# $q$-Identities related to overpartitions and divisor functions

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**Abstract.** We generalize and prove two conjectures of Corteel and Lovejoy, related to overpartitions and divisor functions.

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## 1. Introduction

In this paper, for any pair positive integers $m, n$, we prove the following two identities:

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1}(x+1)\cdots(x+q^{i-1})}{(1-q^i)^m} q^{mi} = \sum_{i=1}^{n} \frac{(-1)^{i-1}(x^i-(-1)^i)}{1-q^i} q^i \sum_{i\le i_2\le\cdots\le i_m\le n} \frac{q^{\sum_{j=2}^{m} i_j}}{\prod_{j=2}^{m}(1-q^{i_j})}, \quad (1.1)$$

$$\frac{(z;q)_{n+1}}{(q;q)_n} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1}(x+1)\cdots(x+q^{i-1})}{1-zq^i} q^i = \sum_{i=0}^{n} (-1)^{i-1} \frac{(z;q)_i}{(q;q)_i} x^i q^i, \quad (1.2)$$

using the classical notations $(z;q)_i = (1-z)\cdots(1-zq^{i-1})$, and $\begin{bmatrix} n \\ i \end{bmatrix} = \frac{(q;q)_n}{(q;q)_i(q;q)_{n-i}}$.

In the next section, we shall show that (1.1) and (1.2) can be obtained from the Newton interpolation in points $\{-1, -q, -q^2, \ldots\}$, using the complete symmetric function in the variables $\{q/(1-q), q^2/(1-q^2), \ldots\}$.

Given $\mathbb{X} = \{x_1, x_2, \ldots\}$, Newton gave the following interpolation formula, for any function $f(x)$:

$$f(x) = f(x_1) + f\partial_1(x - x_1) + f\partial_1\partial_2(x - x_1)(x - x_2) + \cdots,$$

where $\partial_i$, acting on its left, is defined by

$$f(x_1, \ldots, x_i, x_{i+1}, \ldots)\partial_i = \frac{f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.$$

Taking $f(x) = x^n$, we have,

$$x_1^n \partial_1 \cdots \partial_i = h_{n-i}(x_1, x_2, \ldots, x_{i+1}), \tag{1.3}$$

where $h_k$ is the complete symmetric function of degree $k$ defined by

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Recall the following properties of $h_k$:

1. Given an alphabet $\mathbb{X}$, the generating function of $h_k$ is

$$\sum_{k=0}^{\infty} h_k(\mathbb{X}) t^k = \frac{1}{\prod_{x \in \mathbb{X}} (1 - xt)}. \tag{1.4}$$

In particular, Take $\mathbb{X} = \{z, zq, zq^2, \ldots\}$, $\mathbb{X}_n = \{z, zq, zq^2, \ldots, zq^{n-1}\}$. We have

$$\sum_{k=0}^{\infty} h_k(z, zq, zq^2, \ldots) t^k = \frac{1}{\prod_{i=0}^{\infty} (1 - tzq^i)},$$

and

$$\sum_{k=0}^{\infty} h_k(z, zq, zq^2, \ldots, zq^{n-1}) t^k = \frac{1}{\prod_{i=0}^{n-1} (1 - tzq^i)}.$$

In consequence of the $q$-binomial theorem

$$\sum_{i=0}^{\infty} \frac{(a;q)_i}{(q;q)_i} t^i = \frac{(at;q)_\infty}{(t;q)_\infty},$$

it follows that

$$h_k(z, zq, zq^2, \ldots) = \frac{z^k}{(q;q)_k}. \tag{1.5}$$

Putting $a = q^n$ in the $q$-binomial theorem gives

$$\sum_{i=0}^{\infty} \begin{bmatrix} n+i-1 \\ i \end{bmatrix} t^i = \frac{1}{(t;q)_n}.$$

Thus we have

$$h_k(z, zq, zq^2, \ldots, zq^{n-1}) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix} z^k. \tag{1.6}$$

2. More generally, given two alphabets $\mathbb{X}$ and $\mathbb{Y}$, the generating functions of $h_k(\mathbb{X}+\mathbb{Y})$ and $h_k(\mathbb{X}-\mathbb{Y})$ are

$$\sum_{k=0}^{\infty} h_k(\mathbb{X}+\mathbb{Y})t^k = \frac{1}{\prod_{x\in\mathbb{X}}(1-xt)\prod_{y\in\mathbb{Y}}(1-yt)},$$
$$\sum_{k=0}^{\infty} h_k(\mathbb{X}-\mathbb{Y})t^k = \frac{\prod_{y\in\mathbb{Y}}(1-yt)}{\prod_{x\in\mathbb{X}}(1-xt)}. \quad (1.7)$$

As a consequence, one has

$$h_n(\mathbb{X}+\mathbb{Y}) = \sum_{k=0}^{n} h_k(\mathbb{X})h_{n-k}(\mathbb{Y}). \quad (1.8)$$

3. Given $\{x_1, x_2, \ldots, x_n\}$, and a positive integer $m$, we have

$$\sum_{i=1}^{n} x_i h_{m-1}(x_i, x_{i+1}, \ldots, x_n) = h_m(x_1, x_2, \ldots, x_n). \quad (1.9)$$

Taking $\mathbb{X} = \{-1, -q, -q^2, \ldots\}$, it is easy to check from (1.3) and (1.6):

$$x_1^n \partial_1 \cdots \partial_i = h_{n-i}(-1, -q, \ldots, -q^i) = (-1)^{n-i}\begin{bmatrix} n \\ i \end{bmatrix}.$$

# 2. Proofs of (1.1) and (1.2)

The Gauss polynomials $\begin{bmatrix} n \\ k \end{bmatrix}$ satisfy the following recursion (cf.[1]):

$$\sum_{j=0}^{n} \begin{bmatrix} m+j \\ m \end{bmatrix} q^j = \begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix}. \quad (2.1)$$

In this paper, we need the following more general relations.

**Lemma 2.1** *Let $k$, $m$ and $n$ be nonnegative integers. Then we have the following formulas:*

$$\sum_{i=k}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} \sum_{i\le i_2\le\cdots\le i_m\le n} \frac{q^{\sum_{j=2}^{m} i_j}}{\prod_{j=2}^{m}(1-q^{i_j})} = \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{km}}{(1-q^k)^m}, \quad (2.2)$$

*and*

$$\sum_{i=0}^{n} \frac{(z;q)_i}{(q;q)_i} q^i = \frac{(zq;q)_n}{(q;q)_n}. \quad (2.3)$$

*Proof.*

Taking $\mathbb{X} = \{1, q, \ldots, q^l\}$ and $\mathbb{Y} = \{q^{l+1}, q^{l+2}, \ldots\}$, we obtain from (1.5), (1.6) and (1.8):

$$\frac{1}{(q;q)_n} = h_n(\mathbb{X}+\mathbb{Y}) = \sum_{i=0}^{n} h_i(\mathbb{Y}) h_{n-i}(\mathbb{X}) = \sum_{i=0}^{n} \frac{1}{(q;q)_i} \begin{bmatrix} n-i+l \\ l \end{bmatrix} q^{(l+1)i}. \tag{2.4}$$

Letting $f(m)$ be the left side of (2.2), we have

$$\begin{aligned}
\sum_{m=1}^{\infty} f(m) z^m &= \sum_{m=1}^{\infty} \sum_{i=k}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} h_{m-1}\left( \frac{q^i}{1-q^i}, \frac{q^{i+1}}{1-q^{i+1}}, \ldots, \frac{q^n}{1-q^n} \right) z^m \\
&\overset{(1.4)}{=} z \sum_{i=k}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} \frac{1}{(1-q^i z/(1-q^i)) \cdots (1-q^n z/(1-q^n))} \\
&= z \sum_{i=k}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} \sum_{l=0}^{\infty} (q^i;q)_{n-i+1} \begin{bmatrix} n-i+l \\ l \end{bmatrix} (q^i(1+z))^l \\
&= z \begin{bmatrix} n \\ k \end{bmatrix} \sum_{i=k}^{n} \frac{(q;q)_{n-k}}{(q;q)_{i-k}} \sum_{l=0}^{\infty} \begin{bmatrix} n-i+l \\ l \end{bmatrix} q^{(l+1)i} \sum_{m=0}^{l} \binom{l}{m} z^m \\
&= z \begin{bmatrix} n \\ k \end{bmatrix} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \binom{l}{m} z^m \sum_{i=k}^{n} \frac{(q;q)_{n-k}}{(q;q)_{i-k}} \begin{bmatrix} n-i+l \\ l \end{bmatrix} q^{(l+1)i} \\
&\overset{(2.4)}{=} \begin{bmatrix} n \\ k \end{bmatrix} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \binom{l}{m} z^{m+1} q^{(l+1)k} \\
&= \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^k z}{1-q^k(1+z)} \\
&= \sum_{m=1}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{km}}{(1-q^k)^m} z^m.
\end{aligned}$$

Taking $\mathbb{X} = \{1\}$, $\mathbb{Y} = \{q, q^2, \ldots\}$ and $\mathbb{Z} = \{zq, zq^2, \ldots\}$, from (1.7), (1.8) and the $q$-binomial theorem, we get the proof of (2.3):

$$\begin{aligned}
\frac{(zq;q)_n}{(q;q)_n} &= h_n((\mathbb{X}+\mathbb{Y}) - \mathbb{Z}) \\
&= h_n(\mathbb{X} + (\mathbb{Y}-\mathbb{Z})) = \sum_{i=0}^{n} h_i(\mathbb{Y}-\mathbb{Z}) h_{n-i}(\mathbb{X}) = \sum_{i=0}^{n} \frac{(z;q)_i}{(q;q)_i} q^i.
\end{aligned}$$

∎

Taking

$$f(x) = \sum_{i=1}^{n} \frac{(-1)^{i-1}(x^i - (-1)^i)}{1-q^i} q^i \sum_{i \le i_2 \le \cdots \le i_m \le n} \frac{q^{\sum_{j=2}^{m} i_j}}{\prod_{j=2}^{m} (1-q^{i_j})},$$

and

$$\mathbb{X} = \{-1, -q, -q^2, \ldots\},$$

we have,

$$\begin{aligned}
f(x) &= f(x_1) + \sum_{k=1}^{n} f(x_1)\partial_1 \cdots \partial_k \,(x+1)\cdots(x+q^{k-1}) \\
&= \sum_{k=1}^{n}\sum_{i=k}^{n}(-1)^{k-1}\begin{bmatrix} i \\ k \end{bmatrix}\frac{q^i}{1-q^i}\sum_{i\le i_2\le\cdots\le i_m\le n}\frac{q^{\sum_{j=2}^{m} i_j}}{\prod_{j=2}^{m}(1-q^{i_j})}(x+1)\cdots(x+q^{k-1}) \\
&= \sum_{k=1}^{n}(-1)^{k-1}(x+1)\cdots(x+q^{k-1})\sum_{i=k}^{n}\begin{bmatrix} i \\ k \end{bmatrix}\frac{q^i}{1-q^i}\sum_{i\le i_2\le\cdots\le i_m\le n}\frac{q^{\sum_{j=2}^{m} i_j}}{\prod_{j=2}^{m}(1-q^{i_j})} \\
&= \sum_{k=1}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(-1)^{k-1}(x+1)\cdots(x+q^{k-1})}{(1-q^k)^m}q^{mk},
\end{aligned}$$

as stated in (1.1). ∎

Taking

$$f(x) = \sum_{i=0}^{n}(-1)^{i-1}\frac{(z;q)_i}{(q;q)_i}x^i q^i, \quad \text{and} \quad \mathbb{X} = \{-1, -q, -q^2, \ldots\},$$

we have,

$$\begin{aligned}
f(x) &= \sum_{k=0}^{n} f(x_1)\partial_1 \cdots \partial_k \,(x+1)\cdots(x+q^{k-1}) \\
&= \sum_{k=0}^{n}(x+1)\cdots(x+q^{k-1})\sum_{i=k}^{n}(-1)^{k-1}\begin{bmatrix} i \\ k \end{bmatrix}\frac{(z;q)_i}{(q;q)_i}q^i \\
&= \sum_{k=0}^{n}(-1)^{k-1}(x+1)\cdots(x+q^{k-1})q^k\frac{(z;q)_k}{(q;q)_k}\sum_{i=k}^{n}\frac{(zq^k;q)_{i-k}}{(q;q)_{i-k}}q^{i-k} \\
&= \sum_{k=0}^{n}(-1)^{k-1}(x+1)\cdots(x+q^{k-1})q^k\frac{(z;q)_k}{(q;q)_k}\frac{(zq^{k+1};q)_{n-k}}{(q;q)_{n-k}} \\
&= \frac{(z;q)_{n+1}}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(-1)^{k-1}(x+1)\cdots(x+q^{k-1})}{1-zq^k}q^k,
\end{aligned}$$

which implies (1.2). ∎

## 3. Special Cases

In their study of overpartitions [3, Theorem 4.4], Corteel and Lovejoy obtained a combinatorial interpretation of the identity:

$$\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(-1;q)_i}{(q;q)_i}\frac{q^i}{1-q^i} = \sum_{i=1}^{\infty}\frac{2q^{2i-1}}{1-q^{2i-1}} = \sum_{i=1}^{\infty}\frac{2q^i}{1-q^{2i}},$$

and formulated, as conjectures, the following finite forms (private communication):

$$\sum_{i=1}^{2n-1} \begin{bmatrix} 2n-1 \\ i \end{bmatrix} \frac{(-1)^{i-1}(-1;q)_i}{(1-q^i)^m} q^{mi} = \sum_{i=1}^{n} \frac{2q^{2i-1}}{1-q^{2i-1}} \sum_{2i-1 \le i_2 \le \cdots \le i_m \le 2n-1} \frac{q^{\sum_{j=2}^m i_j}}{\prod_{j=2}^m (1-q^{i_j})}, \quad (3.1)$$

and

$$\sum_{i=1}^{2n} \begin{bmatrix} 2n \\ i \end{bmatrix} \frac{(-1)^{i-1}(-1;q)_i}{1-q^{i+2}} q^i = \sum_{i=1}^{n} \frac{2q^{2i-1}(1-q)}{(1+q^2)(1-q^{2i-1})(1-q^{2i+1})}. \quad (3.2)$$

In fact, (3.1) is the special case of (1.1) when $x = 1$. Taking $x = 1$, $z = q^2$, (1.2) can be reduced to (3.2).

The case $x = 0$, $m = 1$ of (1.1) is due to Van Hamme [6] (see also [2], [5], [8]):

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1-q^i} = \sum_{i=1}^{n} \frac{q^i}{1-q^i}.$$

Taking $x = 0$, and (1.9), we get the formula of Dilcher [4]:

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^{i-1} \frac{q^{\binom{i}{2}+mi}}{(1-q^i)^m} = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_m \le n} \frac{q^{i_1}}{1-q^{i_1}} \cdots \frac{q^{i_m}}{1-q^{i_m}}.$$

When $x = 0$ and $z = q^m$ in (1.2), we get Uchimura's identity [9]:

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+2}{2}}}{1-q^{i+m}} = \sum_{i=1}^{n} \frac{q^i}{1-q^i} \Big/ \begin{bmatrix} i+m \\ i \end{bmatrix}.$$

Note added in proof: Two different approaches to prove (1.1) and (1.2) were recently given by Prodinger [7] and Zeng [10].

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