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SIMPLE ROOT SYSTEMS AND PRESENTATIONS FOR CERTAIN COMPLEX REFLECTION GROUPS#

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We find representatives of all the equivalence classes of simple root systems (or r.e.s. for brevity) for the complex reflection groups G_{12}, G_{24}, G_{25} and G_{26}. Then we give representatives of all the congruence classes of (essential) presentations (or r.c.p. (r.c.e.p.) for brevity) for these groups by generators and relations. The method used in the paper is applicable to any finite (complex) reflection groups.

Key Words: Complex reflection groups; Presentations; Simple root systems.

Mathematics Subject Classification: 20F55; 20F05.

INTRODUCTION

Shephard and Todd (1954) classified all the finite complex reflection groups in the paper. Later Cohen (1976) gave a more systematic description for these groups in terms of root systems, vector graphs and root graphs. Recently, Howlett and Shi (2000) defined a simple root system (B, w) for the root system of such a group, which is analogous to the corresponding concept for a Coxeter group. In general, for a given finite complex reflection group G, a root system (R, f) is essentially unique but a simple root system (B, w) for (R, f) is not (even up to G-action), e.g., when $G = G_{33}, G_{34}$ (in the notations of Shephard and Todd, 1954, see Broue et al., 1998, Table 4). One can define an equivalence relation on the set of simple root systems for (R, f) (see 1.7). A natural question is to ask:

Problem A. How many equivalence classes of simple root systems are there in total for any irreducible finite complex reflection group G?

It is well known that any Coxeter group can be presented by generators and relations. A finite complex reflection group G can also be presented in a similar way (see Broue et al., 1998). But such a presentation is not unique for G in general. Different presentations of G may reveal various different properties of G (see Shi,

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2002 for example). Then it is worth to define a congruence relation among the presentations of G (see 1.9) and then to ask

Problem B. How many congruence classes of presentations are there for any irreducible finite complex reflection group G?

In the present paper, we solve these problems just for the finite primitive complex reflection groups $G = G_{12}, G_{24}, G_{25}, G_{26}$ (in the notations of Shephard and Todd, 1954). The other finite primitive complex reflection groups except G_{34} will be dealt with subsequently in two papers by my graduate students Li Wang and Peng Zeng (see Wang, 2003; Zeng, 2003). We choose to deal with these four groups as they represent four different cases. Among these groups, G_{12} is the only group the number of whose generating reflections is not equal to the dimension of the space it acts. The group G_{24} is generated by three reflections of order 2, G_{25} is generated by three reflections of order 3, while G_{26} is generated by three reflections of different orders: one of order 2 and two of order 3. Our method in dealing with Problem A is based on the knowledge of the action of G on its root system. Then the results so obtained, together with a known presentation of G, will be used in dealing with Problem B. We shall further show that the reduced forms for the presentations of the groups obtained in the paper are all essential (see 7.1 and Theorem 7.3). It is a relatively easy task if one is only content with getting r.c.e.p. for a given finite complex reflection group. But some of such presentations may have very complicated form and hence is not applicable in practice. So in the present paper we are trying to find the forms of r.c.e.p. as simple as possible. Then it becomes quite subtle in finding such forms and even more subtle in proving that they are indeed non-congruent and essential. The methods used in the paper are applicable to any other finite (complex) reflection group.

The contents are organized as follows. Section 1 is served as preliminaries, some definitions and results are collected there. Then we give r.e.s. for the group G_{12} in Sec. 2, and give r.c.p. for G_{12} in Sec. 3. We do the same for the groups G_{24}, G_{25}, G_{26} in Secs. 4, 5, 6, respectively. Finally we show in Sec. 7 that the reduced forms for all the presentations of the groups obtained in the paper are essential.

1. PRELIMINARIES

We collect some definitions and results concerning irreducible finite reflection groups, where 1.1–1.5 follow from Cohen's paper (Cohen, 1976) except for the definition of a simple root system which follows from Howlett and Shi (2000).

1.1. Let V be a complex vector space of dimension n. A *reflection* on V is a linear transformation on V of finite order with exactly $n-1$ eigenvalues equal to 1. A *reflection group* G on V is a finite group generated by reflections on V. The group G is *reducible* if it is a direct product of two proper reflection subgroups and *irreducible* otherwise. The action of G on V is said to be *irreducible* if V has no nonzero proper G-invariant subspace. In the present paper we shall always assume that G is irreducible and acts irreducibly on V. A reflection group G on V is called *complex*, if there does not exist any G-invariant $\mathbb{R}$-subspace V_0 of V such that the canonical map $\mathbb{C} \otimes_{\mathbb{R}} V_0 \to V$ is bijective. A reflection group G in V is called *primitive*, if there does not exist any decomposition $V = V_1 \oplus \cdots \oplus V_r$ of nontrivial proper subspaces V_i, $1 \leq i \leq r$, of V such that G permutes $\{V_i \mid 1 \leq i \leq r\}$.

Since G is finite, there exists a unitary inner product $(,)$ on V invariant under G. From now on we assume that such an inner product is fixed.

1.2. A *root* of a reflection on V is an eigenvector corresponding to the unique nontrivial eigenvalue of the reflection. A *root of* G is a root of a reflection in G.

Let s be a reflection on V of order $d > 1$. There is a vector $a \in V$ of length 1 and a primitive dth root ζ of unity such that $s = s_{a,\zeta}$, where $s_{a,\zeta}$ is defined by

$$s_{a,\zeta}(v) = v + (\zeta - 1)(v, a)a \quad \text{for all } v \in V. \tag{1.2.1}$$

We also write $s_{a,d}$ for $s_{a,\zeta}$ if $\zeta = e^{2\pi i/d}$. In this case, we often write $s_{a,d}$ simply by s_a if d is clear from the context.

For each $v \in V$, define $o_G(v)$ to be the order of the (necessarily cyclic) group that consists of the identity and the reflections in G which have v as a root. (This group is $G_W = \{g \in G \mid gu = u \text{ for all } u \in W\}$, where $W = v^\perp$.) Thus $o_G(v) > 1$ if and only if v is a root of G. If a is a root of G, then $o_G(a)$ will be called the *order* of a (with respect to G). We shall denote $o_G(a)$ simply by $o(a)$ when G is clear from the context.

The following results can be shown easily.

Lemma 1.3. (1) *$o(gv) = o(v) = o(cv)$ for all $v \in V$, $g \in G$ and $c \in \mathbb{C}\backslash\{0\}$.*

(2) *$gs_ag^{-1} = s_{g(a)}$ for any $g \in G$ and any root a of G.*

1.4. A pair (R, f) is called *a root system* in V, if

(i) R is a finite set of vectors v spanning V with $|(v, v)| = 1$.
(ii) $f : R \to \mathbb{N}\backslash\{1\}$ is a map such that $s_{a,f(a)}R = R$ and $f(s_{a,f(a)}(b)) = f(b)$ for all $a, b \in R$.
(iii) The group G generated by $\{s_{a,f(a)} \mid a \in R\}$ is a finite reflection group, and for all $a \in R$ and $c \in \mathbb{C}$,

$$ca \in R \Longleftrightarrow ca \in Ga.$$

The group G is called the reflection group associated with the root system (R, f). We have $o_G(a) = f(a)$ for any $a \in R$.

We shall denote a root system (R, f) simply by R when f is clear from the context.

1.5. A *simple root system* is a pair (B, w), where B is a finite set of vectors spanning V and w is a map from B to $\mathbb{N}\backslash\{1\}$, satisfying the following conditions:

(i) For all $a, b \in B$, we have $|(a, b)| = 1 \Longleftrightarrow a = b$.
(ii) The group G generated by $S = \{s_{a,w(a)} \mid a \in B\}$ is finite.
(iii) There is a root system (R, f) with $R = GB$ and $f(a) = w(a)$ for all $a \in B$.
(iv) The group G cannot be generated by fewer than $|B|$ reflections.

We call the elements of S *simple reflections*. We also call (R, f) the root system of G *generated by* B, and B a *simple root system* for R (or for G).

Note that we do not require B to be linearly independent. If B is linearly independent, then condition (iv) holds automatically.

The above definition of a simple system is considerably weaker than the usual definition for Coxeter groups; in particular, it is not always true that if B_1 and B_2 are simple root systems for the same root system R then there is an element $g \in G$ with $gB_1 = B_2$.

By Lemma 1.3 we see that if $\alpha \in B$ and $\beta \in \mathbb{C}\alpha \cap R$, then $B' = (B \setminus \{\alpha\}) \cup \{\beta\}$ also forms a simple root system for R.

1.6. Let B (resp. B') be a subset of V and $w : B \longrightarrow \mathbb{N}\setminus\{1\}$ (resp. $w' : B' \longrightarrow \mathbb{N}\setminus\{1\}$) be a function. Let G (resp. G') be the reflection group generated by $\{s_{\alpha,w(\alpha)} \mid \alpha \in B\}$ (resp. $\{s_{\alpha,w'(\alpha)} \mid \alpha \in B'\}$). If there exists a bijection $\phi : B \longrightarrow B'$ such that $w(\alpha) = w'(\phi(\alpha))$ and $(\alpha, \beta) = (\phi(\alpha), \phi(\beta))$ for any $\alpha, \beta \in B$. Then by the theory of linear algebra, we see that the assignment $s_{\alpha,w(\alpha)} \mapsto s_{\phi(\alpha),w'(\phi(\alpha))}$ determines a group isomorphism from G to G'.

1.7. Let (R, f) be a root system with G the associated reflection group and B a simple root system for R. Then it is known that as a root system for G, R is determined by G up to scalar factors (see Howlett and Shi, 2000, 1.9). However, a simple root system B for R is not uniquely determined by G. Two simple root systems (B, w) and (B', w') for (R, f) are *equivalent*, written $B \sim B'$, if there exists a bijection $\phi : B \longrightarrow B'$ such that for any $\alpha, \beta \in B$,

(1) $w(\alpha) = w'(\phi(\alpha))$ and,
(2) $\langle s_{\alpha,w(\alpha)}, s_{\beta,w(\beta)} \rangle \cong \langle s_{\phi(\alpha),w'(\phi(\alpha))}, s_{\phi(\beta),w'(\phi(\beta))} \rangle$, where the notation $\langle s, t \rangle$ stands for the group generated by s, t.

In particular, two simple root systems B, B' for R are equivalent if one of the following cases occurs:

(a) $gB = B'$ for some $g \in G$;
(b) Condition (2) is replaced by condition (2′) below in the above definition of $B \sim B'$:
(2′) $|(\alpha, \beta)| = |(\phi(\alpha), \phi(\beta))|$ for any $\alpha, \beta \in B$.

In general, not all the simple root systems are equivalent for a given reflection group. So it is natural to ask Problem A in Introduction.

Suppose that we have got one simple root system for a given irreducible finite reflection group G. Then the following result provides a criterion for a subset of a root system to be a simple root system.

Proposition 1.8. *Let G be a finite reflection group with (R, f) the associated root system and B a simple root system for R. Let B' be a subset of R and let $G_{B'}$ be the group generated by the reflections $s_{\alpha,f(\alpha)}$, $\alpha \in B'$.*

(1) *B' forms a simple root system for G if and only if $G_{B'}B' = R$ and $|B'| = |B|$.*
(2) *If $B \subseteq G_{B'}B'$ then $G_{B'}B' = R$.*

Proof. (1) The implication "$\Longrightarrow$" is obvious by the definition of a simple root system. For the reverse implication, we need only check conditions (i) and (ii) in 1.5. Since $G_{B'}B' = R$, we have $G_{B'} = G$ by Lemma 1.3. So (ii) follows by the assumption that G is finite. Since (R, f) is a root system, any vector v in R (in particular, when v is in B') satisfies $|(v, v)| = 1$ and hence $|(v, v')| \neq 1$ for any non-proportional $v, v' \in R$. So to show that (B', w') satisfies (i), it is enough to show that B' does not contain two proportional vectors α, β. For otherwise, the reflections with respect to the vectors in the proper subset $B' \setminus \{\beta\}$ of B' would generate the group G, contradicting the assumptions that $|B'| = |B|$ and that B is a simple root system for G.

(2) Any reflection in the group $G_{B'}$ has the form s_α with $\alpha \in G_{B'}B'$. Since $B \subseteq G_{B'}B'$ and $G = \langle s_\beta \mid \beta \in B \rangle$, we have $G_{B'} \supseteq G$ and hence $G_{B'}$ contains all the reflections s_β, $\beta \in R$. This implies the inclusion $R \subseteq G_{B'}B'$ and hence $G_{B'}B' = R$. $\square$

1.9. Given a reflection group G, *a presentation of G by generators and relations* (or *a presentation* in short) is by definition a pair $(S, \mathscr{P})$, where

(1) S is a finite generator set for G which consists of reflections, and S has minimal cardinality with this property.
(2) $\mathscr{P}$ is a finite set of relations on S, and any other relation on S is a consequence of the relations in $\mathscr{P}$.

Two presentations $(S, \mathscr{P})$ and $(S', \mathscr{P}')$ for G are *congruent*, if there exists a bijection $\eta : S \longrightarrow S'$ such that for any $s, t \in S$,

$$\langle s, t \rangle \cong \langle \eta(s), \eta(t) \rangle. \qquad (*)$$

In this case, we see by taking $s = t$ that the order $o(r)$ of r is equal to the order $o(\eta(r))$ of $\eta(r)$ for any $r \in S$.

If there does not exist such a bijection η, then we say that they are *non-congruent*.

For any reflections s, t in G, it is known (see for example Koster, 1975) that there exists a positive integer k such that $sts \cdots = tst \cdots$ (k factors on each side). Denote by $n(s, t)$ the smallest such k. It is easily seen that when $o(s) = o(t) = 2$, condition $(*)$ is equivalent to condition

$$n(s, t) = n(\eta(s), \eta(t)). \qquad (**)$$

However, when either $o(s)$, $o(t) > 2$ or $\max\{o(s), o(t)\} = 5$, one can show that neither of $(*)$ and $(**)$ implies the other in general by checking all the cases where a complex reflection group is irreducible and is generated by exactly two reflections.

1.10. Given a simple root system (B, w) for G, say $B = \{\alpha_1, \ldots, \alpha_r\}$, denote by $\mathbf{n}(B)$ the $\binom{r}{2}$-tuple $(X_{12}, X_{13}, \ldots, X_{1r}, X_{23}, X_{24}, \ldots, X_{2r}, \ldots, X_{r-1,r})$ with $\binom{r}{2} = \frac{r(r+1)}{2}$, where $X_{ij} = \langle s_{\alpha_i, w(\alpha_i)}, s_{\alpha_j, w(\alpha_j)} \rangle$ for $1 \leq i < j \leq r$. In particular, when w is the constant function 2, we can replace the group X_{ij} by the number $n(s_{\alpha_i,2}, s_{\alpha_j,2})$ for any i, j in the definition of $\mathbf{n}(B)$. This is because in that case, X_{ij} becomes a dihedral group which is entirely determined by the number $n(s_{\alpha_i,2}, s_{\alpha_j,2})$.

1.11. For each presentation $(S, \mathscr{P})$ for G, there exists a simple root system (B_S, w_S) for G (called an associated simple root system of $(S, \mathscr{P})$) such that each element of S has the form $s_{\alpha,w_S(\alpha)}^{k_S(\alpha)}$ for some $\alpha \in B_S$ and some $k_S(\alpha) \in \mathbb{N}$ with $k_S(\alpha)$ coprime to $w_S(\alpha)$. By the minimality for the cardinality of S, we see that for any $\alpha, \beta \in B_S$, $s_{\alpha,w_S(\alpha)}^{k_S(\alpha)} = s_{\beta,w_S(\beta)}^{k_S(\beta)}$ in S if and only if $\alpha = \beta$. By properly choosing generators in a presentation $(S, \mathscr{P})$ for G, we can always make an associated simple root system (B_S, w_S) to satisfy $k_S(\alpha) = 1$ for all $\alpha \in B_S$.

Lemma. *Two presentations of G are congruent if and only if their associated simple root systems are equivalent* (see 1.7).

Proof. It follows directly from the definitions. □

Remark 1.12. (1) In the present paper, we shall deal with Problems A and B (see Introduction) for the groups $G = G_{12}$, G_{24}, G_{25}, and G_{26}. As a basic work in our method, we calculate the group $\langle s_\alpha, s_\beta \rangle$ (or the value $n(s_\alpha, s_\beta)$ when $f(\alpha) = f(\beta) = 2$) for any pair of reflections $s_\alpha \neq s_\beta$ in G with respect to $\alpha, \beta \in R$, and also calculate all the permutations of R induced by the action of the reflections s_α, $\alpha \in R$ (these results are not included explicitly in the paper). These can be done by computer in general.

(2) Howlett classified all the congruence classes of simple reflection sets for the groups G_{12} and G_{24} by a certain detailed analysis on the group-theoretic structure of these groups in a private communication. The method used in the paper is quite different from Howlett's, and could be applied without essential change for the other finite complex reflection groups.

2. SIMPLE ROOT SYSTEMS FOR THE GROUP G_{12}

2.1. Let ϵ_1, ϵ_2 be an orthonormal basis in a hermitian space V of dimension 2. Let $e_1 = \epsilon_1$ and $e_2 = ((-1+\sqrt{2}i)/2)\epsilon_1 + ((\sqrt{3}-\sqrt{6}i)/6)\epsilon_2$. Then e_1, e_2 is also a basis of V. Denote by (a, b) a vector $ae_1 + be_2$ in V. Let $R_{12} = \{\pm e_i \mid 1 \leq i \leq 12\}$, where

$$e_1 = (1,0),\quad e_2 = (0,1),\quad e_3 = (1,1),\quad e_4 = (1-\sqrt{2}i, 1),$$
$$e_5 = (-\sqrt{2}i, 1),\quad e_6 = (1, 1+\sqrt{2}i),\quad e_7 = (1, \sqrt{2}i),$$
$$e_8 = (\sqrt{2}i, -1+\sqrt{2}i),\quad e_9 = (1+\sqrt{2}i, \sqrt{2}i),\quad e_{10} = (1+\sqrt{2}i, -1+\sqrt{2}i),$$
$$e_{11} = (1-\sqrt{2}i, 2),\quad e_{12} = (2, 1+\sqrt{2}i).$$

Let s_i be the reflection in V of order 2 with respect to e_i for $1 \leq i \leq 12$. Then the action of s_i on e_j, $1 \leq i, j \leq 3$, are as follows.

e_i	$s_1(e_i)$	$s_2(e_i)$	$s_3(e_i)$
e_1	$-e_1$	e_6	e_8
e_2	e_4	$-e_2$	$-e_9$
e_3	e_5	e_7	$-e_3$

The group generated by the set $S_1 = \{s_1, s_2, s_3\}$ is G_{12}. R_{12} is a root system for G_{12} with $B_1 = \{e_1, e_2, e_3\}$ a simple root system.

2.2. By definition in 1.5, we see that any simple root system B for the group G_{12} contains exactly three vectors. Since R_{12} is transitive under G_{12}, we may assume $e_1 \in B$ up to G_{12}-action. Consider all the possible orders $o(s_1 s_i)$ of the product $s_1 s_i$ for $1 < i \leq 12$. Then $o(s_1 s_{11}) = 2$, $o(s_1 s_i)$ is 3 for $i \in \{6, 8, 10, 12\}$, 4 for $i \in \{7, 9\}$ and 6 for $i \in \{2, 3, 4, 5\}$. Let G_B be the group generated by the reflections with respect to the vectors in B. It is easily checked that no $B \subset R_{12}$ with $e_1, e_{11} \in B$ and $|B| = 3$ satisfies the equation $G_B B = R_{12}$. So no simple root system for G_{12} contains a pair of orthogonal roots. We can also show that any $B \in \{\{e_1, e_6, e_{12}\}, \{e_1, e_8, e_{10}\}\}$ (these are the only triples in R_{12} which have the form $B = \{e_1, e_i, e_j\}$ with $i \neq j$ in $\{6, 8, 10, 12\}$ and $n(s_{e_i}, s_{e_j}) \leq 3$) satisfies the condition $G_B B \subsetneqq R_{12}$. So in a simple root system, there must exist some pair of roots α, β with $n(s_\alpha, s_\beta) > 3$ (see 1.9).

2.3. We can show that any of the following $B_i \subset R_{12}$ satisfies the equation $G_{B_i} B_i = R_{12}$ and hence is a simple root system for G_{12} by Proposition 1.8.

$$B_1 = \{e_1, e_2, e_3\}, \quad B_2 = \{e_1, e_2, e_{10}\}, \quad B_3 = \{e_1, e_2, e_7\},$$
$$B_4 = \{e_1, e_2, e_8\}, \quad B_5 = \{e_1, e_7, e_8\}.$$

Recall the notation $\mathbf{n}(B)$ in 1.10. We have $\mathbf{n}(B_1) = (6, 6, 6)$, $\mathbf{n}(B_2) = (6, 3, 3)$, $\mathbf{n}(B_3) = (6, 3, 4)$, $\mathbf{n}(B_4) = (6, 4, 6)$ and $\mathbf{n}(B_5) = (4, 3, 3)$. This implies that B_i, $1 \leq i \leq 5$, are pairwise inequivalent simple root systems for G_{12}. By 1.6 and by the knowledge of the values $n(s_\alpha, s_\beta)$ for all the non-proportional pairs α, β in R_{12}, we can further show that there is no more simple root system for G_{12} inequivalent to these five.

3. PRESENTATIONS OF THE GROUP G_{12}

In this section, we shall find r.c.p. for the group G_{12} by generators and relations according to the results in 2.4 (see Propositions 3.2–3.6).

3.1. Suppose that we are given a complex reflection group G, the associated root system R of G, and r.e.s. B_k, $1 \leq k \leq r$, for R. The strategy for finding r.c.p. for G is as follows. We start with the presentation of G given by Shephard-Todd which corresponds to a simple root system, say B_1, for R. Let S_k be the reflection set associated to the simple root system B_k. We make use of the relations among the simple root systems B_k, $1 \leq k \leq r$, to establish some transition among the S_k's by Lemma 1.3. Then for each $1 < k \leq r$, we get a presentation of G with the generator set S_k by some related transition. Suppose that we are given a presentation $(S_h, \mathcal{P})$ of G with the generator set S_h and the transition between the sets S_h and S_k. Then the simplest method to produce a presentation of G with the generator set S_k is just to substitute the reflections of S_h by those of S_k in the relations of $\mathcal{P}$ by the related transition. But it may happen that sometimes the resulting relations we get in the new presentation of G are quite complicated and hence are not very useful in practice. So we shall not always use such a method.

3.2. It is well known that the group G_{12} is presented by the generator set $S = \{s, u, t\}$ and the relations (see Broue et al., 1998):

(1) $s^2 = u^2 = t^2 = 1$.
(2) $suts = utsu = tsut$.

The reflections s_1, s_2 and s_3 in V satisfy relations (1), (2) with s_1, s_2, s_3 in the places of s, u, t, respectively. Since $B_1 = \{e_1, e_2, e_3\}$ is a simple root system for G_{12}, the assignment $\rho(s) = s_1$, $\rho(u) = s_2$ and $\rho(t) = s_3$ determines a faithful representation ρ of the group G_{12} to GL(V). This implies part (1) of the following

Proposition 3.3. (1) *The group G_{12} can be presented by the generator set $S_1 = \{s_1, s_2, s_3\}$ and the relations*:

(a1) $s_1^2 = s_2^2 = s_3^2 = 1$.
(a2) $s_1s_2s_3s_1 = s_2s_3s_1s_2$.
(a3) $s_1s_2s_3s_1 = s_3s_1s_2s_3$.

(2) *Under the presentation of G_{12} in* (1), *we have*

(a4) $s_1s_2s_1s_2s_1s_2 = s_2s_1s_2s_1s_2s_1$.
(a5) $s_2s_3s_2s_3s_2s_3 = s_3s_2s_3s_2s_3s_2$.
(a6) $s_1s_3s_1s_3s_1s_3 = s_3s_1s_3s_1s_3s_1$.
(a7) $o(s_1s_2s_3) = 8$.

Proof. We need only show (2). Let $x = s_1s_2s_3s_1$ and $y = s_1s_2s_3$. Then $x^6 = y^8$ and $o(x) = o(s_1s_2) = o(s_1s_3) = o(s_2s_3)$ by (a1)–(a3). Thus to show (a4)–(a7), it suffices to show that $x^6 = 1$, $x^2 \neq 1$ and $x^3 \neq 1$, where 1 is the identity transformation in V. These can be shown by observing the action of x^6, x^2, x^3 on the basis elements e_1, e_2 in V. □

3.4. We have, by Lemma 1.3(2), that

$$s_{10} = s_2s_3s_2s_3s_2 \quad \text{and} \quad s_3 = s_1s_2s_{10}s_1s_{10}s_2s_1. \tag{3.4.1}$$

Proposition. (1) *The group G_{12} can be presented by the generator set $S_2 = \{s_1, s_2, s_{10}\}$ and the relations*:

(b1) $s_1^2 = s_2^2 = s_{10}^2 = 1$.
(b2) $s_1s_2s_1s_2s_1s_2 = s_2s_1s_2s_1s_2s_1$.
(b3) $s_1s_{10}s_1 = s_{10}s_1s_{10}$.
(b4) $s_1s_{10}s_2 \cdot s_1 = s_2 \cdot s_1s_{10}s_2$.

(2) *Under the presentation of G_{12} in* (1), *we have*

(b5) $s_2s_{10}s_2 = s_{10}s_2s_{10}$.

Proof. (b5) follows from (b1), (b3) and (b4). This shows (2) under the assumption of (1). Now we show (1). We must show that relations (a1)–(a3) in Proposition 3.3 are equivalent to (b1)–(b4) here under transition (3.4.1). Clearly, (a1) is equivalent to (b1). Then we can show that relations (a2)–(a3) are equivalent to (b2)–(b4) under the assumption of (3.4.1) and (a1) (hence (b1)). □

Next three presentations of G_{12} correspond to the simple root systems B_i, $3 \leq i \leq 5$, respectively. We just state the results and leave the proofs to the readers.

3.5. We have

$$s_7 = s_2 s_3 s_2 \quad \text{and} \quad s_3 = s_2 s_7 s_2. \tag{3.5.1}$$

Proposition. (1) *The group G_{12} can be presented by the generator set $S_3 = \{s_1, s_2, s_7\}$ and the relations*:

(c1) $s_1^2 = s_2^2 = s_7^2 = 1$.
(c2) $s_1 s_2 s_7 \cdot s_1 = s_2 \cdot s_1 s_2 s_7$.
(c3) $s_7 s_2 s_1 \cdot s_7 = s_2 \cdot s_7 s_2 s_1$.

(2) *Under the presentation of G_{12} in* (1), *we have*

(c4) $s_1 s_7 s_1 s_7 = s_7 s_1 s_7 s_1$.
(c5) $s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1$.
(c6) $s_2 s_7 s_2 s_7 s_2 s_7 = s_7 s_2 s_7 s_2 s_7 s_2$.

3.6. We have

$$s_8 = s_3 s_1 s_3 \quad \text{and} \quad s_3 = s_1 s_2 s_8 s_2 s_1. \tag{3.6.1}$$

Proposition. (1) *The group G_{12} can be presented by the generator set $S_4 = \{s_1, s_2, s_8\}$ and the relations*:

(d1) $s_1^2 = s_2^2 = s_8^2 = 1$.
(d2) $s_2 s_1 s_2 s_8 \cdot s_2 = s_1 \cdot s_2 s_1 s_2 s_8$.
(d3) $s_8 \cdot s_1 s_2 s_8 s_2 = s_1 s_2 s_8 s_2 \cdot s_1$.

(2) *Under the presentation* (1) *of G_{12}, we have*

(d4) $s_1 s_8 s_1 = s_8 s_1 s_8$.
(d5) $s_2 s_8 s_2 s_8 = s_8 s_2 s_8 s_2$.
(d6) $s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1$.

3.7. We have

$$s_8 = s_1 s_7 s_2 s_7 s_1 \quad \text{and} \quad s_2 = s_7 s_1 s_8 s_1 s_7. \tag{3.7.1}$$

Proposition. *The group G_{12} can be presented by the generator set $S_5 = \{s_1, s_7, s_8\}$ and the relations*:

(e1) $s_1^2 = s_7^2 = s_8^2 = 1$.
(e2) $s_1 s_7 s_1 s_7 = s_7 s_1 s_7 s_1$.
(e3) $s_1 s_8 s_1 = s_8 s_1 s_8$.
(e4) $s_7 s_8 s_7 = s_8 s_7 s_8$.
(e5) $s_1 s_7 s_1 s_7 \cdot s_8 = s_8 \cdot s_1 s_7 s_1 s_7$.

4. SIMPLE ROOT SYSTEMS AND PRESENTATIONS FOR THE GROUP G_{24}

In this section, we shall find r.e.s. for the group G_{24} (see 4.2). Then we shall give r.c.p. for the group G_{24} by generators and relations according to these simple root systems (see Propositions 4.3–4.5).

4.1. Let V be a hermitian space of dimension 3 with an orthonormal basis $\epsilon_1, \epsilon_2, \epsilon_3$. Then $e_1 = \epsilon_2$, $e_2 = \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \alpha\epsilon_3)$ and $e_3 = \frac{\bar{\alpha}}{2}(\epsilon_2 + \epsilon_3)$ also form a basis of V, where $\bar{\alpha}$ denotes the complex conjugate of $\alpha = \frac{1}{2}(1 + i\sqrt{7})$, the latter is a root of the equation $x^2 - x + 2 = 0$. Denote by (a, b, c) a vector $ae_1 + be_2 + ce_3$. Let $R_{24} = \{\pm e_i \mid 1 \leq i \leq 21\}$, where

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1), \quad e_4 = (-1, 1, 0),$$
$$e_5 = (-\bar{\alpha}, 0, 1), \quad e_6 = (0, \alpha, 1), \quad e_7 = (1, 0, -\alpha), \quad e_8 = (0, 1, \bar{\alpha}),$$
$$e_9 = (-1, 1, 1), \quad e_{10} = (-\bar{\alpha}, 1, 1), \quad e_{11} = (-1, \alpha, 1), \quad e_{12} = (1, \bar{\alpha}, -\alpha),$$
$$e_{13} = (\alpha, 1, \bar{\alpha}), \quad e_{14} = (-\bar{\alpha}, 1, 1 + \bar{\alpha}), \quad e_{15} = (-1, \alpha, 1 + \alpha), \quad e_{16} = (\alpha, \bar{\alpha}, -\alpha),$$
$$e_{17} = (1, \bar{\alpha}, -1 - \alpha), \quad e_{18} = (\alpha, 1, 1 + \bar{\alpha}), \quad e_{19} = (-\bar{\alpha}, 2, 1 + \bar{\alpha}),$$
$$e_{20} = (-2, \alpha, 1 + \alpha), \quad e_{21} = (\alpha, \bar{\alpha}, \bar{\alpha}).$$

Denote by s_i the reflection in V with respect to the vector e_i, $1 \leq i \leq 21$. The action of s_i on e_j, $i, j = 1, 2, 3$, is as follows.

e_i	$s_1(e_i)$	$s_2(e_i)$	$s_3(e_i)$
e_1	$-e_1$	$-e_4$	e_7
e_2	e_4	$-e_2$	e_8
e_3	e_5	e_6	$-e_3$

The group generated by s_1, s_2, s_3 is G_{24}. R_{24} is the root system for G_{24} with $B_1 = \{e_1, e_2, e_3\}$ a simple root system.

4.2. We want to find r.e.s. for G_{24}. By Proposition 1.8, we have $|B_k| = 3$. Since R_{24} is transitive under G_{24}, we may assume $e_1 \in B_k$ up to G_{24}-action. The number $n(s_1, s_i)$ (see 1.9) is 2 for $i \in \{7, 17, 19, 21\}$, 3 for $i \in \{2, 4, 6, 11, 14, 15, 18, 20\}$ and 4 for $i \in \{3, 5, 8, 9, 10, 12, 13, 16\}$.

Any finite reflection group generated by s_1, s_i, s_j with $i \in \{7, 17, 19, 21\}$ is a homomorphic image of the finite Coxeter group of type A_3, B_3 or of the affine Coxeter group of type $\widetilde{C}_2$, hence it has order ≤ 48 and is not isomorphic to G_{24} (G_{24} has order 336). So a simple root system B_k for G_{24} contains no pair of orthogonal roots. By 1.6, Proposition 1.8 and the knowledge of the values $n(s_i, s_j)$, $1 \leq i, j \leq 21$, we see that the following are r.e.s. for R_{24}:

$$B_1 = \{e_1, e_2, e_3\}, \quad B_2 = \{e_1, e_3, e_4\} \quad \text{and} \quad B_3 = \{e_1, e_3, e_8\}.$$

We have $\mathbf{n}(B_1) = (3, 4, 4)$, $\mathbf{n}(B_2) = (4, 3, 3)$ and $\mathbf{n}(B_3) = (4, 4, 4)$ (see 1.10).

4.3. Next we shall find r.c.p. for G_{24} by generators and relations.

Proposition. *The group G_{24} can be presented by the generator set $S_1 = \{s_1, s_2, s_3\}$ and the relations*:

(a1) $s_1^2 = s_2^2 = s_3^2 = 1$.
(a2) $s_1s_3s_1s_3 = s_3s_1s_3s_1$.
(a3) $s_1s_2s_1 = s_2s_1s_2$.
(a4) $s_2s_3s_2s_3 = s_3s_2s_3s_2$.
(a5) $s_3 \cdot s_2s_1s_2s_3s_2 = s_2s_1s_2s_3s_2 \cdot s_1$.

Proof. This follows by Broue et al. (1998, Table 4) and by the argument as for Proposition 3.3(1). □

4.4. We have

$$s_4 = s_1s_2s_1 \quad \text{and} \quad s_2 = s_1s_4s_1. \tag{4.4.1}$$

Proposition. *The group G_{24} can be presented by the generator set $S_2 = \{s_1, s_3, s_4\}$ and the relations*:

(b1) $s_1^2 = s_3^2 = s_4^2 = 1$.
(b2) $s_1s_3s_1s_3 = s_3s_1s_3s_1$.
(b3) $s_1s_4s_1 = s_4s_1s_4$.
(b4) $s_3s_4s_3 = s_4s_3s_4$.
(b5) $s_1s_3s_1 \cdot s_4s_1s_3s_1s_4 = s_4s_1s_3s_1s_4 \cdot s_1s_3s_1$.

Proof. Under transition (4.4.1), relation (a1) is equivalent to (b1). Also, (a2) is the same as (b2). Then under relations (4.4.1) and (a1) (hence (b1)), relation (a3) (resp. (a4)) is equivalent to (b3) (resp. (b5)). Finally, under relations (4.4.1), (a1), (a3), (b1) and (b3), relation (a5) is equivalent to (b4). So our result follows by Proposition 4.3. □

4.5. We have

$$s_8 = s_3s_2s_3 \quad \text{and} \quad s_2 = s_3s_8s_3. \tag{4.5.1}$$

Proposition. (1) *The group G_{24} can be presented by the generator set $S_3 = \{s_1, s_3, s_8\}$ and the relations*:

(c1) $s_1^2 = s_3^2 = s_8^2 = 1$.
(c2) $s_1s_3s_1s_3 = s_3s_1s_3s_1$.
(c3) $s_3s_8s_3s_8 = s_8s_3s_8s_3$.
(c4) $s_3s_1s_3s_8s_3 \cdot s_1 = s_8 \cdot s_3s_1s_3s_8s_3$.
(c5) $s_8s_1s_8s_3s_8 \cdot s_1 = s_3 \cdot s_8s_1s_8s_3s_8$.

(2) *Under the presentation of G_{24} in* (1), *we have*

(c6) $s_1s_8s_1s_8 = s_8s_1s_8s_1$.

The proof of the Proposition is left to the readers.

5. SIMPLE ROOT SYSTEMS AND PRESENTATIONS FOR THE GROUP G_{25}

In this section, we deal with the complex reflection group G_{25}. Comparing with the groups G_{12}, G_{24}, a notable difference of G_{25} is that all of its reflections have order 3, rather than 2. There are two equivalence classes of the simple root systems and two congruence classes of the presentations for G_{25} (see 5.2 and Propositions 5.3, 5.4). The presentation given in Corollary 5.4 is congruent to that in Proposition 5.4.

5.1. Let V, ϵ_i be as in 4.1. Let $e_1 = \bar{\omega}\epsilon_3$, $e_2 = 3^{-1/2}i(\epsilon_1 + \epsilon_2 + \epsilon_3)$ and $e_3 = \bar{\omega}\epsilon_2$, where ω is a primitive cubic root of unity and $\bar{\omega}$ its complex conjugate. Then $B_1 = \{e_1, e_2, e_3\}$ forms a basis of V. Denote by (a, b, c) a vector $ae_1 + be_2 + ce_3$. Let $R_{25} = \{(-\omega)^k e_i \mid 1 \le i \le 12, 0 \le k < 6\}$, where

$$\begin{aligned} &e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1), \quad e_4 = (-\bar{\omega},1,0),\\ &e_5 = (1,1,0), \quad e_6 = (0,1,1), \quad e_7 = (0,1,-\bar{\omega}), \quad e_8 = (-\bar{\omega},1,1),\\ &e_9 = (1,1,1), \quad e_{10} = (1,1,-\bar{\omega}), \quad e_{11} = (-\bar{\omega},1,-\bar{\omega}),\\ &e_{12} = (-\bar{\omega},1-\bar{\omega},-\bar{\omega}). \end{aligned}$$

Let s_i be the reflection in V with respect to e_i, $1 \le i \le 12$. Then the group generated by $S_1 = \{s_1, s_2, s_3\}$ is the complex reflection group G_{25}. The set R_{25} forms a root system of G_{25} with B_1 a simple root system. The action of S_1 on B_1 is as follows.

e_i	$s_1(e_i)$	$s_2(e_i)$	$s_3(e_i)$
e_1	ωe_1	$-\omega e_4$	e_1
e_2	e_5	ωe_2	e_6
e_3	e_3	$-\omega e_7$	ωe_3

5.2. We want to find r.e.s. for G_{25}. By Proposition 1.8, we have $|B_k| = 3$. Since R_{25} is transitive under G_{25}, we may assume $e_1 \in B_k$ up to G_{25}-action. Let H_{ij} be the subgroup of G_{25} generated by s_i, s_j for $1 \le i \ne j \le 12$. Then by the action on R_{25} of the subgroups H_{1i} for any $1 < i \le 12$, we see that H_{1i} is isomorphic to the elementary 3-group $\mathbb{Z}_3^2$ for $i = 3, 12$, and is isomorphic to the group G_4 (in the notation of Shephard and Todd, 1954) for all the other i. From this fact, it is easily seen that r.e.s. for R_{25} is as follows.

$$B_1 = \{e_1, e_2, e_3\}, \quad B_2 = \{e_1, e_2, e_6\}.$$

We have $\mathbf{n}(B_1) = (G_4, G_4, \mathbb{Z}_3^2)$ and $\mathbf{n}(B_2) = (G_4, G_4, G_4)$ (see 1.10).

5.3. Next we shall give r.c.p. for G_{25} by generators and relations according to the result in 5.2.

Proposition. *The group G_{25} can be presented by the generator set $S_1 = \{s_1, s_2, s_3\}$ and the relations*:

(a1) $s_1^3 = s_2^3 = s_3^3 = 1$.
(a2) $s_1 s_2 s_1 = s_2 s_1 s_2$.
(a3) $s_2 s_3 s_2 = s_3 s_2 s_3$.
(a4) $s_1 s_3 = s_3 s_1$.

Proof. This follows by Broue et al. (1998, Table 1). □

5.4. We have

$$s_6 = s_2^{-1} s_3 s_2 \quad \text{and} \quad s_3 = s_2 s_6 s_2^{-1}. \tag{5.4.1}$$

Proposition. (1) *The group G_{25} can be presented by the generator set $S_2 = \{s_1, s_2, s_6\}$ and the relations*:

(b1) $s_1^3 = s_2^3 = s_6^3 = 1$.
(b2) $s_1 s_2 s_1 = s_2 s_1 s_2$.
(b3) $s_2 s_6 s_2 = s_6 s_2 s_6$.
(b4) $s_6 s_1 s_2 s_6 = s_2 s_6 s_1 s_2$.

(2) *Under the presentation of G_{25} in* (1), *we have*

(b5) $s_1 s_6 s_1 = s_6 s_1 s_6$.

With the relation $s_1 s_2 s_6 s_1 = s_6 s_1 s_2 s_6$ instead of (b3) in the proposition, we get another congruent presentation of G_{25}.

Corollary. *The group G_{25} can be presented by the generator set $S_2 = \{s_1, s_2, s_6\}$ and the relations*:

(b1) $s_1^3 = s_2^3 = s_6^3 = 1$.
(b2) $s_1 s_2 s_1 = s_2 s_1 s_2$.
(b3) $s_1 s_2 s_6 s_1 = s_2 s_6 s_1 s_2$.
(b4) $s_6 s_1 s_2 s_6 = s_2 s_6 s_1 s_2$.

The proof is left to the readers. Clearly, the presentation in the corollary is congruent to that in the proposition. With (b2) replaced by either (b3) or (b5) in the corollary, we get two more congruent presentations of G_{25}. One might notice that relations (b3′), (b4) are similar to (a2), (a3) in Proposition 3.3 for the group G_{12} with s_3 instead of s_6. But it should be indicated that the orders of the generators for G_{25} are different from those for G_{12}.

6. SIMPLE ROOT SYSTEMS AND PRESENTATIONS FOR THE GROUP G_{26}

In this section, we do the same for the complex reflection group G_{26}. Unlike the previous three groups, the generators of G_{26} have two different orders, and the root system of G_{26} consists of two G_{26}-orbits. There are two equivalence classes of simple root systems and two congruence classes of presentations for G_{26} (see 6.2 and Propositions 6.3, 6.4).

6.1. Let V, ϵ_i be as in 4.1. The vectors $e_1 = 2^{-1/2}(\epsilon_2 - \epsilon_3)$, $e_2 = \epsilon_3$ and $e_3 = 3^{-1/2}i(\epsilon_1 + \epsilon_2 + \epsilon_3)$ form a basis in V. Denote by (a, b, c) a vector $ae_1 + be_2 + ce_3$. Let $R_{26} = \{(-\omega)^j e_i \mid 1 \le i \le 21, j \in J\}$ be with

$$\begin{aligned} &e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1), \quad e_4 = (1,y,0), \\ &e_5 = (\sqrt{2},1,0), \quad e_6 = (0,1,-\bar{\omega}), \quad e_7 = (0,\bar{\omega},1), \quad e_8 = (-\omega,y,0), \\ &e_9 = (1,y,-\bar{\omega}y), \quad e_{10} = (\sqrt{2},1,-\bar{\omega}), \quad e_{11} = (\sqrt{2},1,\omega), \quad e_{12} = (-\omega,y,-\bar{\omega}y), \\ &e_{13} = (1,y,\omega y), \quad e_{14} = (\sqrt{2},1-\omega,-\bar{\omega}), \quad e_{15} = (\sqrt{2},2,\omega), \quad e_{16} = (-\omega,y,\omega y), \\ &e_{17} = (1,3/\sqrt{2},\omega y), \quad e_{18} = (-\sqrt{2}\omega,1-\omega,-\bar{\omega}), \quad e_{19} = (\sqrt{2},2,-\bar{\omega}), \\ &e_{20} = (2,3/\sqrt{2},\omega y), \quad e_{21} = (\sqrt{2},2,\sqrt{3}i), \end{aligned}$$

where $J = \{0, 1, 2, 3, 4, 5\}$ and $y = (1/\sqrt{2})(1 - \omega)$. Let s_i be the reflection in V with respect to the vector e_i. Then the action of s_i on e_j, $i, j = 1, 2, 3$, is as follows.

e_i	$s_1(e_i)$	$s_2(e_i)$	$s_3(e_i)$
e_1	$-e_1$	e_4	e_1
e_2	e_5	ωe_2	e_6
e_3	e_3	e_7	ωe_3

The group generated by s_1, s_2, s_3 is G_{26}. R_{26} is the root system for G_{26} with $B_1 = \{e_1, e_2, e_3\}$ a simple root system.

6.2. Now we consider r.e.s. for the group G_{26}. By Proposition 1.8, we have $|B_k| = 3$. The set R_{26} consists of two G_{26}-orbits: $R_1 = \{\omega^j e_i \mid i \in I, j \in J\}$ and $R_2 = \{\omega^j e_k \mid j \in J, k \in K\}$, where $I = \{1, 4, 8, 9, 12, 13, 16, 17, 20\}$ and $K = \{2, 3, 5, 6, 7, 10, 11, 14, 15, 18, 19, 21\}$. By Lemma 1.3 and condition $G_{26}B_k = R_{26}$, we have $B_k \cap R_i \neq \emptyset$ for $i = 1, 2$. We may assume $e_1 \in B_k$ up to G_{26}-action. By the action of the s_i's on R_{26}, we see that each B_k contains one root in R_1 and two roots in R_2. r.e.s. for R_{26} are as follows.

$$B_1 = \{e_1, e_2, e_3\}, \quad B_2 = \{e_1, e_2, e_6\}.$$

We have $\mathbf{n}(B_1) = (G(3,1,2), G_4, \mathbb{Z}_2 \times \mathbb{Z}_3)$ and $\mathbf{n}(B_2) = (G(3,1,2), G(3,1,2), G_4)$ (see 1.10 for $\mathbf{n}(B)$ and Shephard and Todd, 1954 for $G(3,1,2)$).

6.3. Next we shall give r.c.p. for G_{26} by generators and relations according to the result in 6.2.

Proposition. *The group G_{26} can be presented by the generator set $S_1 = \{s_1, s_2, s_3\}$ and the relations:*

(a1) $s_1^2 = s_2^3 = s_3^3 = 1$.
(a2) $s_1s_2s_1s_2 = s_2s_1s_2s_1$.
(a3) $s_2s_3s_2 = s_3s_2s_3$.
(a4) $s_1s_3 = s_3s_1$.

Proof. This follows by Broue et al. (1998, Table 1). □

6.4. We have

$$s_6 = s_2^{-1}s_3s_2 \quad \text{and} \quad s_3 = s_2s_6s_2^{-1}. \tag{6.4.1}$$

Proposition. (1) *The group G_{26} can be presented by the generator set $S_2 = \{s_1, s_2, s_6\}$ and the relations*:

(b1) $s_1^2 = s_2^3 = s_6^3 = 1$.
(b2) $s_1s_2s_1s_2 = s_2s_1s_2s_1$.
(b3) $s_2s_6s_2 = s_6s_2s_6$.
(b4) $s_2s_6s_1s_2 = s_6s_1s_2s_6$.

(2) *Under the presentation of G_{26} in* (1), *we have*

(b5) $s_1s_6s_1s_6 = s_6s_1s_6s_1$.

7. ESSENTIAL PRESENTATIONS

7.1. According to the definition in 1.9, if $(S, \mathscr{P})$ is a presentation of a reflection group G, and if $\mathscr{P}'$ is a set of relations satisfied in G, including $\mathscr{P}$ as a subset, then $(S, \mathscr{P}')$ is also a presentation of G.

A presentation $(S, \mathscr{P})$ of G is *essential* if $(S, \mathscr{P}_0)$ is not a presentation of G for any proper subset $\mathscr{P}_0$ of $\mathscr{P}$.

Checking the essentiality for a presentation of a group is usually a subtle task.

In a presentation $(S, \mathscr{P})$ of G, a relation $P \in \mathscr{P}$ is *redundant* if P is a consequence of the relations in $\mathscr{P}\backslash\{P\}$. Thus a presentation $(S, \mathscr{P})$ of G is essential if and only if $\mathscr{P}$ contains no redundant relation.

Let $(S, \mathscr{P})$ be one of the presentations we got so far for the groups $G = G_{12}, G_{24}, G_{25}, G_{26}$. We define $\mathscr{P}^{\text{re}}$, which is obtained from $\mathscr{P}$ in the following way: For any $s_i \in S$, let $\mathscr{C}_{s_i}$ be the set of all the elements in S which are conjugate to s_i. If $|\mathscr{C}_{s_i}| = k > 1$, let $h_i = \min\{j \mid s_j \in \mathscr{C}_{s_i}\}$. Then we remove $k-1$ relations $s_j^{o(s_j)} = 1$, $j > h_i$.

Example 7.2. (1) Let $(S, \mathscr{P})$ be the presentation of G_{12} given in Proposition 3.3. Then $\mathscr{P}^{\text{re}}$ is obtained from $\mathscr{P}$ by removing relations $s_2^2 = 1$ and $s_3^2 = 1$.

(2) Let $(S, \mathscr{P})$ be the presentation of G_{26} in Proposition 6.3. Then $\mathscr{P}^{\text{re}}$ is obtained from $\mathscr{P}$ by removing relation $s_3^3 = 1$.

The main result of the present section is as follows.

Theorem 7.3. *For any presentation $(S, \mathscr{P})$ we got so far for the group $G = G_{12}, G_{24}, G_{25}$ or G_{26}, the above defined $(S, \mathscr{P}^{\text{re}})$ is again a presentation of G. Moreover, $(S, \mathscr{P}^{\text{re}})$ is essential.*

We shall show the theorem in the remaining part of the section. The following is a simple fact in the group theory.

Lemma 7.4. *In a group G, if $x, y, z \in G$ satisfy $xy = yz$, then $o(x) = o(z)$.*

Then the conclusion that $(S, \mathscr{P}^{\mathrm{re}})$ is a presentation of the group G follows by Lemma 7.4. In Example 7.2(1), relations $s_2^2 = s_3^2 = 1$ are the consequence of relations $s_1^2 = 1$, $s_1 \cdot s_2s_3s_1 = s_2s_3s_1 \cdot s_2$ and $s_3 \cdot s_1s_2s_3 = s_1s_2s_3 \cdot s_1$ in $\mathscr{P}^{\mathrm{re}}$ (see Proposition 3.3, (a1)–(a3)). In Example 7.2(2), relation $s_3^3 = 1$ is a consequence of relations $s_2^3 = 1$ and $s_2 \cdot s_3s_2 = s_3s_2 \cdot s_3$ in $\mathscr{P}^{\mathrm{re}}$ (see Proposition 6.3, (a1)–(a4)). So in each of these two examples, $(S, \mathscr{P}^{\mathrm{re}})$ is a presentation of the respective group. All the other cases could be checked similarly. Thus it makes sense to call $(S, \mathscr{P}^{\mathrm{re}})$ the *reduced form* for the presentation $(S, \mathscr{P})$.

Let $(x1^{\mathrm{re}})$ be the set of relation(s) in the presentation $(S, \mathscr{P}^{\mathrm{re}})$ obtained from $(x1)$ in the presentation $(S, \mathscr{P})$ by removing certain relation(s) for $x = a, b, c, \ldots$. For example, when $(S, \mathscr{P})$ is the presentation of the group G_{12} (resp., G_{26}) in Proposition 3.3 (resp., Proposition 6.3), $(\mathrm{a}1^{\mathrm{re}})$ in $(S, \mathscr{P}^{\mathrm{re}})$ contains a single relation $s_1^2 = 1$ (resp., two relations $s_1^2 = 1$ and $s_2^3 = 1$).

The following fact is useful in checking the essentiality for a presentation of a complex reflection group.

Lemma 7.5. *Let G be a complex reflection group with $(S, \mathscr{P})$ its presentation.*

(1) *The relation set $\mathscr{P}$ contains at least one relation which either involves more than two generators, or has the form $s^m = 1$ with $m > 2$ for some $s \in S$.*
(2) *The relation(s) contained in $(x1^{\mathrm{re}})$, $x = a, b, \ldots$, is (are) not redundant in the reduced form of $(S, \mathscr{P})$ when $(S, \mathscr{P})$ is one of those we got in the previous sections.*
(3) *Let $(S, \mathscr{P}')$ be another presentation of G with $\mathscr{P}' \subseteq \mathscr{P}$. For any $P \in \mathscr{P}'$, if P is redundant in $\mathscr{P}'$, then P is redundant in $\mathscr{P}$.*

Proof. (1) is true since a complex reflection group is not a Coxeter group. Then (2) follows by the following fact: substituting $s_i = c$ for all $1 \leq i \leq 3$ and any fixed $c \in \mathbb{C}\backslash\{0\}$, all the relations in $\mathscr{P}\backslash\{P \mid P \in (x1)\}$ remain valid. Finally, as a consequence of relations in $\mathscr{P}'\backslash\{P\}$, P is clearly a consequence of relations in $\mathscr{P}\backslash\{P\}$. So we get (3). □

Thus it remains to show the essentiality of the reduced form $(S, \mathscr{P}^{\mathrm{re}})$ for any presentation $(S, \mathscr{P})$ we got so far for the group $G = G_{12}, G_{24}, G_{25}$ or G_{26}. The following result will be used for this purpose.

Lemma 7.6. *Suppose that a reflection group G has two presentations $(S, \mathscr{P})$ and $(S', \mathscr{P}')$, where $\mathscr{P} = \{P_i \mid i \in I\}$ and $\mathscr{P}' = \{P'_i \mid i \in I'\}$ are relation sets with $(I, \leq_I)$, $(I', \leq_{I'})$ finite posets (the word "poset" means "partially ordered set"). Suppose that there exists an order-preserving surjection $\phi : (I, \leq_I) \longrightarrow (I', \leq_{I'})$ such that for any $k \in I$, relation set $\mathscr{P}_k = \{P_i \mid i \leq_I k\}$ is equivalent to $\mathscr{P}'_{\phi(k)} = \{P'_{i'} \mid i' \leq_{I'} \phi(k)\}$ (i.e., $\mathscr{P}_k$ implies and is implied by $\mathscr{P}'_{\phi(k)}$). Suppose that $m, \phi(m)$ are maximal elements in I, I', respectively with $\phi^{-1}(\phi(m)) = \{m\}$. Then relation P_m is not redundant in $\mathscr{P}$ if and only if $P'_{\phi(m)}$ is not redundant in $\mathscr{P}'$.*

Proof. First show the implication "$\Longrightarrow$". We argue by contradiction. Suppose that $P'_{\phi(m)}$ is a consequence of all the other relations in $\mathscr{P}'$. By our assumption, we see that relation P_m is a consequence of the relations in $\{P'_i \mid i \leq_{I'} \phi(m)\}$ and hence a consequence of the relations in $\mathscr{P}'\backslash\{P'_{\phi(m)}\}$. Since relation set $\mathscr{P}\backslash\{P_m\}$ is equivalent to relation set $\mathscr{P}'\backslash\{P'_{\phi(m)}\}$ by our condition, this implies that P_m is a consequence of the relations in $\mathscr{P}\backslash\{P_m\}$, a contradiction.

The implication "$\Longleftarrow$" can be shown similarly by interchanging the roles of $m, I, \mathscr{P}$ with $\phi(m), I', \mathscr{P}'$, respectively. □

7.7. By Lemma 7.5(2), we see that relation $(x1^{\text{re}})$, $x = a, b, \ldots$, is not redundant in the respective reduced presentation for the group $G = G_{12}, G_{24}, G_{25}$ or G_{26}. In the following discussion, when we talk about the redundancy of a relation P in a (or reduced) presentation $(S, \mathscr{P})$ (or $(S, \mathscr{P}^{\text{re}})$) of G, we often simply state "P is not redundant" for "P is not redundant in $(S, \mathscr{P})$ (or $(S, \mathscr{P}^{\text{re}})$)" if $(S, \mathscr{P})$ (or $(S, \mathscr{P}^{\text{re}})$) is clear from the context. By Lemma 7.5(3), when a relation P is in $\mathscr{P}^{\text{re}}$, to show that P is not redundant in $\mathscr{P}^{\text{re}}$, we need only show that P is not redundant in $\mathscr{P}$ (we always do it this way in the subsequent discussion).

7.8. Let us first consider the group G_{24}. We can show that the generators s_1, s_2 are symmetric for the presentation of G_{24} in Proposition 4.3 in the following sense: suppose that relations (ai'), $1 \leq i \leq 3$, are obtained from (ai) by interchanging s_1, s_2, then two relation sets (a1)–(a3) and (a1′)–(a3′) can be obtained from one to another. We can also show that s_1, s_3 (resp. s_1, s_3, s_8) are symmetric for the presentation of G_{24} in Proposition 4.4 (resp. Proposition 4.5). Since the reflection group G_{24} is complex, relations (a5), (b5) are not redundant by Lemma 7.5(1), and neither are relations (a4) and (b4) by the proof of Proposition 4.4 and by Lemma 7.6. By symmetry, relations (a2) and (b3) are not redundant. Relations (a2), (b2) and (c2) are the same. So neither of (b2) and (c2) is redundant by the proof of Propositions 4.4, 4.5 and by Lemma 7.6. Again by the proof of Proposition 4.5 and by Lemma 7.6, relation (c5) is not redundant. By the symmetry of s_1, s_8 for the presentation of G_{24} in Proposition 4.5, neither of relations (c3) and (c4) is redundant. Finally, by the proof of Proposition 4.5 and by Lemma 7.6, relation (a3) is not redundant. This proves the essentiality of the reduced forms for all the presentations of G_{24} in Propositions 4.3–4.5.

7.9. Now we consider the group G_{25}. We claim that relation (a2) is not redundant. For, relation $s_1s_2 = s_2s_1$ together with relations (a1), (a3) and (a4) determine a reducible reflection group $\langle s_1 \rangle \times \langle s_2, s_3 \rangle$, meaning that (a2) is not a consequence of relations (a1), (a3), (a4). Similarly, (a3) is not redundant. We claim that relation (a4) is not redundant. For, by replacing s_6 by s_3 for the presentation of G_{25} in Proposition 5.4, relations (b1)–(b3) becomes (a1)–(a3), respectively, and (b5) becomes $s_1s_3s_1 = s_3s_1s_3$, differing from (a4), which shows that relation (a4) is not a consequence of relations (a1)–(a3). So the reduced form for the presentation of G_{25} in Proposition 5.3 is essential. Then Lemma 7.6, we see that neither of relations (b2) and (b4) is redundant. Then by the symmetry of s_1, s_2, s_6 for the presentation of G_{25} in Proposition 5.4, we see that relation (b3) is not redundant, too. This implies that the reduced form for the presentation of G_{25} in Proposition 5.4 is essential. Finally, to show that the reduced form for the presentation of G_{25} in Corollary 5.5 is essential, we need only show that relation (b3′) is not redundant. But this follows by the symmetry of s_1, s_2, s_6 for the presentation of G_{25} in Corollary 5.5.

The proof is similar for the essentiality of the reduced form for the presentations of the group G_{26} in Propositions 6.3 and 6.4.

7.10. Finally we consider the group G_{12}. Relation (a3) is not redundant, since (b1) and (b4) become (a1) and (a2), respectively, by replacing s_{10} by s_3, while (b3) does not become (a6) under the same substitution. By symmetry of s_1, s_2, s_3 for the presentation of G_{12} in Proposition 3.3, we see that relation (a2) is not redundant, too. So the reduced form for the presentation of G_{12} in Proposition 3.3 is essential. By Lemma 7.5(1), relation (b4) is not redundant. We claim that relation (b3) is not redundant. For, relation $s_1s_{10}s_1s_{10}s_1s_{10} = s_{10}s_1s_{10}s_1s_{10}s_1$ does not conflict with (b1), (b2) and (b4) by observing the presentation of G_{12} in Proposition 3.3 with s_3 in the place of s_{10}. Next we claim that relation (b2) is not redundant. To see this, one need only notice that with respect to the presentation of G_{12} in Proposition 3.3, the images $\bar{s}_1, \bar{s}_2, \bar{s}_3$ of the generators s_1, s_2, s_3 in the quotient group of G_{12} over its center Z (Z has order two) satisfy relations (b1), (b3), (b4) with $\bar{s}_1, \bar{s}_2, \bar{s}_3$ in the places of s_1, s_2, s_{10}, respectively, and relation $\bar{s}_1\bar{s}_2\bar{s}_1 = \bar{s}_2\bar{s}_1\bar{s}_2$. Hence the reduced form for the presentation of G_{12} in Proposition 3.4 is essential. The non-redundancy of relation (c3) follows by Lemma 7.6. Then by the symmetry of s_1, s_7 for the presentation of G_{12} in Proposition 3.5, we see that relation (c2) is not redundant, too. This implies that the reduced form for the presentation of G_{12} in Proposition 3.5 is essential. By Lemma 7.6, we see that neither of relations (d2) and (d3) is redundant and hence the reduced form for the presentation of G_{12} in Proposition 3.6 is essential. Finally consider the presentation of G_{12} in Proposition 3.7. Relation (e5) is not redundant by Lemma 7.5(1). Relation (e2) is not redundant since one gets a presentation of the quotient group G_{12}/Z (isomorphic to the symmetric group S_4) by replacing (e2) by $s_1s_7 = s_7s_1$ in Proposition 3.7. Next we claim that relation (e3) is not redundant. For, substituting s_1, s_7, s_8 by s_2, s_8, s_1 in relations (e1)–(e5), respectively, followed by replacing a resulting relation $s_1s_2s_1 = s_2s_1s_2$ (which comes from (e3)) by (d6), all the relations we get are the consequence of the presentation of G_{12} in Proposition 3.6. This shows the claim. By symmetry of s_1, s_7 in the presentation, we see that relation (e4) is not redundant, too. So the reduced form for the presentation of G_{12} in Proposition 3.7 is essential.

APPENDIX

Here we record the results of L. Wang and P. Zeng on the numbers $N(G)$ of the equivalence classes of simple root systems for the primitive complex reflection groups G_7, G_{11}, G_{15}, G_{19}, G_{22}, G_{27}, G_{29}, G_{31}, G_{32}, G_{33}.

G	$N(G)$	G	$N(G)$
G_7	2	G_{27}	6
G_{11}	4	G_{29}	9
G_{15}	4	G_{31}	61
G_{19}	6	G_{32}	5
G_{22}	18	G_{33}	14

Since any complex reflection group generated by one or two reflections has a unique simple root system up to equivalence, so far G_{34} is the only group left unknown concerning the equivalence classes of simple root systems.

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