ARTICLE IN PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT

Discrete Mathematics ▮▮▮ (▮▮▮▮) ▮▮▮–▮▮▮

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Elementary proofs of some q-identities of Jackson and Andrews–Jain

Victor J.W. Guo

Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, People's Republic of China

Received 25 November 2003; received in revised form 28 November 2004; accepted 1 March 2005

Abstract

We present elementary proofs of three q-identities of Jackson. They are Jackson's terminating q-analogue of Dixon's sum, Jackson's q-analogue of Clausen's formula, and a generalization of both of them. We also give an elementary proof of Jain's q-analogue of terminating Watson's summation formula, which is actually equivalent to Andrews's q-analogue of Watson's formula.

Keywords: Jackson; Basic hypergeometric series; q-analogue of Dixon's sum; q-analogue of Clausen's formula

1. Introduction

We employ the notation and terminology in [13], and we always assume that $0 < |q| < 1$. The *q-shifted factorial* is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n \in \mathbb{N}.$$

For brevity, we adopt the usual notation

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n.$$

E-mail address: jwguo@eyou.com

0012-365X/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2005.03.001

The *q-binomial coefficient* is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Jackson [18] established the following three interesting theorems:

Theorem 1.1. *For $n \geqslant 0$, we have*

$$\sum_{k=0}^{2n}(-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}(a,b;q)_k(a,b;q)_{2n-k}q^{k(2n-k+1)/2} = (a,b,q^{n+1},abq^n;q)_n. \quad (1.1)$$

Theorem 1.2. *For $n \geqslant 0$, we have*

$$\sum_{k=0}^{n} \frac{(a,b;q)_k(a,b;q)_{n-k}}{(q,abq^{1/2};q)_k(q,abq^{1/2};q)_{n-k}} q^{k/2} = \frac{(a,b;q^{1/2})_n(ab;q)_n}{(ab,q^{1/2};q^{1/2})_n(abq^{1/2};q)_n}. \quad (1.2)$$

Theorem 1.3. *For $n \geqslant 0$, we have*

$$\sum_{k=0}^{2n}(-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}(a,b,c,d;q)_k(a,b,c,d;q)_{2n-k}q^{k(6n-3k+1)/2}$$
$$= (-1)^n(a,b,c,d,q^{n+1},abq^n,bcq^n,caq^n;q)_n d^n q^{n(3n-1)/2}, \quad (1.3)$$

where $abcd = q^{1-3n}$.

Jackson obtained (1.1) and (1.3) by a rather curious transformation of his q-analogue of Dougall's theorem [16], and derived (1.2) as a special case of a q-theorem in his paper [17]. Note that Jackson had already stated (1.1) without proof in [15]. It was pointed out by Bailey [3] that (1.2) with n even is a special case of (1.3).

As usual, the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1,a_2,\ldots,a_{r+1} \\ b_1,b_2,\ldots,b_r \end{matrix};q,z\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{r+1};q)_n z^n}{(q,b_1,b_2,\ldots,b_r;q)_n}.$$

Andrews [2] obtained a q-analogue of a terminating version of Watson's summation formula for ${}_3F_2[1]$:

$${}_4\phi_3\left[\begin{matrix} a,b,c^{1/2},-c^{1/2} \\ (abq)^{1/2},-(abq)^{1/2},c \end{matrix};q,q\right] = \frac{(aq,bq,q/c,abq/c;\ q^2)_\infty}{(q,abq,bq/c,aq/c;\ q^2)_\infty}, \quad (1.4)$$

where $b = q^{-n}$ and n is a nonnegative integer.

Jain [19] obtained a q-analogue of terminating Watson's summation formula due to Bailey [5] in the form

$$\sum_{k=0}^{n}\frac{(a,b;q)_k(q^{-2n};q^2)_k q^k}{(q;q)_k(abq;q^2)_k(q^{-2n};q)_k} = \frac{(aq,bq;q^2)_n}{(q,abq;q^2)_n}, \quad (1.5)$$

and pointed out that (1.5) for $n \to \infty$ gives the q-analogue of Gauss's second summation theorem due to Andrews [1]. It is not difficult to see that (1.4) and (1.5) are equivalent to each other.

The objective of this paper is to give elementary proofs of the above three theorems of Jackson and Jain's q-identity (1.5). We will show that (1.1)–(1.3) and (1.5) can be recovered from their special cases. In particular, (1.2) in the case n is even implies (1.3). The main idea is that to prove a terminating q-identity with parameters it is enough to prove it for infinitely many values of these parameters. In this way, we also obtain an interesting q-identity (4.5) which generalizes (1.2) in the case n is odd. Note that Chen and Liu [7,8] and Liu [21] developed a method of deriving hypergeometric identities by parameter augmentation, which is powerful to deal with nonterminating hypergeometric series.

The following lemma plays an important role throughout the paper.

Lemma 1.4. *Let* $f(a,b)=b(1-ax_1)(1-ax_2)(1-ax_3)(1-ab^2x_1x_2x_3)$. *Then the divided difference*

$$\frac{f(a,b)-f(b,a)}{a-b}$$

can be factorized into

$$-(1-abx_1x_2)(1-abx_1x_3)(1-abx_2x_3).$$

2. Jackson's terminating q-analogue of Dixon's sum

When considered as a basic hypergeometric series, Eq. (1.1) reads

$${}_3\phi_2\left[\begin{matrix} q^{-2n}, & a, & b \\ & q^{1-2n}/a, & q^{1-2n}/b \end{matrix}; q, q^{2-n}/ab\right] = \frac{(a,b;q)_n(q,ab;q)_{2n}}{(q,ab;q)_n(a,b;q)_{2n}},$$

which is now called Jackson's terminating q-analogue of Dixon's sum. See Bailey [3], Carlitz [6], and Gasper and Rahman [13, p. 50].

In this section we give an elementary proof of (1.1). The following lemma is its special case $b=q$.

Lemma 2.1. *For* $n \geqslant 0$, *we have*

$$\sum_{k=0}^{2n}(-1)^k(a;q)_k(a;q)_{2n-k}q^{k(2n-k+1)/2}=(a,aq^{n+1};q)_n.$$

Proof. Since

$$(1-a^2q^{2n})=(1-aq^{2n-k})+aq^{2n-k}(1-aq^k),$$

we have

$$
\begin{aligned}
(1-a^2q^{2n}) \sum_{k=0}^{2n} (-1)^k (a;q)_k (a;q)_{2n-k} q^{k(2n-k+1)/2} \\
&= \sum_{k=0}^{2n} (-1)^k (a;q)_k (a;q)_{2n+1-k} q^{k(2n-k+1)/2} \\
&\quad + aq^n \sum_{k=0}^{2n} (-1)^k (a;q)_{k+1} (a;q)_{2n-k} q^{(k+1)(2n-k)/2}.
\end{aligned} \tag{2.1}
$$

Noticing that $(a;q)_k(a;q)_{2n+1-k}q^{k(2n-k+1)/2}$ is symmetric in k and $2n+1-k$, we have

$$
\sum_{k=1}^{2n} (-1)^k (a;q)_k (a;q)_{2n+1-k} q^{k(2n-k+1)/2} = 0,
$$

or equivalently,

$$
\sum_{k=0}^{2n-1} (-1)^k (a;q)_{k+1} (a;q)_{2n-k} q^{(k+1)(2n-k)/2} = 0.
$$

Therefore, the right-hand side of (2.1) is equal to

$$
(a;q)_{2n+1} + aq^n (a;q)_{2n+1} = (1+aq^n)(a;q)_{2n+1}.
$$

Namely,

$$
\sum_{k=0}^{2n} (-1)^k (a;q)_k (a;q)_{2n-k} q^{k(2n-k+1)/2} = (a;q)_{2n+1}/(1-aq^n),
$$

as desired. □

Proof of Theorem 1.1. It is enough to prove the cases $b=q^m$, $m\geqslant 1$. We proceed by induction on m. For $m=1$, Eq. (1.1) holds by Lemma 2.1.

Suppose (1.1) holds for b $(=q^m)$. Substituting a with aq in (1.1) and multiplying by $(1-a)^2$ on its both sides, we obtain

$$
\begin{aligned}
&\sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} (a;q)_{k+1} (a;q)_{2n-k+1} (b;q)_k (b;q)_{2n-k} q^{k(2n-k+1)/2} \\
&\quad = (1-a)(a;\ q)_{n+1} (b, q^{n+1}, abq^{n+1}; q)_n.
\end{aligned} \tag{2.2}
$$

Since

$$
(1-bq^k)(1-bq^{2n-k}) = (1-b/a)(1-abq^{2n}) + (b/a)(1-aq^k)(1-aq^{2n-k}),
$$

we get

$$(a, bq; q)_k (a, bq; q)_{2n-k} = \frac{(1-b/a)(1-abq^{2n})}{(1-b)^2}(a, b; q)_k (a, b; q)_{2n-k} + \frac{b/a}{(1-b)^2}(a; q)_{k+1}(a; q)_{2n-k+1}(b; q)_k (b; q)_{2n-k}.$$

Therefore, by the inductive hypothesis and (2.2), we have

$$\begin{aligned}\sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}(a, bq; q)_k (a, bq; q)_{2n-k} q^{k(2n-k+1)/2} &= \frac{(1-b/a)(1-abq^{2n})}{(1-b)^2}(a, b, q^{n+1}, abq^n; q)_n \\ &\quad + \frac{b/a}{(1-b)^2}(1-a)(a;\ q)_{n+1}(b, q^{n+1}, abq^{n+1}; q)_n \\ &= (a, bq, q^{n+1}, abq^{n+1}; q)_n.\end{aligned}$$

Namely, Eq. (1.1) holds for bq $(=q^{m+1})$. This completes the proof. □

It is easy to see that

$$\begin{aligned}\sum_{k=0}^{2n+1}(-1)^k (a; q)_k (a; q)_{2n+1-k} q^{k(2n-k+3)/2} &= \sum_{k=0}^{2n+1}(-1)^k (a; q)_k (a; q)_{2n+2-k} q^{k(2n-k+3)/2} \\ &\quad + aq^{2n+1}\sum_{k=0}^{2n+1}(-1)^k (a; q)_k (a; q)_{2n+1-k} q^{k(2n-k+1)/2}.\end{aligned} \tag{2.3}$$

By Lemma 2.1, we have

$$\sum_{k=0}^{2n+2}(-1)^k (a; q)_k (a; q)_{2n+2-k} q^{k(2n-k+3)/2} = (a; q)_{n+1}(aq^{n+2}; q)_{n+1}. \tag{2.4}$$

By symmetry, it is obvious that

$$\sum_{k=0}^{2n+1}(-1)^k (a; q)_k (a; q)_{2n+1-k} q^{k(2n-k+1)/2} = 0. \tag{2.5}$$

Combining (2.3)–(2.5) yields

$$\sum_{k=0}^{2n+1}(-1)^k (a; q)_k (a; q)_{2n+1-k} q^{k(2n-k+3)/2} = (1-q^{n+1})(a; q)_{n+1}(aq^{n+2}; q)_n.$$

Then, by the same argument as in the proof of Theorem 1.1, we obtain

Theorem 2.2. *For $n\geqslant 0$, we have*

$$\sum_{k=0}^{2n+1}(-1)^k\begin{bmatrix}2n+1\\k\end{bmatrix}(a,b;q)_k(a,b;q)_{2n+1-k}q^{k(2n-k+3)/2} = (a,b,q^{n+1};q)_{n+1}(abq^{n+1};q)_n. \tag{2.6}$$

Setting $a=b=q^{-2n-1}$ in (2.6), we get

$$\sum_{k=0}^{2n+1}(-1)^k\begin{bmatrix}2n+1\\k\end{bmatrix}^3 q^{(k-n)(3k-3n-1)/2}=(-1)^n\frac{(q;q)_{3n+1}}{(q,q,q;q)_n},$$

an identity due to Bailey [4].

3. Jackson's q-analogue of Clausen's formula

It is clear that (1.2) is equivalent to the following product formula [17,18]:

$$ {}_2\phi_1\left[\begin{matrix}a,b\\abq^{1/2}\end{matrix};q,z\right]{}_2\phi_1\left[\begin{matrix}a,b\\abq^{1/2}\end{matrix};q,zq^{1/2}\right] = {}_4\phi_3\left[\begin{matrix}a,b,a^{1/2}b^{1/2},-a^{1/2}b^{1/2}\\ab,a^{1/2}b^{1/2}q^{1/4},-a^{1/2}b^{1/2}q^{1/4}\end{matrix};\quad q^{1/2},z\right],\quad |z|<1, \tag{3.1}$$

which is a q-analogue of Clausen's formula [9]:

$$\left\{{}_2F_1\left[\begin{matrix}a,b\\a+b+1/2\end{matrix};z\right]\right\}^2={}_3F_2\left[\begin{matrix}2a,2b,a+b\\2a+2b,a+b+1/2\end{matrix};z\right],\quad |z|<1.$$

See for example [11,12].

Several proofs of (3.1) have been given by Singh [23], Nassrallah [22], and Jain and Srivastava [20]. In this section we add a new one.

Lemma 3.1. *For $n\geqslant 0$, we have*

$$\sum_{k=0}^{n}\frac{(q^{1/2};q)_k(q^{1/2};q)_{n-k}}{(1-aq^k)(q;q)_k(q;q)_{n-k}}q^{k/2}=\frac{(aq^{1/2};q)_n}{(a;q)_{n+1}}. \tag{3.2}$$

Proof. Clearly, Eq. (3.2) is equivalent to

$$\sum_{k=0}^{n}(a;q)_k(aq^{k+1};q)_{n-k}\frac{(q^{1/2};q)_k(q^{1/2};q)_{n-k}}{(q;q)_k(q;q)_{n-k}}q^{k/2}=(aq^{1/2};q)_n. \tag{3.3}$$

When $a=q^{-m}$ $(0\leqslant m\leqslant n)$, the left-hand side of (3.3) has only one nonzero term

$$(q^{-m};q)_m(q;q)_{n-m}\frac{(q^{1/2};q)_m(q^{1/2};q)_{n-m}}{(q;q)_m(q;q)_{n-m}}q^{m/2}=(q^{-m+1/2};q)_n.$$

Noticing that both sides of (3.3) are polynomials in a, we complete the proof. $\square$

Remark. Eq. (3.3) is in fact the Lagrange interpolation formula for $(aq^{1/2};q)_n$ at the values q^{-i} $(0\leqslant i\leqslant n)$ of a. See Fu and Lascoux [10] and Zeng [24].

Proof of Theorem 1.2. It suffices to show that (1.2) holds for all $b=q^{m+1/2}$, $m\geqslant 1$. We proceed by induction on m. For $m=0$, $b=q^{1/2}$, Eq. (1.2) reduces to

$$\sum_{k=0}^{n}\frac{(q^{1/2};q)_k(q^{1/2};q)_{n-k}}{(1-aq^k)(1-aq^{n-k})(q;q)_k(q;q)_{n-k}}q^{k/2}=\frac{(aq^{1/2};q)_n}{(1-aq^{n/2})(a;q)_{n+1}}. \qquad (3.4)$$

Changing k to $n-k$ in (3.2), we obtain

$$\sum_{k=0}^{n}\frac{(q^{1/2};q)_k(q^{1/2};q)_{n-k}}{(1-aq^{n-k})(q;q)_k(q;q)_{n-k}}q^{(n-k)/2}=\frac{(aq^{1/2};q)_n}{(a;q)_{n+1}}. \qquad (3.5)$$

Then (3.2) + (3.5) yields

$$\sum_{k=0}^{n}\frac{(q^{k/2}+q^{(n-k)/2})(1-aq^{n/2})(q^{1/2};\,q)_k(q^{1/2};\,q)_{n-k}}{(1-aq^k)(1-aq^{n-k})(q;\,q)_k(q;q)_{n-k}}=2\frac{(aq^{1/2};q)_n}{(a;q)_{n+1}}.$$

Since the above summation is symmetric in k and $n-k$, we get (3.4).

Now suppose (1.2) holds for $b=q^{m+1/2}$. Substituting a with aq in (1.2), we obtain

$$\sum_{k=0}^{n}\frac{(aq,b;q)_k(aq,b;q)_{n-k}}{(q,abq^{3/2};q)_k(q,abq^{3/2};q)_{n-k}}q^{k/2}=\frac{(aq,b;q^{1/2})_n(abq;q)_n}{(abq,q^{1/2};q^{1/2})_n(abq^{3/2};q)_n}. \qquad (3.6)$$

Putting $\{x_1,x_2,x_3\}=\{q^k,q^{n-k},q^{1/2}\}$ in Lemma 1.4, we have

$$\begin{aligned}&\frac{b(1-aq^{1/2})(1-ab^2q^{n+1/2})(1-a)^2}{(1-abq^n)(b-a)}\sum_{k=0}^{n}\frac{(aq,b;q)_k(aq,b;q)_{n-k}}{(q,abq^{3/2};q)_k(q,abq^{3/2};q)_{n-k}}q^{k/2}\\&\quad+\frac{a(1-bq^{1/2})(1-a^2bq^{n+1/2})(1-b)^2}{(1-abq^n)(a-b)}\sum_{k=0}^{n}\frac{(a,bq;\,q)_k(a,bq;q)_{n-k}}{(q,abq^{3/2};q)_k(q,abq^{3/2};q)_{n-k}}q^{k/2}\\&\quad=(1-abq^{1/2})^2\sum_{k=0}^{n}\frac{(a,b;q)_k(a,b;q)_{n-k}}{(q,abq^{1/2};q)_k(q,abq^{1/2};q)_{n-k}}q^{k/2}, \qquad (3.7)\end{aligned}$$

while putting $\{x_1,x_2,x_3\}=\{1,q^{n/2},q^{(n+1)/2}\}$ in Lemma 1.4, we have

$$\begin{aligned}&\frac{b(1-aq^{1/2})(1-ab^2q^{n+1/2})(1-a)^2}{(1-abq^n)(b-a)}\frac{(aq,b;q^{1/2})_n(abq;q)_n}{(abq,q^{1/2};q^{1/2})_n(abq^{3/2};q)_n}\\&\quad+\frac{a(1-bq^{1/2})(1-a^2bq^{n+1/2})(1-b)^2}{(1-abq^n)(a-b)}\frac{(a,bq;q^{1/2})_n(abq;q)_n}{(abq,q^{1/2};q^{1/2})_n(abq^{3/2};q)_n}\\&\quad=(1-abq^{1/2})^2\frac{(a,b;q^{1/2})_n(ab;q)_n}{(ab,q^{1/2};q^{1/2})_n(abq^{1/2};q)_n}. \qquad (3.8)\end{aligned}$$

Comparing (3.7) and (3.8) and using (3.6) and (1.2), we obtain

$$\sum_{k=0}^{n}\frac{(a,bq;q)_k(a,bq;q)_{n-k}}{(q,abq^{3/2};q)_k(q,abq^{3/2};q)_{n-k}}q^{k/2}=\frac{(a,bq;q^{1/2})_n(abq;q)_n}{(abq,q^{1/2};q^{1/2})_n(abq^{3/2};q)_n},$$

i.e., Eq. (1.2) holds for bq. This completes the proof. □

For the case n is even in (1.2), we have

$$\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}\frac{(a,b;q)_k(a,b;q)_{2n-k}}{(abq^{1/2};q)_k(abq^{1/2};q)_{2n-k}}q^{k/2}=\frac{(a,b,q^{n+1},abq^n,aq^{1/2},bq^{1/2};q)_n}{(abq^{1/2},q^{1/2};q)_n(abq^{1/2};q)_{2n}} \tag{3.9}$$

(see Jackson [18, Eq. (3)]), while for the odd case, we have

$$\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\k\end{bmatrix}\frac{(a,b;q)_k(a,b;q)_{2n+1-k}}{(abq^{1/2};q)_k(abq^{1/2};q)_{2n+1-k}}q^{k/2}=\frac{(a,b,q^{n+1};q)_{n+1}(abq^{n+1},aq^{1/2},bq^{1/2};q)_n}{(abq^{1/2};q)_n(q^{1/2};q)_{n+1}(abq^{1/2};q)_{2n+1}}. \tag{3.10}$$

4. Jackson's Theorem 1.3

We have seen that (1.1) and (1.2) can be recovered from their special cases $b=q$ and $b=q^{1/2}$, respectively. We will show that (1.3) can also be derived from the case $b=q^{1/2-n}$, which reduces to Jackson's q-analogue of Clausen's formula.

Substituting for $d=q^{1-3n}/abc$, Eq. (1.3) may be written as

$$\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\k\end{bmatrix}\frac{(a,b,c;q)_k(a,b,c;q)_{2n-k}}{(abcq^n;q)_k(abcq^n;q)_{2n-k}}q^{k(2n-k+1)/2}=\frac{(a,b,c,abq^n,bcq^n,acq^n,q^{n+1};q)_n}{(abcq^n;q)_n(abcq^n;q)_{2n}}. \tag{4.1}$$

It is clear that (1.1) can be obtained from (4.1) by taking $c=0$.

Proof of Theorem 1.3. It suffices to show that (4.1) holds for all $b=q^{1/2+m}$, $m\geqslant -n$. It is easy to see that the case $b=q^{1/2-n}$ of (4.1) reduces to (3.9) on a and c, as was pointed out by Bailey [3].

Suppose (4.1) holds for $b=q^{1/2+m}$. Substituting a with aq in (4.1), we obtain

$$\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\k\end{bmatrix}\frac{(aq,b,c;q)_k(aq,b,c;q)_{2n-k}}{(abcq^{n+1};q)_k(abcq^{n+1};q)_{2n-k}}q^{k(2n-k+1)/2}$$
$$=\frac{(aq,b,c,abq^{n+1},bcq^n,acq^{n+1},q^{n+1};q)_n}{(abcq^{n+1};q)_n(abcq^{n+1};q)_{2n}}. \tag{4.2}$$

Putting $\{x_1,x_2,x_3\}=\{q^k,q^{2n-k},cq^n\}$ in Lemma 1.4, we have

$$\frac{b(1-acq^n)(1-ab^2cq^{3n})(1-a)^2}{(b-a)(1-abq^{2n})}$$
$$\times\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\k\end{bmatrix}\frac{(aq,b,c;q)_k(aq,b,c;q)_{2n-k}}{(abcq^{n+1};q)_k(abcq^{n+1};q)_{2n-k}}q^{k(2n-k+1)/2}$$
$$+\frac{a(1-bcq^n)(1-a^2bcq^{3n})(1-b)^2}{(a-b)(1-abq^{2n})}$$
$$\times\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\k\end{bmatrix}\frac{(a,bq,c;q)_k(a,bq,c;q)_{2n-k}}{(abcq^{n+1};q)_k(abcq^{n+1};q)_{2n-k}}q^{k(2n-k+1)/2}$$
$$=(1-abcq^n)^2\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\k\end{bmatrix}\frac{(a,b,c;q)_k(a,b,c;q)_{2n-k}}{(abcq^n;q)_k(abcq^n;q)_{2n-k}}q^{k(2n-k+1)/2}, \tag{4.3}$$

while putting $\{x_1,x_2,x_3\}=\{1,q^n,cq^{2n}\}$ in Lemma 1.4, we have

$$\frac{b(1-acq^n)(1-ab^2cq^{3n})(1-a)^2}{(b-a)(1-abq^{2n})}\frac{(aq,b,c,abq^{n+1},bcq^n,acq^{n+1},q^{n+1};q)_n}{(abcq^{n+1};q)_n(abcq^{n+1};q)_{2n}}$$
$$+\frac{a(1-bcq^n)(1-a^2bcq^{3n})(1-b)^2}{(a-b)(1-abq^{2n})}\frac{(a,bq,c,abq^{n+1},bcq^{n+1},acq^n,q^{n+1};q)_n}{(abcq^{n+1};q)_n(abcq^{n+1};q)_{2n}}$$
$$=(1-abcq^n)^2\frac{(a,b,c,abq^n,bcq^n,acq^n,q^{n+1};q)_n}{(abcq^n;q)_n(abcq^n;q)_{2n}}. \tag{4.4}$$

It follows from (4.1)–(4.4) that

$$\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\k\end{bmatrix}\frac{(a,bq,c;q)_k(a,bq,c;q)_{2n-k}}{(abcq^{n+1};q)_k(abcq^{n+1};q)_{2n-k}}q^{k(2n-k+1)/2}$$
$$=\frac{(a,bq,c,abq^{n+1},bcq^{n+1},acq^n,q^{n+1};q)_n}{(abcq^{n+1};q)_n(abcq^{n+1};q)_{2n}}.$$

Namely, Eq. (4.1) holds for $bq=q^{3/2+m}$. This completes the proof. □

It is interesting that there holds a q-identity similar to (4.1) as follows:

Theorem 4.1. *For $n \geqslant 0$, we have*

$$\sum_{k=0}^{2n+1} (-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix} \frac{(a,b,c;q)_k (a,b,c;q)_{2n+1-k}}{(abcq^{n+1};q)_k (abcq^{n+1};q)_{2n+1-k}} q^{k(2n-k+3)/2}$$
$$= \frac{(a,b,c,q^{n+1};q)_{n+1} (abq^{n+1}, bcq^{n+1}, acq^{n+1};q)_n}{(abcq^{n+1};q)_n (abcq^{n+1};q)_{2n+1}}. \tag{4.5}$$

When $c = 0$, Eq. (4.5) reduces to (2.6), and when $c = q^{-1/2-n}$, Eq. (4.5) reduces to (3.10). The proof of Theorem 4.1 is exactly the same as that of Theorem 1.3. Namely, we need to use Lemma 1.4 twice, taking $\{x_1, x_2, x_3\} = \{q^k, q^{2n+1-k}, cq^{n+1}\}$ and $\{x_1, x_2, x_3\} = \{1, q^n, cq^{2n+1}\}$, respectively.

It should be mentioned that Theorem 4.1 can also be deduced from Jackson's q-analogue of Dougall's theorem [16] by utilizing a curious transformation, just as the way that Theorem 1.3 was obtained by Jackson [18].

5. Jain's q-analogue of terminating Watson's formula

The case $b = q^2$ of (1.5) reduces to

$$\sum_{k=0}^{n} \frac{(q^{-2n};q^2)_k (a;q)_k (1-q^{k+1}) q^k}{(q^{-2n};q)_k (aq;q^2)_{k+1}} = \frac{1-q^{2n+1}}{1-aq^{2n+1}}. \tag{5.1}$$

It is easy to check that, for $k = 0, 1, \ldots, n$,

$$\frac{(q^{-2n};q^2)_k (a;q)_k (1-q^{k+1}) q^k}{(q^{-2n};q)_k (aq;q^2)_{k+1}}$$
$$= \frac{q^{2n+1}}{1-aq^{2n+1}} \left[\frac{(q^{-2n};q^2)_{k+1} (a;q)_{k+1}}{(q^{-2n};q)_k (aq;q^2)_{k+1}} - \frac{(q^{-2n};q^2)_k (a;q)_k}{(q^{-2n};q)_{k-1} (aq;q^2)_k} \right].$$

Hence, summing over k from 0 to n gives (5.1).

Proof of (1.5). It suffices to show that (1.5) holds for all $b = q^{2m}$ ($m \geqslant 1$). The case $b = q^2$ reduces to (5.1) and has been proved.

Suppose (1.5) holds for $b = q^{2m}$. Substituting a with aq^2 in (1.5), we obtain

$$\sum_{k=0}^{n} \frac{(aq^2, b;q)_k (q^{-2n};q^2)_k q^k}{(q;q)_k (abq^3;q^2)_k (q^{-2n};q)_k} = \frac{(aq^3, bq;q^2)_n}{(q, abq^3;q^2)_n}, \tag{5.2}$$

Letting $\{x_1, x_2, x_3\} = \{0, q^k, q^{k+1}\}$ in Lemma 1.4, we have

$$\frac{b(1-a)(1-aq)}{b-a}\sum_{k=0}^{n}\frac{(aq^2, b; q)_k(q^{-2n}; q^2)_k q^k}{(q;q)_k(abq^3;q^2)_k(q^{-2n};q)_k} - \frac{a(1-b)(1-bq)}{b-a}\sum_{k=0}^{n}\frac{(a, bq^2; q)_k(q^{-2n}; q^2)_k q^k}{(q;q)_k(abq^3;q^2)_k(q^{-2n};q)_k} = (1-abq)\sum_{k=0}^{n}\frac{(a, b; q)_k(q^{-2n}; q^2)_k q^k}{(q;q)_k(abq;q^2)_k(q^{-2n};q)_k}. \tag{5.3}$$

On the other hand, letting $\{x_1, x_2, x_3\} = \{0, 1, q^{2n+1}\}$ in Lemma 1.4, we have

$$\frac{b(1-a)(1-aq)}{b-a}\frac{(aq^3, bq; q^2)_n}{(abq^3; q^2)_n} - \frac{a(1-b)(1-bq)}{b-a}\frac{(aq, bq^3; q^2)_n}{(abq^3; q^2)_n} = (1-abq)\frac{(aq, bq; q^2)_n}{(abq; q^2)_n}. \tag{5.4}$$

From (5.2)–(5.4) and the inductive hypothesis (1.5) it follows that

$$\sum_{k=0}^{n}\frac{(a, bq^2; q)_k(q^{-2n}; q^2)_k q^k}{(q;q)_k(abq^3;q^2)_k(q^{-2n};q)_k} = \frac{(a, bq^3; q^2)_n}{(abq^3; q^2)_n}.$$

Namely, Eq. (1.5) holds for $bq^2 = q^{2m+2}$. This completes the proof. □

The same method can also be used to prove a symmetric q-Pfaff-Saalschütz identity. See Guo and Zeng [14].

Acknowledgements

The author is grateful to one of the referees for providing me Lemma 1.4, which unifies eight identities in a previous version of this paper.

References

[1] G.E. Andrews, On the q-analog of Kummer's theorem and applications, Duke Math. J. 40 (1973) 525–528.
[2] G.E. Andrews, On q-analogues of the Watson and Whipple summations, SIAM J. Math. Anal. 7 (1976) 332–336.
[3] W.N. Bailey, A note on certain q-identities, Quart. J. Math., Oxford Ser. 12 (1941) 173–175.
[4] W.N. Bailey, On the analogue of Dixon's theorem for bilateral basic hypergeometric series, Quart. J. Math., Oxford Ser. (2) 1 (1950) 318–320.
[5] W.N. Bailey, On the sum of a terminating ${}_3F_2(1)$, Quart. J. Math., Oxford Ser. (2) 4 (1953) 237–240.
[6] L. Carlitz, Some formulas of F.H. Jackson, Monatsh. für Math. 73 (1969) 193–198.
[7] W.Y.C. Chen, Z.-G. Liu, Parameter augmentation for basic hypergeometric series, II, J. Combin. Theory Ser. A 80 (1997) 175–195.

[8] W.Y.C. Chen, Z.-G. Liu, Parameter augmentation for basic hypergeometric series I, in: B.E. Sagan, R.P. Stanley (Eds.), Mathematical Essays in Honor of Gian-Carlo Rota, Birkhäuser Boston, Boston, MA, 1998, pp. 111–129.

[9] T. Clausen, Ueber die Fälle, wenn die Reihe von der Form $y = 1 + x\alpha\beta/1.\gamma + \cdots$ ein Quadrat von der Form $z = 1 + x\alpha'\beta'\gamma'/1.\delta'\varepsilon' + \cdots$ hat, J. Reine Angew. Math. 3 (1828) 89–91.

[10] A.M. Fu, A. Lascoux, q-Identities from Lagrange and Newton interpolation, Adv. Appl. Math. 31 (2003) 527–531.

[11] G. Gasper, q-Extensions of Clausen's formula and of the inequalities used by de Branges in his proof of the Bieberbach Robertson, and Milin Conjectures, SIAM J. Math. Anal. 20 (1989) 1019–1034.

[12] G. Gasper, M. Rahman, A nonterminating q-Clausen formula and some related product formulas, SIAM J. Math. Anal. 20 (1989) 1270–1282.

[13] G. Gasper, M. Rahman, Basic hypergeometric series, Encyclopedia of Mathematics and Its Applications, vol. 35, Cambridge University Press, Cambridge, 1990.

[14] V.J.W. Guo, J. Zeng, A combinatorial proof of a symmetric q-Pfaff-Saalschütz identity, Electron. J. Combin. 12 (2005) #N2.

[15] F.H. Jackson, The application of basic numbers to Bessel's and Legendre's functions, Proc. London Math. Soc. (2) 3 (1905) 1–23.

[16] F.H. Jackson, Summation of q-hypergeometric series, Messenger Math. 57 (1921) 101–112.

[17] F.H. Jackson, The q^{θ} equations whose solutions are products of q^{θ} equations of lower order, Quart. J. Math., Oxford Ser. 11 (1940) 1–17.

[18] F.H. Jackson, Certain q-identities, Quart. J. Math., Oxford Ser. 12 (1941) 167–172.

[19] V.K. Jain, Some transformations of basic hypergeometric functions, II, SIAM J. Math. Anal. 12 (1981) 957–961.

[20] V.K. Jain, H.M. Srivastava, q-Series identities and reducibility of basic double hypergeometric functions, Canad. J. Math. 38 (1986) 215–231.

[21] Z.-G. Liu, Some operator identities and q-series transformation formulas, Discrete Math. 265 (2003) 119–139.

[22] B. Nassrallah, Ph.D. Thesis, Carleton University, Ottawa, Ontario, 1982.

[23] V.N. Singh, The basic analogues of identities of the Cayley-Orr type, J. London Math. Soc. 34 (1959) 15–22.

[24] J. Zeng, On some q-identities related to divisor functions, Adv. Appl. Math. 34 (2005) 313–315.