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Note

A short proof of the q-Dixon identity

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Abstract

We give a simple proof of Jackson's terminating q-analogue of Dixon's identity.

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In the last twenty years several short proofs of Dixon's identity have been published [3–5]. However, there are not so many proofs of Jackson's terminating q-analogue of Dixon's identity [2,6]:

$$\sum_k (-1)^k q^{(3k^2+k)/2} \begin{bmatrix} a+b \\ a+k \end{bmatrix} \begin{bmatrix} a+c \\ c+k \end{bmatrix} \begin{bmatrix} b+c \\ b+k \end{bmatrix} = \frac{[a+b+c]!}{[a]![b]![c]!}, \tag{1}$$

where $[n]! = \prod_{i=1}^n (1-q^i)/(1-q)$ and the q-binomial coefficient $\begin{bmatrix} x \\ k \end{bmatrix}$ is defined by

$$\begin{bmatrix} x \\ k \end{bmatrix} = \begin{cases} \prod_{i=1}^{k} \dfrac{1-q^{x-i+1}}{1-q^i} & \text{if } k \geqslant 0, \\ 0 & \text{if } k < 0. \end{cases}$$

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The aim of this note is to give a short proof of (1) by generalizing the argument of [5]. Note that $\begin{bmatrix} n \\ k \end{bmatrix} = [n]!/([k]![n-k]!)$ for $0 \leqslant k \leqslant n$. So (1) can be written as follows:

$$\sum_k (-1)^k q^{(3k^2+k)/2} \begin{bmatrix} a+b \\ b-k \end{bmatrix} \begin{bmatrix} a+c \\ c+k \end{bmatrix} \begin{bmatrix} b+c \\ b+k \end{bmatrix} = \begin{bmatrix} b+c \\ b \end{bmatrix} \begin{bmatrix} a+b+c \\ b+c \end{bmatrix}. \tag{2}$$

Clearly, both sides of (2) are polynomials in q^a of degree $b+c$. It suffices to verify (2) for $b+c+1$ distinct values of a. Suppose $b \leqslant c$.

For $a=0$ the two sides of (2) are equal to $\begin{bmatrix} b+c \\ b \end{bmatrix}$. For $a=-p$ with $1 \leqslant p \leqslant b+c$ the right-hand side of (2) vanishes, while the left-hand side of (2) is equal to

$$L = \sum_k (-1)^k q^{(3k^2+k)/2} \begin{bmatrix} b-p \\ b-k \end{bmatrix} \begin{bmatrix} c-p \\ c+k \end{bmatrix} \begin{bmatrix} b+c \\ b+k \end{bmatrix}.$$

We now show that $L=0$ for $1 \leqslant p \leqslant b+c$ as follows:

- If $p \in [1, b]$, then $\begin{bmatrix} b-p \\ b-k \end{bmatrix} = 0$ for $k<0$ and $\begin{bmatrix} c-p \\ c+k \end{bmatrix} = 0$ for $k \geqslant 0$.
- If $p \in [b+1, c]$, then $\begin{bmatrix} c-p \\ c+k \end{bmatrix} = 0$ for any k such that $-b \leqslant k \leqslant b$.
- If $p \in [c+1, b+c]$, since $\begin{bmatrix} -x \\ k \end{bmatrix} = (-1)^k q^{-kx-\binom{k}{2}} \begin{bmatrix} x+k-1 \\ k \end{bmatrix}$ we have

$$\begin{aligned} L &= \sum_k (-1)^{k+b+c} q^{\binom{c-k}{2}+A} \begin{bmatrix} p-k-1 \\ p-b-1 \end{bmatrix} \begin{bmatrix} p+k-1 \\ p-c-1 \end{bmatrix} \begin{bmatrix} b+c \\ b+k \end{bmatrix} \\ &= \sum_k (-1)^{k+p+c-1} q^{\binom{c-k}{2}+B} \begin{bmatrix} k-b-1 \\ p-b-1 \end{bmatrix} \begin{bmatrix} p+k-1 \\ p-c-1 \end{bmatrix} \begin{bmatrix} b+c \\ b+k \end{bmatrix}, \end{aligned} \tag{3}$$

where

$$A = b(b+1)/2 - bp - c(p-1) + ck$$

and

$$B = A + (p-k-1)(p-b-1) - \binom{p-b-1}{2}.$$

Since

$$q^B \begin{bmatrix} k-b-1 \\ p-b-1 \end{bmatrix} \begin{bmatrix} p+k-1 \\ p-c-1 \end{bmatrix}$$

is a polynomial in q^k of degree

$$c - (p-b-1) + 2p - b - c - 2 = p - 1 \leqslant b+c-1,$$

the right-hand side of (3) vanishes if we can show that

$$\sum_k (-1)^k q^{\binom{c-k}{2}} \begin{bmatrix} b+c \\ b+k \end{bmatrix} q^{ik} = 0 \quad \text{for } 0 \leqslant i \leqslant b+c-1. \tag{4}$$

Applying the well-known identity

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} z^k = \prod_{i=0}^{n-1}(1 - zq^i) \tag{5}$$

(see, for example, [1, p. 36]) with $z = q^{-i}$ and replacing k by $n-k$, we obtain

$$\sum_{k=0}^{n}(-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}+ik} = 0 \quad \text{for } 0 \leqslant i \leqslant n-1 \tag{6}$$

which gives (4) by setting $n = b + c$ and shifting k to $k + b$.

Remark. The q-Dixon identity (1) is usually derived from the q-Pfaff–Saalschütz identity [7]:

$$\begin{bmatrix} a+b \\ a+k \end{bmatrix} \begin{bmatrix} a+c \\ c+k \end{bmatrix} \begin{bmatrix} b+c \\ b+k \end{bmatrix} = \sum_{n} q^{n^2-k^2} \frac{[a+b+c-n]!}{[a-n]![b-n]![c-n]![n+k]![n-k]!}. \tag{7}$$

Indeed, substituting (7) into the left-hand side of (1) we get

$$\sum_{n} \frac{[a+b+c-n]!q^{n^2}}{[a-n]![b-n]![c-n]![2n]!} \sum_{k=-n}^{n} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} 2n \\ n-k \end{bmatrix},$$

but (5) implies that

$$\sum_{k=-n}^{n} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} 2n \\ n-k \end{bmatrix} = (-1)^n q^{n(n-1)/2} \sum_{k=0}^{2n} (-1)^k (q^{1-n})^k q^{\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix} = \delta_{n0}.$$

Hence, our polynomial argument is somehow equivalent to the role played by the q-Pfaff–Saalschütz identity in the proof of the q-Dixon identity. Note that Zeilberger [7] has given a nice combinatorial proof of (7).

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