On Zero-sum sequences of prescribed length

WEIDONG GAO AND RAVINDRANATHAN THANGADURAI

Summary. Let $k \geq 1$ be any integer. Let G be a finite abelian group of exponent n. Let $s_k(G)$ be the smallest positive integer t such that every sequence S in G of length at least t has a zero-sum subsequence of length kn. We study this constant for groups $G \cong \mathbb{Z}_n^d$ when $d = 3$ or 4. In particular, we prove, as a main result, that $s_k(\mathbb{Z}_p^3) = kp + 3p - 3$ for every $k \geq 4$, $5p + \frac{p-1}{2} - 3 \leq s_2(\mathbb{Z}_p^3) \leq 6p - 3$ and $6p - 3 \leq s_3(\mathbb{Z}_p^3) \leq 8p - 7$ for every prime $p \geq 5$.

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1. Introduction

Let G be an, additively written, finite abelian group. From the structure theorem of finite abelian groups, we know that $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ with $1 < n_1|n_2|\cdots|n_d$, where $n_d = \exp(G) = n$ is the exponent of G and d is the rank of G. A sequence in G is a formal product $S = \prod_{i=1}^{\ell} g_i$ of elements $g_i \in G$ (that is, an element of the free abelian monoid with basis G). We denote by $|S| = \ell$, the length of S, by $v_g(S)$ the number of times $g \in G$ appears in S, by $\sigma(S) = \sum_{i=1}^{\ell} g_i$, the sum of S and by $T|S$, a subsequence T of S. We say that the sequence is a **zero-sum sequence**, if $\sigma(S) = 0$ in G. Also, if $T|S$, then the deleted sequence ST^{-1}, we mean the sequence after removing the elements of T from S. Let $R|S$ and $T|S$ be two subsequences of $S = \prod_{i=1}^{\ell} g_i$. We say R and T are **disjoint** subsequences of S, if there exists two disjoint non-empty subsets I and J of $\{1, 2, \cdots, \ell\}$ such that $R = \prod_{i\in I} g_i$ and $T = \prod_{j\in J} g_j$.

Definition 1.1. For any positive integer k, we define $s_k(G)$ as the smallest positive integer t such that every sequence S in G of length at least t has a zero-sum subsequence of length $k\exp(G)$.

This constant was first studied by the first author [6] and by Adhikari and Rath [1].

Let $\mathbb{Z}_n$ be the cyclic group of order n. Let $\mathbb{Z}_n^d$ be the finite abelian group of order n^d such that it is isomorphic to the direct sum of d copies of $\mathbb{Z}_n$.

The study of $s_1(\mathbb{Z}_n^d)$ stems from an integer lattice point problem (See, e.g., [2] and [9]). In 1961, Erdős, Ginzburg and Ziv [4] proved that $s_1(\mathbb{Z}_n) = 2n - 1$ and hence $s_k(\mathbb{Z}_n) = kn + n - 1$ for all integers $k > 1$. Recently, C. Reiher [13] proved that $s_1(\mathbb{Z}_n^2) = 4n - 3$ which together with a result in [8] ([8], Theorem 3.7) implies $s_k(\mathbb{Z}_n^2) = kn + 2n - 2$ for all integers $k > 1$.

In this paper, we shall mainly investigate $s_k(\mathbb{Z}_n^3)$ and $s_k(\mathbb{Z}_n^4)$. For $k > 1$, we obtain the following main results.

Theorem 1.1. *(1) Let $p \geq 5$ be an odd prime number. Then we have, (i) $5p + \frac{p-1}{2} - 3 \leq s_2(\mathbb{Z}_p^3) \leq 6p - 3$; (ii) $6p - 3 \leq s_3(\mathbb{Z}_p^3) \leq 8p - 7$, and (iii) $s_k(\mathbb{Z}_p^3) = kp + 3p - 3$ for every $k \geq 4$.*

(2) We have, $s_2(\mathbb{Z}_3^3) = 13$; $15 \leq s_3(\mathbb{Z}_3^3) \leq 17$ and $s_k(\mathbb{Z}_3^3) = 3k + 6\ \forall\ k \geq 4$.

(3) We have $s_k(\mathbb{Z}_2^3) = 2k + 3$ for every integer $k \geq 2$.

Theorem 1.2. *For every integer $k \geq 1$ and every prime $p \geq 7$, we have*

$$s_{6k}(\mathbb{Z}_p^4) \leq 6(k+1)p - 4.$$

Concerning the lower bound of $s_1(\mathbb{Z}_n^d)$, recently, C. Elsholtz [3] proved the following:

$$s_1(\mathbb{Z}_n^d) \geq \left(\frac{9}{8}\right)^{[d/3]} (n-1)2^d + 1$$

for $d > 2$ and odd $n > 2$. Thus, when $d = 3$, the above lower bound implies $s_1(\mathbb{Z}_n^3) \geq 9n - 8$ for odd $n > 2$, which is seemingly the optimal one and so we formally write this as the following conjecture.

Conjecture 0. For any odd integer $n > 1$, we have

$$s_1(\mathbb{Z}_n^3) = 9n - 8.$$

Note that Conjecture 0 is proved for $n = 3$ by Harborth [9]. Also, Conjecture 0 is multiplicative, that is, it is enough to prove Conjecture 0 for all primes $p > 2$. However, an easy observation shows that $s_1(\mathbb{Z}_{2^a}^3) = 8 \cdot 2^a - 7$. We shall prove the following theorem which is related to conjecture 0.

Theorem 1.3. *Let $p \geq 5$ be a prime number. Let S be a sequence in $\mathbb{Z}_p^3$ of length $9p - 3$. Suppose S has at most two disjoint zero-sum subsequences of length $2p$. Then S has a zero-sum subsequence of length p.*

Remark 1.1. Since $s_2(\mathbb{Z}_p^3) > 5p - 3$ for every prime $p \geq 5$, there exists a class of sequences of length $5p - 3$ which do not have any zero-sum subsequence of length $2p$. Thus, Theorem 1.3 is valid in this class.

2. Preliminaries

Definition 2.1. Davenport's constant, $D(G)$, stands for the smallest positive integer t such that every sequence S in G of length at least t has a nonempty zero-sum subsequence in it.

It is clear that $D(G) \leq |G|$. The constant $D(G)$ was coined by H. Davenport in connection with non-unique factorization in the ring of integers of number fields. Finding the exact values of $D(G)$ for all groups G seems to be a very difficult problem. Till now, we know the exact value of $D(G)$ only for very few groups. For example, $D(\mathbb{Z}_n) = n$, $D(\mathbb{Z}_m \oplus \mathbb{Z}_n) = m + n - 1$ (where $m|n$), $D(\mathbb{Z}_{2p^\ell}^3) = 6p^\ell - 2$, $D(\mathbb{Z}_{32^\ell}^3) = 92^\ell - 2$, $D(\oplus_{i=1}^k \mathbb{Z}_{p^{e_i}}) = 1 + \sum_{i=1}^k (p^{e_i} - 1)$. For more information and conjectures, we refer to [5]. The best known upper bound for $D(\mathbb{Z}_n^d)$ with $d \geq 3$ is $n(1 + (d-1)\log n)$ and the following conjecture is well-known,

Conjecture 1. $D(\mathbb{Z}_n^d) = d(n-1) + 1$ for any integers $n > 1$ and $d \geq 3$.

W. D. Gao [6] proved that

$$s_k(G) \geq kn + D(G) - 1, \tag{1}$$

and if $k < D(G)/n$, then $s_k(G) \geq kn + D(G)$. Moreover, he proved that equality of (1) holds for all k such that $k \geq |G|/n$. We discuss the problem to determine for which k equality holds in (1), and related questions, in more detail at the end of this paper.

Lemma 2.1. *Let $n \geq 2$ be an integer and d be a positive integer. If $D(\mathbb{Z}_n^{d+1}) = (d+1)(n-1)+1$, then any sequence S in $\mathbb{Z}_n^d$ of length $(d+1)(n-1)+1$ has a zero-sum subsequence T of length kn for some integer k satisfying $1 \leq k \leq d$.*

Proof. Assume that $D(\mathbb{Z}_n^{d+1}) = (d+1)(n-1)+1$. Let $S = \prod_i a_i$ be any sequence in $\mathbb{Z}_n^d$ of length $(d+1)(n-1)+1$. Set $b_i = (1, a_i)$ in $\mathbb{Z}_n^{d+1}$ for every $i = 1, 2, \cdots, (d+1)(n-1)+1$. Then $W = \prod_i b_i$ is a sequence in $\mathbb{Z}_n^{d+1}$ of length $(d+1)(n-1)+1$. Since $D(\mathbb{Z}_n^{d+1}) = (d+1)(n-1)+1$, we have, W has a nonempty zero-sum subsequence T of length t with $1 \leq t \leq (d+1)(n-1)+1$. That is, if necessary by renaming the indices, we see that

$$0 = \sigma(T) = \sum_{i=1}^t b_i = \left(\sum_{i=1}^t 1, \sum_{i=1}^t a_i\right) = \left(t, \sum_{i=1}^t a_i\right) \quad \text{in} \quad \mathbb{Z}_n^{d+1}.$$

This implies, $t = kn$ and $T' = \prod_{i=1}^{kn} a_i$ is a zero-sum subsequence of S of length kn with $1 \leq k \leq d$. $\square$

Corollary 2.1.1. *Let p be any prime number and r be any positive integer. Let S be a sequence in $\mathbb{Z}_{p^r}^d$ of length $(d+1)(p^r-1)+1$. Then S has a zero-sum subsequence of length kp^r with $1 \le k \le d$.*

Proof. Since $D(\mathbb{Z}_{p^r}^d) = d(p^r-1)+1$ for any positive integer d, the result follows from Lemma 2.1. $\square$

Definitions 2.2. Let $S = \prod_{i=1}^{\ell} g_i$ be a sequence in $\mathbb{Z}_p^d$. Then

$$f_E(S) = \left| \left\{ I \subset \{1, 2, \cdots, \ell\} \} \;\middle|\; \sum_{i \in I} g_i = 0,\ |I| \text{ even} \right\} \right|,$$

$$f_O(S) = \left| \left\{ I \subset \{1, 2, \cdots, \ell\} \} \;\middle|\; \sum_{i \in I} g_i = 0,\ |I| \text{ odd} \right\} \right|$$

and

$$r(S; l) = \left| \left\{ I \subset \{1, 2, \cdots, \ell\} \} \;\middle|\; \sum_{i \in I} g_i = 0,\ |I| = lp \right\} \right|.$$

Here, we follow the usual convention that the empty sequence (that is, when $I = \emptyset$) is a zero-sum sequence and hence $f_E(S) \ge 1$.

Theorem A. (Olson, [12]) *Let S be a sequence in $\mathbb{Z}_p^d$ such that $|S| \ge d(p-1)+1$. Then $f_E(S) \equiv f_O(S) \pmod{p}$.*

The following Lemma 2.2, Theorem 2.1 and Theorem 2.3 are interesting in itself; but we need these results for our main results.

Lemma 2.2. *Let $d \ge 2$ be a positive integer, and let l be an integer such that $1 \le l \le d$. Let $p \ge d+2$ be a prime number. Let T be a sequence in $\mathbb{Z}_p^d$ with $(d+1)(p-1)+1 \le |T| \le (d+2)p-1$. Suppose that T has no zero-sum subsequences of length kp for every $k \in \{1, 2, \cdots, d+1\} \setminus \{l\}$. Then*

$$r(T; l) \equiv (-1)^{l+1} \pmod{p}.$$

Proof. Set $t = |T|$, and suppose $T = \prod_{i=1}^{t} a_i$ with $(d+1)(p-1)+1 \le t \le (d+2)p-1$. Set $b_i = (1, a_i) \in \mathbb{Z}_p^{d+1}$ for every $i = 1, 2, \cdots, t$. Put $W = \prod_{i=1}^{t} b_i$. Let V' be a non-empty zero-sum subsequence of W. Such a sequence exists, as $t \ge D(\mathbb{Z}_p^{d+1}) = (d+1)(p-1)+1$. By the making of b_i, it is clear that $p||V'|$. Let V be corresponding zero-sum subsequence of T, then $p||V|$ and $|V| = kp$ with $k \in \{1, 2, \cdots, d+1\}$. Since T contains no zero-sum subsequence of length kp with $k \in \{1, 2, \cdots, d+1\} \setminus \{l\}$, we have $|V| = lp$. Therefore, either $r(T; l) = f_E(W) - 1$, if $2|l$ or $r(T; l) = f_O(W)$, if $2 \not| l$. By Theorem A, we know that $f_O(W) \equiv f_E(W) \pmod{p}$ which implies that either $r(T; l) + 1 = f_E(W) \equiv f_O(W) = 0 \pmod{p}$

provided that $2|l$, or $r(T;l) = f_O(W) \equiv f_E(W) = 1 \pmod p$ provided that $2 \not| l$. Therefore, $r(T;l) \equiv (-1)^{l+1} \pmod p$. □

Note. In the statement of Lemma 2.2, we have assumed an upper bound for $|T|$ to ensure that $|V| \neq (d+2)p$.

Theorem 2.1. *Let $d \geq 2$ be an integer and let $p \geq d+2$ be a prime number. Let l be an integer such that $1 \leq l \leq d$. Let S be a sequence in $\mathbb{Z}_p^d$ of length at least $(d+2)(p-1)+2$. Then S contains a zero-sum subsequence of length kp for some integer $k \in \{1, 2, \cdots, d+1\} \setminus \{l\}$. Moreover, for every $l \in \{1, 2, \cdots, d\} \setminus \{\frac{d+1}{2}\}$, S contains a zero-sum subsequence of length kp with $k \in \{1, 2, \cdots, d\} \setminus \{l\}$.*

Proof. Assume to the contrary that, there is a sequence S in $\mathbb{Z}_p^d$ with $|S| = (d+2)(p-1)+2$ and S contains no zero subsequences of length kp for every integer $k \in \{1, 2, \cdots, d+1\} \setminus \{l\}$. By Lemma 2.1, we have

$$r(T;l) \equiv (-1)^{l+1} \pmod p$$

holds for every subsequence T of S with $|T| \geq (d+1)(p-1)+1$. We, clearly, have

$$\sum_{T|S,\ |T|=(d+1)(p-1)+1} r(T;l) = \binom{(d+2)(p-1)+2-lp}{(d+1)(p-1)+1-lp} r(S;l).$$

Therefore,

$$\sum_{T|S,\ |T|=(d+1)(p-1)+1} (-1)^{l+1} \equiv \binom{(d+2-l)p-d}{(d+1-l)p-d} (-1)^{l+1} \pmod p.$$

This gives that

$$\binom{(d+2)(p-1)+2}{(d+1)(p-1)+1} \equiv \binom{(d+2-l)p-d}{(d+1-l)p-d} (\bmod p).$$

Since $p \geq d+2$,

$$\begin{aligned} d+1 &\equiv \binom{(d+2)(p-1)+2}{p} \equiv \binom{(d+2)(p-1)+2}{(d+1)(p-1)+1} \\ &\equiv \binom{(d+2-l)p-d}{(d+1-l)p-d} \equiv \binom{(d+2-l)p-d}{p} \\ &\equiv d+1-l \pmod p, \end{aligned}$$

which is a contradiction. This proves the first part of the theorem.

To prove the moreover part of the theorem, suppose $l \neq \frac{d+1}{2}$. By the first part of the theorem, there is a zero-sum subsequence V with $|V| = kp$ and

$k \in \{1, 2, \cdots, d+1\} \setminus \{l\}$. If $k \leq d$ then we are done. Otherwise, $|V| = (d+1)p$ and by Corollary 2.1.1 the sequence V contains a zero-sum subsequence W with $|W| = hp$ and $1 \leq h \leq d$. Therefore, VW^{-1} is also a zero-sum subsequence of $|T|$ with $|VW^{-1}| = (d+1-h)p$. By assuming that $h = l$ and $d+1-h = l$, we get $l = \frac{d+1}{2}$, a contradiction. Hence the proof completes. □

Definition 2.3. Let k be any positive integer. By $E_k(G)$, we denote the smallest positive integer t such that every sequence in G of length at least t contains a zero-sum subsequence T with $k \not| |T|$.

Theorem B. *If p is an odd prime and k is any positive integer such that $(k,p) = 1$, then*

$$E_k(\mathbb{Z}_p^d) = \left[\frac{k}{k-1}d(p-1)\right] + 1.$$

For $k = 2$, this was first proved by the first author in [7] and for general k by Wolfgang A. Schmid [15].

Theorem 2.2. *If p is an odd prime and k is any positive integer such that $(k,p) = 1$, then every sequence of length $\left[\frac{k}{k-1}(d+1)(p-1)\right] + 1$ in $\mathbb{Z}_p^d$ has a zero-sum subsequence of length rp with $k \not| r$.*

Proof. Let $\ell = \left[\frac{k}{k-1}(d+1)(p-1)\right] + 1$ and let $S = \prod_{i=1}^{\ell} a_i$ be a sequence in $\mathbb{Z}_p^d$ of length ℓ. Let $b_i = (1, a_i) \in \mathbb{Z}_p^{d+1}$ for $i = 1, 2, \cdots, \ell$. By Theorem B, we see that there exists a zero-sum subsequence T of $\prod_{i=1}^{\ell} b_i$ such that $k \not| |T|$. Set $l = |T|$. That is, by rearranging the indices, if necessary, we have,

$$0 = \sum_{i=1}^{l} b_i = \sum_{i=1}^{l} (1, a_i) = \left(l, \sum_{i=1}^{l} a_i\right) \text{ in } \mathbb{Z}_p^{d+1},$$

which implies, p divides l and $T' = \prod_{i=1}^{l} a_i$ is a zero-sum subsequence of S. Therefore, it is clear that $|T'| = rp$ for some integer r with $k \not| r$. □

Lemma 2.3. *Let S be a sequence in $\mathbb{Z}_3^3$ of length 12. Suppose S is not a zero-sum sequence. Then S contains a zero-sum subsequence of length 6.*

Proof. It is enough to assume that $v_g(S) \leq 5$ for every $g \in \mathbb{Z}_3^3$. Otherwise, we obviously have a zero subsequence of length 6. Then there exists a subsequence T of S of length 9 such that T is not a zero-sum subsequence. Now, by Corollary 2.2.1, T has a zero-sum subsequence T_1 of length 3 or 6. Assume that $|T_1| = 3$. Consider the sequence ST_1^{-1} which is of length 9. Since S is not a zero-sum sequence, ST_1^{-1} is not a zero-sum subsequence of S. Once again by Corollary

2.2.1, there exists a zero-sum subsequence T_2 of ST_1^{-1} of length 3 or 6. If $|T_2| = 3$, then T_1T_2 is the required zero-sum subsequence of length 6. Otherwise T_2 does the job. This completes the proof of the lemma. □

Lemma 2.4. *Let $d > 1$ be an integer and let ℓ be an integer such that $1 \leq \ell \leq d-1$. Then for any positive integer n we have*

$$s_\ell(\mathbb{Z}_n^d) \geq n(d+\ell) + \left[\frac{(d-\ell)n-1}{d-1}\right] - d.$$

Proof. Let

$$T = (1, 1, \cdots, 1)^s \prod_{i=1}^{d} e_i^{n-1},$$

where $e_i = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$ for all $i = 1, 2, \cdots, d$ and $s = [\frac{(d-\ell)n-1}{d-1}]$. Note that any zero-sum subsequence W of T will be of the form

$$W = (1, 1, \cdots, 1)^i \prod_{j=1}^{d} e_j^{n-i}$$

and hence $|W| = d(n-i)+i = dn-(d-1)i$. Since $s = [\frac{(d-\ell)n-1}{d-1}]$, it is clear that $|W| > \ell n$. Now, let $S = T(0, 0, \cdots, 0)^{\ell n-1}$ be a sequence in $\mathbb{Z}_n^d$ whose length is $|T| + \ell n - 1 = d(n-1) + s + n\ell - 1 = (d+\ell)n + s - d - 1$. Clearly, by the construction of S, we see that S doesn't have a zero-sum subsequence of length ℓn. Hence we have the desired inequality. □

Lemma 2.5. *Let $k, \ell \geq 1$ be integers. Then*

$$s_{k\ell}(G) \leq (\ell-1)k \exp(G) + s_k(G).$$

Proof. Let $m = (\ell-1)k\exp(G) + s_k(G)$ and let $S = \prod_{i=1}^{m} g_i$ be any sequence in G of length m. To prove the lemma, we shall prove that S has a zero-sum subsequence of length $k\ell \exp(G)$. By the definition of m, we can extract ℓ disjoint zero-sum subsequences, say, $T_1, T_2, \cdots, T_\ell$ of S such that $|T_i| = k\exp(G)$ for each i. Hence the sequence $T_1T_2\cdots T_\ell$ is the desired zero-sum subsequence of S. □

3. Proof our main results

Proof of Theorem 1.1. (1) (i) Put $d = 3$, $\ell = 2$ and $n = p$ in Lemma 2.4, we get $5p + \frac{p-1}{2} - 3 \leq s_2(\mathbb{Z}_p^3)$.

Now we shall prove that $s_2(\mathbb{Z}_p^3) \leq 6p-3$. Let S be a sequence in $\mathbb{Z}_p^3$ of length $6p-3$. Put $d = l = 3$ in Theorem 2.1. We get a zero-sum subsequence T of S of length p or $2p$. Assume that $|T| = p$. Then the deleted sequence $S_1 = ST^{-1}$, which is of length $5p-3$, has a zero-sum subsequence T_1 of length either p or $2p$ by Theorem 2.1, with $l = 3$. Assuming that $|T_1| = p$, we get a zero-sum sequence $T_2 = TT_1$ which is of length $2p$. Thus, $s_2(\mathbb{Z}_p^3) \leq 6p-3$.

(ii) In view of Equation (1), it is enough to prove that $s_3(\mathbb{Z}_p^3) \leq 8p-7$ for all prime $p \geq 5$. Let S be a sequence in $\mathbb{Z}_p^3$ of length $8p-7$. By Theorem 2.2, there exists a zero-sum subsequence T of S with $|T| = p, 3p, 5p$ or $7p$.

If $|T| = p$, then the deleted sequence ST^{-1} is of length $7p-7$. Applying $s_2(\mathbb{Z}_p^3) \leq 6p-3$, we see that the sequence ST^{-1} has a zero-sum subsequence T_1 of length $2p$. Thus TT_1 is the required zero-sum subsequence of S of length $3p$.

If $|T| = 5p$, then by putting $d = 3$ and $l = 1$ in Theorem 2.1, we get, T has zero-sum subsequence T_5 of length $2p$, or $3p$. Assume that $|T_5| = 2p$. Then look at the deleted sequence TT_5^{-1} which is a zero-sum sequence of length $3p$.

If $|T| = 7p$, then as $s_2(\mathbb{Z}_p^3) \leq 6p-3$, there exists a zero-sum subsequence T_2 of T of length $2p$. That is, T breaks into two zero-sum subsequences T_2 and T_3 of lengths $2p$ and $5p$ respectively. Since $|T_3| = 5p$, by the previous case, we are done again. Thus we have proved that $s_3(\mathbb{Z}_p^3) \leq 8p-7$ for all primes $p \geq 5$.

(iii) First we shall prove that $s_{2k}(\mathbb{Z}_p^3) = 2kp + 3p - 3$ and then prove that $s_{2k+1}(\mathbb{Z}_p^3) = (2k+1)p + 3p - 3$ for every integer $k \geq 2$.

Let S be a sequence in $\mathbb{Z}_p^3$ of length $2kp + 3p - 3$. If $k = 2$, then $|S| = 7p-3$. Since $s_2(\mathbb{Z}_p^3) \leq 6p-3$, S contains a zero-sum subsequence T_1 of length $2p$. Note that $|ST_1^{-1}| = 5p-3$. Using Theorem 2.1 with $l = 3$, we see that ST_1^{-1} has a zero-sum subsequence T_2 of length p or $2p$. If $|T_2| = 2p$, then T_1T_2 is a zero-sum subsequence of S of length $4p$ and we are done. So, we may assume that $|T_2| = p$. Since $|ST_1^{-1}T_2^{-1}| = 4p-3$, by Corollary 2.1.1, there is a zero subsequence T_3 of $ST_1^{-1}T_2^{-1}$ of length $p, 2p$ or $3p$. Therefore, $T_1T_2T_3$, T_1T_3 or T_2T_3 is a zero subsequence of S of length $4p$. Hence $s_4(\mathbb{Z}_p^3) \leq 7p-3$. Thus, by the inequality (1), we see that $s_4(\mathbb{Z}_p^3) = 4p + 3p - 3$.

Now, we shall assume the result is true for any $k \geq 2$ and prove it for $k+1$. By the virtue of inequality (1), it is enough to prove that $s_{2(k+1)}(\mathbb{Z}_p^3) \leq 2(k+1)p + 3p - 3$. Consider a sequence S_4 in $\mathbb{Z}_p^3$ of length $2(k+1)p + 3p - 3$. As $k \geq 2$, one can find a zero-sum subsequence T_4 of S_4 with $|T_4| = 2p$, as $s_2(\mathbb{Z}_p^3) \leq 6p-3$. Now, since the deleted sequence $S_5 = S_4T_4^{-1}$ has length $2kp + 2p + 3p - 3 - 2p = 2kp + 3p - 3$, by induction hypothesis, S_5 has a zero-sum subsequence W such that $|W| = 2kp$. Then T_4W is a zero-sum subsequence of S_4 with $|TW| = 2(k+1)p$. Thus it follows that $s_{2k}(\mathbb{Z}_p^3) = 2kp + 3p - 3$ for every integer $k \geq 2$.

First we shall prove that $s_5(\mathbb{Z}_p^3) = 8p-3$. It is enough to prove that $s_5(\mathbb{Z}_p^3) \leq 8p-3$. Let S be a sequence in $\mathbb{Z}_p^3$ of length $8p-3$. By Theorem 2.2, S contains a zero-sum subsequence T of length lp with $l \in \{1, 3, 5, 7\}$. Therefore it is enough to assume that $|T| = p, 3p$ or $7p$. If $|T| = p$, then apply $s_4(\mathbb{Z}_p^3) = 7p - 3$ to get a zero-sum subsequence T_1 of ST^{-1} of length $4p$ and we are done. Hence it is enough to assume that $|T| = 3p$ or $7p$. If $|T| = 7p$, again by using $s_4(\mathbb{Z}_p^3) = 7p-3$, one can get a zero-sum subsequence T_2 of T length $4p$ and its complement is of length $3p$. Thus, we may assume that S contains a zero-sum subsequence T of length $3p$. Note that $|ST^{-1}| = 5p - 3$, by Theorem 2.1, (by putting $d = l = 3$), there is a zero-sum subsequence W of ST^{-1} such that $|W| = kp$ with $k \in \{1, 2\}$. If $|W| = 2p$, then $|TW| = 5p$ and we are done. Otherwise, $|W| = p$ and it reduces to the above case. Thus $s_5(\mathbb{Z}_p^3) = 8p - 3$.

Now to prove $s_k(\mathbb{Z}_p^3) = kp + 3p - 3$ for every odd integer $k \geq 7$, consider a sequence S in $\mathbb{Z}_p^3$ of length $kp + 3p - 3$. Since $k \geq 7$, as $s_2(\mathbb{Z}_p^3) \leq 6p - 3$, S has a zero-sum subsequence T of length $2p$. Since the sequence ST^{-1} has length $(k-2)p+3p-3$, by the induction hypothesis, ST^{-1} has a zero-sum subsequence T_1 of length $(k-2)p$ (as $k-2 \geq 5$ and odd). Thus TT_1 is the required zero-sum subsequence of length kp.

(2) From the inequality (1), it is clear that $s_2(\mathbb{Z}_3^3) \geq 13$ and hence it is enough to prove that $s_2(\mathbb{Z}_3^3) \leq 13$. Let S be a sequence in $\mathbb{Z}_3^3$ of length 13. If $v_g(S) \geq 6$ for some $g \in \mathbb{Z}_3^3$, then we are done. So, we can assume that $v_g(S) \leq 5$ for every $g \in \mathbb{Z}_3^3$. Then one can find a subsequence T of S such that $|T| = 12$ and T is not a zero-sum subsequence of S. Therefore, by Lemma 2.3, we have a zero-sum subsequence of length 6. Thus, $s_2(\mathbb{Z}_3^3) = 13$.

Now, we shall prove that $s_3(\mathbb{Z}_3^3) \leq 17$. Let S be a sequence in $\mathbb{Z}_3^3$ of length 17. By putting $k = 2$ in Theorem 2.2, we see that S does have a zero-sum subsequence T of length $3, 9$ or 15. It is enough to assume that $|T| = 3$ or 15. If $|T| = 3$, then consider $S_1 = ST^{-1}$ which is of length 14. Since $s_2(\mathbb{Z}_3^3) = 13$, there exists a zero-sum subsequence of length 6 in ST^{-1} and hence there is a zero-sum subsequence of length 9 in S. Now, it remains to consider the case $|T| = 15$. Again by the value $s_2(\mathbb{Z}_3^3) = 13$, there exists a zero-sum subsequence T_1 of T of length 6 and hence TT_1^{-1} is a zero-sum subsequence of S and is of length 9. Hence $s_3(\mathbb{Z}_3^3) \leq 17$.

To complete the proof, we shall proceed by induction on k. When $k = 4$, by the inequality (1), it suffices to prove that $s_4(\mathbb{Z}_3^3) \leq 18$. Let S be a sequence in $\mathbb{Z}_3^3$ of length 18. We have to prove that S contains a zero-sum subsequence of length 12. As $s_2(\mathbb{Z}_3^3) = 13$, S contains a zero-sum subsequence T of length 6. If ST^{-1} is a zero-sum subsequence, then we are done as its length is 12. If ST^{-1} is not a zero-sum subsequence, then by Lemma 2.3, we have a zero-sum subsequence T_1 of ST^{-1} of length 6. Thus TT_1 is the required zero-sum subsequence of S of length 12.

So, we shall assume that $s_k(\mathbb{Z}_3^3) = 3k + 6$ for some $k \geq 4$ and prove it for $k + 1$. Let S be a sequence in $\mathbb{Z}_3^3$ of length $3(k + 1) + 6$. Since (see for instance, [9] and [10]) $s_1(\mathbb{Z}_3^3) = 19 < 3(k + 1) + 6$, S contains a zero-sum subsequence

T of length 3. As the length of the sequence ST^{-1} is $3k+6$, by the induction hypothesis, we see that ST^{-1} has a zero-sum subsequence of length $3k$. Hence S has a zero-sum subsequence of length $3k+3 = 3(k+1)$. Thus $s_k(\mathbb{Z}_3^3) = 3k+6$ for every $k \geq 4$.

(3) By inequality (1), we have $s_2(\mathbb{Z}_2^3) \geq 7$. So, we shall prove that $s_2(\mathbb{Z}_2^3) \leq 7$. Let S be a sequence in $\mathbb{Z}_2^3$ of length 7. By Corollary 2.1.1, we see that S contains a zero-sum subsequence T_1 of length 2 or 4. Assume that $|T_1| = 2$. Since ST_1^{-1} is of length 5, once again by Corollary 2.1.1, we get a zero-sum subsequence T_2 of length 2 or 4. If $|T_2| = 2$, then T_1T_2 is the required zero-sum subsequence of length 4 of S. Otherwise T_2 will do. Thus, $s_2(\mathbb{Z}_2^3) = 7$. Now, $s_3(\mathbb{Z}_2^3) = 9$ follows easily because we know that $s_1(\mathbb{Z}_2^3) = 9$ (see for instance, [9]) and $s_2(\mathbb{Z}_2^3) = 7$. Now the rest follows by a straight forward induction. □

Proof of Theorem 1.2. First let us prove that $s_6(\mathbb{Z}_p^4) \leq 12p-4$. Then by Lemma 2.5, the result follows. Let p be any prime with $p \geq 7$. Let S be a sequence in $\mathbb{Z}_p^4$ of length $12p-4$. By Theorem 2.1, we know that every sequence in $\mathbb{Z}_p^4$ of length $6p-4$ has a zero-sum subsequence of length ℓp with $\ell \in \{1,2,3,4\} \setminus \{r\}$ for every $r \in \{1,2,3,4\}$. We distinguish cases as follows:

Case 1. (S has two disjoint zero-sum subsequences T_1 and T_2 of length $3p$.)

In this case, it is clear that T_1T_2 forms a zero-sum subsequence of S of length $6p$ and we are done.

Case 2. (Case 1 doesn't hold but S has a zero-sum subsequence T of length $3p$.)

Then consider the deleted sequence ST^{-1} which is of length $9p-4$. Clearly ST^{-1} does not have zero-sum subsequence of length $3p$. By letting $l = 4 = d$ in Theorem 2.1, we get, ST^{-1} has disjoint zero-sum subsequences of lengths p, p, p or $p, 2p$ or $2p, 2p$. For the first two cases, we clearly have the desired zero-sum subsequence of length $6p$ of S. So, we may assume that ST^{-1} has two disjoint zero-sum subsequences T_1 and T_2 each of length $2p$. Note that $|ST^{-1}T_1^{-1}T_2^{-1}| = 5p-4$. By Corollary 2.1.1, the sequence $ST^{-1}T_1^{-1}T_2^{-1}$ has a zero-sum subsequence of length rp with $r \in \{1,2,3,4\}$ and we always get a zero-sum subsequence of length $6p$ of S for whatever value of r.

Case 3. (S does not have any zero-sum subsequence of length $3p$.)

By the assumption, it is only possible that S has disjoint zero subsequences of lengths $2p, 2p, 2p$ by letting $l = 4 = d$ in Theorem 2.1. Hence S has a zero-sum subsequence of length $6p$. □

Proof of Theorem 1.3. Let $p \geq 5$ be any prime and let S be a sequence in $\mathbb{Z}_p^3$ of length $9p-3$. Suppose S has at most two disjoint zero-sum subsequences of length $2p$. By Theorem 1.1 (1), we know that $s_6(\mathbb{Z}_p^3) = 9p-3$. Hence there exists a zero-sum subsequence T of S of length $6p$. Again using the value $s_2(\mathbb{Z}_p^3) \leq 6p-3$, there exists a zero-sum subsequence T_1 of T of length $2p$. Thus $T_2 = TT_1^{-1}$ is a zero-sum subsequence of T of length $4p$. By Corollary 2.1.1, we know that T_2 has a zero-sum subsequence T_3 of length p or $2p$ or $3p$. If $|T_3| = 2p$, then $T_2T_3^{-1}$ is also a zero subsequence of T_2 of length $2p$. Thus S has $T_1, T_2T_3^{-1}, T_3$ disjoint zero-sum subsequence of length $2p$ which is a contradiction to the assumption. Hence $|T_3| = p$ or $3p$. In either case, we have a zero-sum subsequence T_3 or $T_2T_3^{-1}$ of length p of S. This completes the proof of the theorem. □

Before we conclude this section, we shall discuss the following open problems and applications of our results.

Definition 3.1. By $\ell(G)$, we denote the smallest positive integer t such that $s_k(G) - k\exp(G) = D(G) - 1$ for every $k \geq t$.

Gao [6] proved that

$$\frac{D(G)}{\exp(G)} \leq \ell(G) \leq \frac{|G|}{\exp(G)}. \tag{2}$$

It is clear from the upper bound of the inequality (2) that the sequence $\{s_k(G) - k\exp(G)\}_{k=1}^{\infty}$ is eventually constant. Since $\ell(\mathbb{Z}_n) = 1$, the sequence $\{s_k(\mathbb{Z}_n) - kn\}$ is a constant sequence. From the introduction, it follows that $\ell(\mathbb{Z}_n^2) = 2$ and we see that the $s_1(\mathbb{Z}_n^2) - n > s_2(\mathbb{Z}_n^2) - 2n$ is strictly decreasing. So, the following conjecture seems to be plausible.

Conjecture 2. The sequence $\{s_k(G) - k\exp(G)\}_{k=1}^{\ell(G)-1}$ is strictly decreasing.

In [6], the following two conjectures have been posed.

Conjecture 3. (W. D. Gao, [6]) If $k \leq \ell(G) - 1$, then $s_k(G) - k\exp(G) \geq D(G)$.

We mentioned in the Preliminaries that Conjecture 3 is true for every $k < D(G)/n$. Also, one can easily see that if Conjecture 2 is true, then so is Conjecture 3.

Conjecture 4. (W. D. Gao, [6]) If $G \notin \{\mathbb{Z}_n, \mathbb{Z}_2^2\}$, then $\ell(G) < |G|/\exp(G)$.

Referee pointed out that the following recent work of S. Kubertin [11] related to this problem. Indeed, S. Kubertin [11] conjectured the following.

Conjecture 5. (S. Kubertin, [11]) For positive integers $k \geq d$ and n we have

$$s_k(\mathbb{Z}_n^d) = (k+d)n - d.$$

Conjecture 5 has been verified for all prime powers n and $k \geq n^{d-1}$ by Gao [6]. Also, Conjecture 5 has been verified in [11] for all $k = \ell p$, $n = p^r$ and for any integer $d > 1$. Also, he verifies Conjecture 5 for $n = p^r$ when $d = 3$ or 4.

If both Conjecture 1 and Conjecture 5 are true, then one easily see that $\ell(\mathbb{Z}_n^d) \leq d$. Therefore, Conjecture 4 is true for $G = \mathbb{Z}_n^d$.

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W. D. Gao
Center for Combinatorics
Nankai University
Tianjin 300071
China
email: gao@cfc.nankai.edu.cn

R. Thangadurai
School of Mathematics
Harish-Chandra Research Institute
Chhatnag Road, Jhunsi
Allahabad - 211019, India.
email: thanga@mri.ernet.in