y$ in W, then $a(x) \geqslant a(y)$. So $x \underset{LR}{\sim} y$ implies $a(x) = a(y)$ (see [10, Theorem 5.4]).

(c) If $w = x \cdot y$ then $w \underset{L}{\leqslant} y$ and $w \underset{R}{\leqslant} x$.

(d) If $x \underset{L}{\leqslant} y$ and if either $x \underset{LR}{\sim} y$ or $a(x) = a(y)$ then $x \underset{L}{\sim} y$ (see [11, Corollary 1.9], [12, Subsection 1.7 (i)] and [1, Corollary 3.3]).

(e) The relation $x \underset{L}{\leqslant} y$ (resp., $x \underset{R}{\leqslant} y$) implies $\mathcal{R}(x) \supseteq \mathcal{R}(y)$ (resp., $\mathcal{L}(x) \supseteq \mathcal{L}(y)$). In particular, the relation $x \underset{L}{\sim} y$ (resp., $x \underset{R}{\sim} y$) implies $\mathcal{R}(x) = \mathcal{R}(y)$ (resp., $\mathcal{L}(x) = \mathcal{L}(y)$) (see [8, Proposition 2.4]).

By the notation x—y in W, we mean that $\max\{\deg P_{x,y}, \deg P_{y,x}\} = \frac{1}{2}(|\ell(x)-\ell(y)|-1)$, where $P_{x,y}$ is the celebrated Kazhdan–Lusztig polynomial associated to the ordered pair (x, y) in W, and we stipulate that the degree of the zero polynomial is $-\infty$.

(f) If $x, y \in W$ with x—y are in some right $\{s, t\}$-strings (not necessarily in the same right string) for some $s, t \in S$ with $st \neq ts$, then there exist some $x', y' \in W$ which are obtained from x, y respectively by a right $\{s, t\}$-star operation and satisfy x'—y' (see [10 Subsection 10.4]).

1.5. By a *graph*, we mean a finite set of nodes together with a finite set of edges. Two nodes of a graph are *adjacent* if they are joined by an edge. A *directed graph* (or a *digraph* for brevity) is a graph with each edge oriented. A *directed edge* (i.e., an edge with orientation) with two incident nodes $\mathbf{v}, \mathbf{v}'$ is denoted by an ordered pair $(\mathbf{v}, \mathbf{v}')$, if the orientation is from $\mathbf{v}$ to $\mathbf{v}'$. A node $\mathbf{s}$ of a digraph $\mathbf{G}$ is a *source* (resp., a *sink*) if $(\mathbf{s}, \mathbf{s}')$ (resp., $(\mathbf{s}', \mathbf{s})$) is a directed edge of $\mathbf{G}$ for any node $\mathbf{s}'$ adjacent to $\mathbf{s}$. A source or a sink of $\mathbf{G}$ is also called an *extreme node*. A *directed path* ξ of $\mathbf{G}$ is a sequence of nodes $\mathbf{v}_0, \mathbf{v}_1, ..., \mathbf{v}_r$ in $\mathbf{G}$ with $r \geqslant 0$ such that $(\mathbf{v}_{i-1}, \mathbf{v}_i)$ is a directed edge of $\mathbf{G}$ for $1 \leqslant i \leqslant r$. A *subdigraph* of a digraph $\mathbf{G}$ is a digraph which can be obtained from $\mathbf{G}$ by removing some nodes and all the directed edges incident to these removed nodes.

1.6. To an expression

$$\chi: \quad w = s_1 s_2 ... s_r \tag{1.6.1}$$

(not necessarily reduced) of any $w \in W$ with $s_i \in S$, we associate a digraph $\mathbf{G}(\chi)$ as follows. The node set $\mathbf{V}(\chi)$ of $\mathbf{G}(\chi)$ is $\{\mathbf{s}_i \mid 1 \leqslant i \leqslant r\}$ (with the convention that $\mathbf{s}_i \neq \mathbf{s}_j$ for any $i \neq j$), and the directed edge set $\mathbf{E}(\chi)$ of $\mathbf{G}(\chi)$ consists of all the ordered pairs $(\mathbf{s}_i, \mathbf{s}_j)$ satisfying the conditions $i < j$, $s_i s_j \neq s_j s_i$ and that there does not exist any $i = h_0 < h_1 < ... < h_t = j$ with $t > 1$ such that $s_{h_{p-1}} s_{h_p} \neq s_{h_p} s_{h_{p-1}}$ for every $1 \leqslant p \leqslant t$. The digraph $\mathbf{G}(\chi)$ so obtained usually depends on the choice of an expression χ of w. However, if two expressions of w can be obtained from each other by only applying the relations of the form $st = ts$ for some $s, t \in S$ with $o(st) = 2$, then their corresponding digraphs are the same. In particular, when w is in W_{c} and an expression χ of w in (1.6.1) is reduced, the digraph $\mathbf{G}(\chi)$ only depends on the element w, but not on the particular choice of a reduced expression χ of w. In this case, it makes sense to denote $\mathbf{G}(\chi)$, $\mathbf{V}(\chi)$, $\mathbf{E}(\chi)$ by $\mathbf{G}(w)$, $\mathbf{V}(w)$, $\mathbf{E}(w)$, respectively. Call $\mathbf{G}(w)$ the *associated digraph* of w.

By the above construction of a digraph $\mathbf{G}(w)$ for $w \in W_{\mathrm{c}}$, there exists a natural map $\phi : \mathbf{s}_i \mapsto s_i$ from $\mathbf{V}(w)$ to S and hence $\mathbf{V}(w)$ can be regarded as a multi-set in S.

Note that the above definition of the digraph $\mathbf{G}(w)$ can be regarded as a reformulation of Viennot's notion of a heap (see [19]). The digraph $\mathbf{G}(w)$ is also a certain kind of dependence graph (see [5] for example).

1.7. By [18, Proposition 2.3], we see that an element w of W is in W_{c} if and only if there exists some (and hence any) reduced expression $\chi : w = s_1 s_2 ... s_r$ such that the following two conditions hold:

(1.7.1) for any pair $i < j$ with $s_i = s_j$, there exists a directed path in $\mathbf{G}(\chi)$ connecting the nodes $\mathbf{s}_i$ and $\mathbf{s}_j$.

(1.7.2) for any directed path $\mathbf{s}_{i_1}, \mathbf{s}_{i_2}, ..., \mathbf{s}_{i_m}$ in $\mathbf{G}(\chi)$ with $s_{i_h} = s_{i_{h+2}}$ for $1 \leqslant h \leqslant m-2$ and $m = o(s_{i_1} s_{i_2}) > 2$, there always exists another directed path with $\mathbf{s}_{i_1}, \mathbf{s}_{i_m}$ two extreme nodes.

For $w \in W_{\rm c}$, $\mathcal{L}(w)$ (resp., $\mathcal{R}(w)$) (see 1.3) is exactly the set of all $s \in S$ with $\phi^{-1}(s)$ containing a source (resp., a sink) of $\mathbf{G}(w)$. Let $s \in \mathcal{L}(w)$ (resp., $s \in \mathcal{R}(w)$). Then both $\mathcal{L}(w) \not\supseteq \mathcal{L}(sw)$ and $\mathcal{L}(w) \not\subseteq \mathcal{L}(sw)$ (resp., $\mathcal{R}(w) \not\supseteq \mathcal{R}(ws)$ and $\mathcal{R}(w) \not\subseteq \mathcal{R}(ws)$) hold if and only if the removal of the source (resp., sink) $\mathbf{s}$ from $\mathbf{G}(w)$ yields a new source (resp., sink) in the resulting digraph.

For $w \in W_{\rm c}$, there is an expression $w = x \cdot w_J \cdot y$ for some $J \subseteq S$ and $x, y \in W$ if and only if there is a node set $\mathbf{J}$ of $\mathbf{G}(w)$ with $\phi(\mathbf{J}) = J$ and $|\mathbf{J}| = |J|$ such that

(1.7.3) for any $\mathbf{s} \neq \mathbf{t}$ in $\mathbf{J}$, there is no directed path connecting $\mathbf{s}$ and $\mathbf{t}$ in $\mathbf{G}(w)$.

Denote by $n(w)$ the maximum possible cardinality of a node set $\mathbf{J}$ of $\mathbf{G}(w)$ satisfying condition (1.7.3). Then $n(w)$ is also the maximum possible value of $\ell(w_J)$ in an expression $w = x \cdot w_J \cdot y$, or equivalently, the maximum size of an antichain in the corresponding heap.

1.8. Let $F_{\rm c}$ be the set of all the elements w in $W_{\rm c}$ such that $\mathcal{L}(sw) \subset \mathcal{L}(w)$ (or equivalently, $\mathcal{L}(sw) = \mathcal{L}(w) \setminus \{s\}$) for any $s \in \mathcal{L}(w)$. Denote by $F'_{\rm c}$ the set of all the elements w in $F_{\rm c}$ such that $n(sw) < n(w) = |\mathcal{L}(w)|$ for any $s \in \mathcal{L}(w)$. Let $F''_c = F_{\rm c} \setminus F'_{\rm c}$.

Proposition. *Assume that W is a finite or an affine Coxeter group.*

(1) If $w \in F'_{\rm c}$ then any right retraction of w is also in $F'_{\rm c}$ (see [17, Lemma 3.14]*).*

(2) If W is irreducible and $w \in F''_{\rm c}$, then $s \leqslant w$ for any $s \in S$ (see [17, Lemma 3.11]*).*

Now assume that W is a Weyl or an affine Weyl group.

(3) $a(w) = n(w)$ for any $w \in W_{\rm c}$ (see [17, Theorem 3.1], [4, Theorem 4.1]*).*

(4) $a(w) = |\mathcal{L}(w)|$ for any $w \in F_{\rm c}$ (see [17, Corollary 3.18]*).*

The next result is concerned with some further properties of $w \in F'_{\mathrm{c}}$.

Lemma 1.9. (see [17, Lemma 3.15]) *Let W be a Weyl or an affine Weyl group.*

(1) For any $w \in F'_{\mathrm{c}}$, there exists a sequence of elements $x_0 = w, x_1, ..., x_r = w_K$ in F'_{c} with $K = \mathcal{L}(w)$ such that x_i can be obtained from x_{i-1} by a right star operation and $x_i < x_{i-1}$ for every $1 \leqslant i \leqslant r$.

(2) For any $w \in W_{\mathrm{c}}$, there exists some $y \in F'_{\mathrm{c}}$ such that y is a left retraction of w with $y \underset{L}{\sim} w$ and $n(y) = n(w)$.

§**2. Left cells of W containing some element of W_{c}.**

In this section, we consider all the left cells of W containing some $w \in W_{\mathrm{c}}$. Since any $w \in W_{\mathrm{c}}$ has the form $w = x \cdot y$ with $w \underset{L}{\sim} y$ for some $y \in F'_{\mathrm{c}}$ and $x \in W_{\mathrm{c}}$ (see Lemma 1.9 (2)), we may assume $w \in F'_{\mathrm{c}}$ without loss of generality. Say an element $x \in W$ satisfies *condition (A)*, if

(A) $x \underset{L}{<} sx$ (i.e., $x \underset{L}{\leqslant} sx$ but $x \underset{L}{\not\sim} y$) for any $s \in \mathcal{L}(x)$.

The main result of the paper is to prove

Theorem 2.1. *Let W be an irreducible finite or affine Coxeter group. Let $w \in F'_{\mathrm{c}}$ and $z \in W$ satisfy $z \underset{L}{\sim} w$.*

(1) If the element z satisfies condition (A), then $z = w$.

(2) If $z \in F'_{\mathrm{c}}$ then $z = w$.

(3) In general, we have $z = x \cdot w$ for some $x \in W_{\mathrm{c}}$.

We break the proof of Theorem 2.1 up into some lemmas.

Lemma 2.2. *Suppose that we are given $w \in F'_{\mathrm{c}}$ with $m = \ell(w)$. Let $z \in W$ satisfy $z \underset{L}{\sim} w$ and condition (A). Assume that we are in the following case:*

(1) W is an irreducible Weyl or affine Weyl group;

(2) $m > |\mathcal{L}(w)|$ and the assertion of Theorem 2.1 has been proved when $\ell(w) < m$;

(3) $w = w_J \cdot y$ satisfies $n(w) = J$ and $y \in W_{\mathrm{c}}$;

(4) w, z can be transformed to $w' = ws', z' = zs$ respectively by a right $\{s, s'\}$-star operation with $s' \in \mathcal{R}(y)$ and $w' \underset{L}{\sim} z'$, where $s, s' \in S$ satisfy $ss' \neq s's$.

Then we have

(a) $z = x \cdot w_{J'} \cdot y$ *with* $J' = J \setminus \{s\}$ *for some* $x \in W$*;*

(b) the element s *commutes with but is not equal to any* $v \in S$ *with* $v \leqslant w_{J'} \cdot ys'$*.*

Proof. Note that any $x \in F'_{\mathrm{c}}$ satisfies condition (A) by Proposition 1.8 (3) and 1.4 (d). We have $w' \in F'_{\mathrm{c}}$ by Proposition 1.8 (1).

We claim that z' does not satisfy condition (A). For otherwise, one would have $z' = w' = ws'$ by the assumption (2) with w', z' in the place of w, z respectively. By the condition $s' \in \mathcal{R}(y) \subseteq \mathcal{R}(w)$, we have $s \in \mathcal{R}(ws')$ since the transformation of sending w to ws' is a right $\{s, s'\}$-star operation. So $z = z's = ws's$ is a right retraction of w and hence is in F'_{c} by Proposition 1.8 (1). Since $\ell(z) < \ell(w) = m$, we have $z = w$ by the assumption (2) with z, w in the place of w, z respectively, which is impossible.

So there exists some $t \in \mathcal{L}(z')$ satisfying $tz' \underset{L}{\sim} z'$. We have $\mathcal{R}(z) = \mathcal{R}(w)$ by 1.4 (e) and the condition $z \underset{L}{\sim} w$. Then $s' \in \mathcal{R}(z)$ and $z' = z \cdot s$. There is a reduced expression of z':

(2.2.1) $$z' = s_1 ... s_a s' s \qquad \text{with } s_i \in S.$$

We claim that we have a reduced expression of tz':

(2.2.2) $$tz' = s_1 ... s_a s.$$

For otherwise, we would have a reduced expression either $tz' = s_1 ... s_a s'$ or $tz' = s_1 ... \widehat{s_i} ... s_a s' s$ for some $1 \leqslant i \leqslant a$ by the exchange condition on W, where $\widehat{s_i}$ means the deletion of the factor s_i. If $tz' = s_1 ... s_a s'$, then we would have $z' \underset{L}{<} tz'$ by [10 Corollary 5.5], a contradiction. Also, if $tz' = s_1 ... \widehat{s_i} ... s_a s' s$, then $tz = s_1 ... \widehat{s_i} ... s_a s'$, which can be obtained from tz' by a right $\{s, s'\}$-star operation, and hence $tz \underset{R}{\sim} tz' \underset{L}{\sim} z' \underset{R}{\sim} z$. This would imply $tz \underset{LR}{\sim} z$ and hence $tz \underset{L}{\sim} z$ by 1.4 (c)–(d), contradicting condition (A) on z since $t \in \mathcal{L}(z') = \mathcal{L}(z)$ by 1.4 (e).

By 1.4 (d), we can write $tz' = x \cdot z''$ for some $x, z'' \in W$, where z'' satisfies conditions $z'' \underset{L}{\sim} tz'$ and (A). Since $z'' \underset{L}{\sim} w'$, $w' \in F'_{\mathrm{c}}$ and $\ell(w') < \ell(w)$, we have $z'' = w'$ by the assumption (2) with w', z'' in the place of w, z respectively. So $tz' = x \cdot w'$. We can write

$y = y' \cdot s'$ for some $y' \in W$ by the fact that $s' \in \mathcal{R}(y)$. Then

(2.2.3) $$z' = t \cdot x \cdot w_J \cdot y'$$

by the assumption (3). Hence

(2.2.4) $$z = (t \cdot x \cdot w_J \cdot y')s.$$

By the fact $s \in \mathcal{R}(ws') = \mathcal{R}(w_J y')$ and by the exchange condition on W, we have either

(2.2.5) $$z = t \cdot x \cdot w_{J'} \cdot y' \quad \text{with } J' \subset J \text{ and } |J'| = |J| - 1$$

or

(2.2.6) $$z = t \cdot x \cdot w_J \cdot y'' \quad \text{with } y'' < y' \text{ and } \ell(y'') = \ell(y') - 1.$$

We claim that the case (2.2.6) could not occur. For otherwise, since $|J| = a(w) = a(z)$ by 1.4 (b) and Proposition 1.8 (3), we would have $z \underset{L}{\sim} tz$ by 1.4 (d), contradicting the assumption that z satisfies condition (A) since $t \in \mathcal{L}(z)$. In particular, we have

(2.2.7) $$w_{J'} y' s = w_J \cdot y' = w' \in F'_{\mathrm{c}} \subseteq W_{\mathrm{c}}.$$

This implies $J \setminus J' = \{s\}$ by Lemma 1.1 (3). So we get from (2.2.3) and (2.2.7) that

(2.2.8) $$z' = t \cdot x \cdot w_{J'} \cdot y' \cdot s$$

and hence

(2.2.9) $$tz' = x \cdot w_{J'} \cdot y' \cdot s.$$

On the other hand, we have

(2.2.10) $$tz' = (t \cdot x \cdot w_{J'} \cdot y')s' \cdot s$$

by (2.2.8) and (2.2.1)–(2.2.2). Comparing (2.2.9) with (2.2.10), we get

$$t\cdot x\cdot w_{J'}\cdot y'=x\cdot w_{J'}\cdot y'\cdot s'. \tag{2.2.11}$$

This implies by (2.2.5) and (2.2.11) that

$$z=t\cdot x\cdot w_{J'}\cdot y'=x\cdot w_{J'}\cdot y'\cdot s'=x\cdot w_{J'}\cdot y \tag{2.2.12}$$

So (a) is proved.

For (b), the conclusion that s commutes with any $v\in S$ with $v\leqslant w_{J'}\cdot y'$ follows by (2.2.7) and Lemma 1.1 (2). If $s\leqslant w_{J'}\cdot y'$, then $s\leqslant y'$ and thus $\ell(w_J\cdot y')<\ell(w_J)+\ell(y')$ by $J=J'\cup\{s\}$, which is absurd. So $s\not\leqslant w_{J'}\cdot y'=w_{J'}\cdot ys'$. □

Lemma 2.3. *Keep all the assumptions of Lemma 2.2 on the elements $w\in F'_{\rm c}$ and $z\in W$ (in particular, $w=w_J\cdot y$ and $J=J'\cup\{s\}=\mathcal{L}(w)$). Then the element w can also be transformed to wu' by a right $\{u,u'\}$-star operation for some $u\in S$, $u'\in\mathcal{R}(y)$ with $uu'\neq u'u$ and $s\notin\{u,u'\}$.*

Proof. We have a reduced expression $y=y'\cdot s'$ of y for some $y'\in W$. Let $w_1=sw=w_{J'}\cdot y$. Then $w_1\in W_{\rm c}$. By (2.2.7), we have $w=s\cdot w_1=w_{J'}y'ss'$. We claim that w_1 can be transformed to w_1u' by a right $\{u,u'\}$-star operation for some $u'\in\mathcal{R}(y)$ and $u\in S$ with $uu'\neq u'u$. For otherwise, we would have $w_1^{-1}\in F_{\rm c}$. Then $|\mathcal{L}(w_1)|+1=|\mathcal{L}(w)|>|\mathcal{R}(w)|=|\mathcal{R}(w_1)|=|\mathcal{L}(w_1^{-1})|\geqslant|\mathcal{R}(w_1^{-1})|=|\mathcal{L}(w_1)|$ by the assumptions that $w\in F'_{\rm c}$ and $\ell(w)>|\mathcal{L}(w)|$, by the fact that $\mathbf{s}$ is not a sink of $\mathbf{G}(w)$ (since $\{s,s'\}\cap\mathcal{R}(w)=\{s'\}$ by the assumption in Lemma 2.2 (4)), and by Proposition 1.8 (4). This implies $|\mathcal{L}(w_1)|=|\mathcal{R}(w_1)|$ and hence $w_1^{-1}\in F''_{\rm c}$ (see 1.8) by the fact that $\ell(w_1^{-1})>|\mathcal{L}(w_1^{-1})|$. On the other hand, we have $s\not\leqslant w_1=w_{J'}\cdot y=w_{J'}\cdot y'\cdot s'$ by Lemma 2.2 (b). This is impossible by Proposition 1.8 (2). Our claim is proved. The claim implies that $u,u'\leqslant w_1$. Since $s\not\leqslant w_1$, we have $s\notin\{u,u'\}$. Hence $w=s\cdot w_1$ also can be transformed to wu' by a right $\{u,u'\}$-star operation. □

Lemma 2.4. *Theorem 2.1 is true when $W=H_3,H_4,I_2(m)$ ($m=5$ or $\geqslant 7$).*

Proof. By [9, Proposition 3.8], we see that the result is true in the case where $w \in F_{\rm c}'$ has a unique reduced expression. In particular, the result is true when $W = I_2(m)$. It remains to consider the case where $W \in \{H_3, H_4\}$, and $w \in F_{\rm c}'$ has more than one reduced expression. Now assume that we are in such a case. Let W_1 be the set of all the fully commutative elements of W each of which has more than one reduced expression. Then we get the following facts:

(i) First we claim that W_1 is a single two-sided cell of W.

By [15, 3.5], we know that W_1 is a union of two-sided cells of W. Then the claim follows by [1, Section 3] for $W = H_4$, where W_1 is the two-sided cell E in the notation of [1]; and by a direct calculation for $W = H_3$, where W_1 consists of 25 elements, which is a union of 5 left (resp., right) cells.

(ii) Next we claim that $W_1 = \{y \in W_{\rm c} \mid n(y) = 2\}$.

This is because $n(w) \leqslant 2$ for any $w \in W$, $n(x) = 1$ for any $x \in W_{\rm c} \setminus W_1$ and $n(y) > 1$ for any $y \in W_1$.

(iii) Let $w \in W_1$. When $W = H_3$, let $S = \{s_1, s_2, s_3\}$ satisfy $(s_1s_2)^5 = (s_2s_3)^3 = 1$. Define $z = s_1s_3 \cdot s_2 \cdot s_1 \cdot s_2 \cdot s_3$. Then we claim that w is in $F_{\rm c}'$ if and only if w is a right retraction of z with $\mathcal{L}(w) = \{s_1, s_3\}$. When $W = H_4$, let $S = \{s_1, s_2, s_3, s_4\}$ satisfy $(s_1s_2)^5 = (s_2s_3)^3 = (s_3s_4)^3 = 1$. Define $z_1 = s_1s_3 \cdot s_2 \cdot s_1 \cdot s_2 \cdot s_3 \cdot s_4$, $z_2 = s_1s_4$ and $z_3 = s_2s_4 \cdot s_3$. Then we claim that w is in $F_{\rm c}'$ if and only if w is a right retraction of z_i with $|\mathcal{L}(w)| = 2$ for some $1 \leqslant i \leqslant 3$. For, by (ii), we see that $w \in W_1$ is in $F_{\rm c}'$ if and only if $|\mathcal{L}(w)| = 2$ and $n(sw) = 1$ for any $s \in \mathcal{L}(w)$. Then the above two claims follow by a direct calculation.

By 1.4 (c)–(d), the above (i)–(ii) and Lemma 1.9 (2), we see that any $z \in W_1$ can be written in the form $z = x \cdot w$ for some $w \in F_{\rm c}' \cap W_1$ and some $x \in W$ with $z \underset{L}{\sim} w$. This, in particular, implies that $z \in W_1$ satisfies condition (A) only if $z \in F_{\rm c}'$. By (iii), we see that any $w \in F_{\rm c}' \cap W_1$ satisfies condition (A). By comparing their generalized τ-invariants (see [13, Section 4] for the definition), we see that two elements x, y of $F_{\rm c}' \cap W_1$ satisfy $x \underset{L}{\sim} y$ if and only if $x = y$. Hence our result on H_3 and H_4 follows. □

2.5. *Proof of Theorem 2.1.* By Lemma 2.4, we need only consider the case where W is an irreducible Weyl or affine Weyl group. Now assume that we are in such a case.

First we prove (2)–(3) under the assumption of (1). Since any element of F'_{c} satisfies condition (A) by 1.4 (d) and Proposition 1.8 (3)–(4), assertion (2) is an immediate consequence of (1). For (3), we can write $z = x \cdot z'$ for some $x, z' \in W$ with z' satisfying the conditions $z' \underset{L}{\sim} z$ and (A). Then we have $z' = w$ by (1) and hence $z = x \cdot w$.

So it remains to prove (1). We can write $w = w_J \cdot y$ with $J = \mathcal{L}(w)$ for some $y \in W_{\mathrm{c}}$. Hence $|J| = n(w) = a(w)$ by Proposition 1.8 (3)–(4). We apply induction on $\ell(y) \geqslant 0$. When $\ell(y) = 0$, we have $w = w_J$. The condition $z \underset{L}{\sim} w$ implies that $\mathcal{R}(z) = \mathcal{R}(w) = J$ and $|J| = a(z)$ by 1.4 (e), (b), (a). Hence $z = x \cdot w_J$ for some $x \in W$. Then condition (A) on z further implies $z = w_J$. Next assume $\ell(y) > 0$. By Lemma 1.9 (1), the element w can be transformed to $w' = ws'$ by a right $\{s, s'\}$-star operation for some $s \in S$, $s' \in \mathcal{R}(y)$ with $ss' \neq s's$. Then $w' \in F'_{\mathrm{c}}$ by Proposition 1.8 (1). At least one (say z') of zs' and zs is obtained from z by a right $\{s, s'\}$-star operation and satisfies $z' \underset{L}{\sim} w'$ by 1.4 (e)–(f). If $z' = zs'$, then z' is a right retraction of z with $z' \underset{R}{\sim} z$ by the fact that $s' \in \mathcal{R}(w) = \mathcal{R}(z)$. For any $r \in \mathcal{L}(z')$, we have $r \in \mathcal{L}(z)$ by 1.4 (e), hence $z' \underset{R}{\sim} z \underset{L}{<} rz \underset{R}{\leqslant} rzs' = rz'$ by the assumption that z satisfies condition (A). This implies $z' \underset{LR}{<} rz'$ and further $z' \underset{L}{<} rz$, i.e., z' also satisfies condition (A). By the inductive hypothesis, we have $z' = w'$ and hence $z = z's' = w's' = w$, as required.

Now assume $z' = zs$. Then all the assumptions (1)–(4) of Lemma 2.2 on w, z hold, where the assumptions (1)–(3) hold by our inductive hypothesis, while the assumption (4) holds by the above discussion and the assumption $z' = zs$. Hence by Lemmas 2.2 and 2.3, we have $z = x \cdot w_{J'} \cdot y$ with $J' = J \setminus \{s\}$ for some $x \in W$, and w can also be transformed to $w'' = wu'$ by a right $\{u, u'\}$-star operation for some $u \in S$ and $u' \in \mathcal{R}(y)$ with $uu' \neq u'u$ and $s \notin \{u, u'\}$. By the same argument as above, we can prove the following assertions:

(i) $w'' \in F'_{\mathrm{c}}$;

(ii) at least one (say z'') of zu' and zu is obtained from z by a right $\{u, u'\}$-star operation and satisfies $z'' \underset{L}{\sim} w''$;

(iii) if $z'' = zu'$ then $z'' = w''$ and hence $z = w$;

(iv) if $z'' = zu$ then $z = x' \cdot w_{J''} \cdot y$ with $J'' = J \setminus \{u\}$ for some $x' \in W$.

We claim that the cases of $z = x \cdot w_{J'} \cdot y$ and $z = x' \cdot w_{J''} \cdot y$ can't happen simultaneously. For otherwise, we would have $x \cdot u = x' \cdot s$. Since $s \neq u$, this implies $s \in \mathcal{R}(x)$, contradicting the fact that $xw_J = x \cdot w_J$ (see (2.2.3)). So we must have $z = w$ by the assertions (ii)–(iv). This completes our proof. $\square$

Theorem 2.1 tells us that any $w \in F'_c$ is the unique shortest element in the left cell L_w of W containing w and that any $z \in L_w$ has the form $z = x \cdot w$ for some $x \in W$.

Remark 2.6. (1) The proof of Lemma 2.2 follows the line of the corresponding part in the proof of [14, Theorem 4.7]. However, the remaining part in the proof of Theorem 2.1 is new.

(2) Corollaries 4.10 and 4.11 in [17] are the consequence of Theorem 2.1.

References

1. D. Alvis, *The left cells of the Coxeter group of type H_4*, J. Algebra **107** (1987), 160–168.
2. T. Asai et al., *Open problems in algebraic groups.*, Problems from the conference on algebraic groups and representations held at Katata (Ryoshi Hotta (ed.)), August 29–September 3, 1983.
3. S. C. Billey and G. S. Warrington, *Kazhdan–Lusztig polynomials for 321-hexagon-avoiding permutations*, J. Algebraic Combin. **13** (2001), 111–136.
4. K. Bremke and C. K. Fan, *Comparison of a-functions*, J. Algebra **203** (1998), 355–360.
5. V. Diekert and G. Rozenberg (ed.), *The book of traces.*, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.
6. C. K. Fan and J. R. Stembridge, *Nilpotent orbits and commutative elements*, J. Algebra **196** (1997), 490–498.
7. R. M. Green and J. Losonczy, *Fully commutative Kazhdan–Lusztig cells*, Ann. Inst. Fourier (Grenoble) **51** (2001), 1025–1045.
8. D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.
9. G. Lusztig, *Some examples in square integrable representations of semisimple p-adic groups*, Trans. of the Amer. Math. Soc. **277** (1983), 623–653.
10. G. Lusztig, *Cells in affine Weyl groups*, in " Algebraic Groups and Related Topics "(R. Hotta, ed.), Advanced Studies in Pure Math., Kinokuniya and North Holland,(1985), 255–287.
11. G. Lusztig, *Cells in affine Weyl groups, II*, J. Algebra **109** (1987), 536–548.
12. J. Y. Shi, *The Kazhdan–Lusztig cells in certain affine Weyl groups*, vol. 1179, Springer-Verlag, Lecture Notes in Mathematics, 1986.
13. J. Y. Shi, *Left cells in affine Weyl groups*, Tôhoku Math. J. **46** (1994), 105–124.
14. J. Y. Shi, *Coxeter elements and Kazhdan–Lusztig cells*, J. Algebra **250** (2002), 229–251.
15. J. Y. Shi, *Fully commutative elements and Kazhdan–Lusztig cells in the finite and affine Coxeter groups*, Proc. of Amer. Math. Soc. **131** (2003), 3371–3378.
16. J. Y. Shi, *Fully commutative elements and Kazhdan–Lusztig cells in the finite and affine Coxeter groups, II*, Proc. Amer. Math. Soc. **133** (2005), 2525-2531.

17. J. Y. Shi, *Fully commutative elements in the Weyl and affine Weyl groups*, J. Algebra **284 (1)** (2005), 13–36.
18. J. R. Stembridge, *On the fully commutative elements of Coxeter groups*, J. Algebraic Combin. **5** (1996), 353–385.
19. G. X. Viennot, *Heaps of pieces, I: basic definitions and combinatorial lemmas*, Combinatoire Énumerative (G. Labelle and P. Leroux, ed.), Springer-Verlag, Berlin, 1986, pp. 321–350.