A Telescoping Method for Double Summations

William Y.C. Chen[1], Qing-Hu Hou[2] and Yan-Ping Mu[3]

Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P. R. China
E-mail: [1]chen@nankai.edu.cn, [2]hou@nankai.edu.cn, [3]myphb@eyou.com

Dedicated to James D. Louck on the Occasion of His Seventy-Fifth Birthday

Abstract

We present a method to prove hypergeometric double summation identities. Given a hypergeometric term $F(n,i,j)$, we aim to find a difference operator $L = a_0(n)N^0 + a_1(n)N^1 + \cdots + a_r(n)N^r$ and rational functions $R_1(n,i,j), R_2(n,i,j)$ such that $LF = \Delta_i(R_1F) + \Delta_j(R_2F)$. Based on simple divisibility considerations, we show that the denominators of R_1 and R_2 must possess certain factors which can be computed from $F(n,i,j)$. Using these factors as estimates, we may find the numerators of R_1 and R_2 by guessing the upper bounds of the degrees and solving systems of linear equations. Our method is valid for the Andrews-Paule identity, the Carlitz's identities, the Apéry-Schmidt-Strehl identity, the Graham-Knuth-Patashnik identity, and the Petkovšek-Wilf-Zeilberger identity.

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1. Introduction

This paper is concerned with double summations of hypergeometric terms $F(n,i,j)$. A function $F(n,k_1,\ldots,k_m)$ is called a *hypergeometric term* if the quotients

$$\frac{F(n+1,k_1,\ldots,k_m)}{F(n,k_1,\ldots,k_m)},\quad \frac{F(n,k_1+1,\ldots,k_m)}{F(n,k_1,\ldots,k_m)},\quad \ldots,\quad \frac{F(n,k_1,\ldots,k_m+1)}{F(n,k_1,\ldots,k_m)}$$

are rational functions of $n,k_1,\ldots,k_m$. Throughout the paper, we use N to denote the shift operator with respect to the variable n, given by $NF(n) =$

$F(n+1)$ and use Δ_x to denote the difference operator with respect to the variable x, given by $\Delta_x F = F(x+1) - F(x)$. For polynomials a and b, we denote by $\gcd(a,b)$ their monic greatest common divisor. When we express a rational function as a quotient p/q, we always assume that p and q are relatively prime unless it is explicitly stated otherwise.

Zeilberger's algorithm [14, 17, 22], also known as the method of *creative telescoping*, is devised for proving hypergeometric identities of the form

$$\sum_k F(n,k) = f(n), \tag{1.1}$$

where $F(n,k)$ is a hypergeometric term and $f(n)$ is a given function. This algorithm has been used to deal with multiple sums by Wilf and Zeilberger [21]. Given a hypergeometric term $F(n,k_1,\ldots,k_m)$, the approach of Wilf and Zeilberger is to try to find a linear difference operator L with coefficients being polynomials in n

$$L = a_0(n)N^0 + a_1(n)N^1 + \cdots + a_r(n)N^r$$

and rational functions $R_1,\ldots,R_m$ of $n,k_1,\ldots,k_m$ such that

$$LF = \sum_{l=1}^{m} \Delta_{k_l}(R_l F). \tag{1.2}$$

As noted by K. Wegschaider [20], when the boundary conditions are admissible, Equation (1.2) leads to a homogenous recursion for the multi-summations:

$$L \sum_{k_1,\ldots,k_m} F(n,k_1,\ldots,k_m) = 0.$$

When $m = 1$, L and R_1 can be solved by Gosper's algorithm [13, 17]. S.A. Abramov, K.O. Geddes and H.Q. Le also provided a lower bound for the order r [2,3] and found a faster algorithm [4] compared with Zeilberger's algorithm. For a survey on recent developments, see [1]. For $m \geq 2$, constructing the denominators of $R_1,\ldots,R_m$ for the Wilf-Zeilberger approach remains an open problem. In a recent paper [16], M. Mohammed and D. Zeilberger used the denominator of LF/F as the estimate of the denominators of R_i. In an alternative approach, Wegschaider generalized Sister Celine's technique [20] to multiple summations, and proved many double summation identities. A different approach has been proposed by F. Chyzak [11,12] by finding recursions of the summation iteratively starting from the inner sum. C. Schneider [18] presented the Chyzak method from the point of view of Karr's difference field theory.

In this paper, we provide estimates of the denominators of R_1 and R_2 for double summations. These estimates turn out to be good enough for several double summation identities, including the Andrews-Paule identity for which our approach seems to be more suitable than Wegschaider's approach. It should be noted that the algorithm of Mohammed and Zeilberger sometimes gives higher order of recursions. In this sense, our method can be regarded as an improvement of the algorithm of Mohammed and Zeilberger.

To give a sketch of our approach, we first consider Gosper's algorithm for bivariate hypergeometric terms. Suppose that $F(i,j)$ is a hypergeometric term and $p_1/q_1, p_2/q_2$ are rational functions such that

$$F(i,j) = \Delta_i \left(\frac{p_1(i,j)}{q_1(i,j)} F(i,j) \right) + \Delta_j \left(\frac{p_2(i,j)}{q_2(i,j)} F(i,j) \right).$$

We show that under certain hypotheses (Section 2, $(H1)$–$(H3)$), the denominators q_1, q_2 can be written in the form

$$\begin{aligned} q_1(i,j) &= v_1(i)\, v_2(j)\, v_3(i+j)\, v_4(i,j)\, u_1(j)\, u_2(i,j), \\ q_2(i,j) &= v_1(i)\, v_2(j)\, v_3(i+j)\, v_4(i,j)\, w_1(j)\, w_2(i,j), \end{aligned} \tag{1.3}$$

such that v_1, v_2, v_4 and u_2, w_2 are bounded in the sense that they are factors of certain polynomials which can be computed for a given $F(i,j)$, see Theorem 2.1. Then we apply these estimates to the telescoping algorithm for double summations. Suppose that

$$LF(n,i,j) = \Delta_i\big(R_1(n,i,j)F(n,i,j)\big) + \Delta_j\big(R_2(n,i,j)F(n,i,j)\big),$$

where

$$R_1(n,i,j) = \frac{1}{d(n,i,j)} \cdot \frac{f_1(n,i,j)}{g_1(n,i,j)}, \qquad R_2(n,i,j) = \frac{1}{d(n,i,j)} \cdot \frac{f_2(n,i,j)}{g_2(n,i,j)}$$

and $d(n,i,j)$ is the denominator of $LF(n,i,j)/F(n,i,j)$. We may deduce that g_1, g_2 can be factored in the form of (1.3) such that v_1, v_2, v_4 and u_2, w_2

are bounded, see Theorem 3.1. Although we do not have the universal denominators, these bounds can be used to give estimates of the denominators g_1 and g_2. Then by further guessing the bounds of the degrees of the numerators of R_1 and R_2, we get the desired difference operator if we are lucky.

Indeed, our approach works quite efficiently for many identities such as the Andrews-Paule identity, Carlitz's identities, the Apéry-Schmidt-Strehl identity, the Graham-Knuth-Patashnik identity, and the Petkovšek-Wilf-Zeilberger identity.

2. Denominators in Bivariate Gosper's Algorithm

For a given bivariate hypergeometric term $F(i,j)$, we give estimates of the denominators of the rational functions $R_1(i,j), R_2(i,j)$ satisfying

$$F(i,j) = \Delta_i\big(R_1(i,j)F(i,j)\big) + \Delta_j\big(R_2(i,j)F(i,j)\big). \tag{2.1}$$

Let

$$\begin{aligned} R_1(i,j) &= \frac{f_1(i,j)}{g_1(i,j)}, \qquad & R_2(i,j) &= \frac{f_2(i,j)}{g_2(i,j)}, \\ \frac{F(i+1,j)}{F(i,j)} &= \frac{r_1(i,j)}{s_1(i,j)}, & \frac{F(i,j+1)}{F(i,j)} &= \frac{r_2(i,j)}{s_2(i,j)}. \end{aligned} \tag{2.2}$$

Dividing $F(i,j)$ on both sides of (2.1) and substituting (2.2) into it, we derive that

$$1 = \frac{r_1(i,j)}{s_1(i,j)}\frac{f_1(i+1,j)}{g_1(i+1,j)} - \frac{f_1(i,j)}{g_1(i,j)} + \frac{r_2(i,j)}{s_2(i,j)}\frac{f_2(i,j+1)}{g_2(i,j+1)} - \frac{f_2(i,j)}{g_2(i,j)}. \tag{2.3}$$

Let

$$u(i,j) = \gcd(s_1(i,j), s_2(i,j)), \qquad v(i,j) = \gcd(g_1(i,j), g_2(i,j)),$$

and

$$\begin{aligned} s_1'(i,j) &= s_1(i,j)/u(i,j), \qquad & s_2'(i,j) &= s_2(i,j)/u(i,j), \\ g_1'(i,j) &= g_1(i,j)/v(i,j), & g_2'(i,j) &= g_2(i,j)/v(i,j). \end{aligned} \tag{2.4}$$

We find that in many cases we can restrict our attention to those R_1, R_2 whose denominators g_1, g_2 satisfy the following three hypotheses. We see that in the proof of the following theorem, these hypotheses enable us to cancel out unknown factors from the multiples of g_1, g_2 so that we can obtain an upper bound of g_1 and g_2. Thus, these hypotheses come naturally from the requirement of simple divisibility properties. Moreover, it turns out that these divisibility requirements are sufficient in many cases to give good estimates for the denominators g_1 and g_2. The three hypotheses are as follows:

$(H1)$ Suppose $p(i,j)$ and $p(i+h_1, j+h_2)$ are both irreducible factors of $g_1(i,j)$ ($g_2(i,j)$, respectively) for some $h_1, h_2 \in \{-1,0,1\}$. Then they must be coincide.

$(H2)$ $\gcd(g_1'(i,j), v(i,j)) = \gcd(g_2'(i,j), v(i,j)) = 1$.

$(H3)$ For any integers $h_1, h_2 \in \{-1,0,1\}$,

$$\gcd(g_1'(i+h_1, j+h_2), g_2'(i,j)) = 1.$$

For example, the following functions satisfy the above hypotheses:

$$g_1(i,j) = (2n-2i+1)(n-i+1)(j+1)^2, \quad g_2(i,j) = (2n-2i+1)(n-i+1)(i+1)^2.$$

Remarks.

1. Hypothesis $(H1)$ looks like requiring that g_1 and g_2 are shift-free (see Abramov and Petkovšek [5]). However, only the shifts of ± 1 are considered and shift invariant factors are admissible. For example, we allow that $g_1(i,j) = (i+1)(i+3)$ or $g_1(i,j) = i+j$.

2. According to [6], $\gcd(g_1(i,j), g_1(i+h_1, j+h_2))$ and $\gcd(g_2(i,j), g_2(i+h_1, j+h_2))$ can factor into integer-linear factors for h_1, h_2 being not both zero.

3. Hypothesis $(H3)$ is to require that g_1'/g_2' are shift-reduce (see also [5]) respect to the shifts of ± 1.

Under the above hypotheses, we have

Theorem 2.1 *The denominators $g_1(i,j), g_2(i,j)$ can be factored into polynomials:*

$$g_1(i,j) = v_1(i)v_2(j)v_3(i+j)v_4(i,j)u_1(j)u_2(i,j),$$
$$g_2(i,j) = v_1(i)v_2(j)v_3(i+j)v_4(i,j)w_1(i)w_2(i,j),$$

such that

$$v_1(i) \mid r_1(i-1,j)s_2'(i-1,j), \tag{2.5}$$
$$v_2(j) \mid r_2(i,j-1)s_1'(i,j-1), \tag{2.6}$$
$$v_4(i,j) \mid \gcd\big(r_1(i-1,j)s_2'(i-1,j), r_2(i,j-1)s_1'(i,j-1)\big), \tag{2.7}$$
$$u_2(i,j) \mid \gcd\big(s_1(i,j)s_2'(i,j), r_1(i-1,j)s_2'(i-1,j)\big), \tag{2.8}$$
$$w_2(i,j) \mid \gcd\big(s_2(i,j)s_1'(i,j), r_2(i,j-1)s_1'(i,j-1)\big). \tag{2.9}$$

Proof. Substituting (2.4) into (2.3), we get

$$\begin{aligned} 1 \;=\; & \frac{r_1(i,j)}{s_1'(i,j)u(i,j)}\frac{f_1(i+1,j)}{g_1'(i+1,j)v(i+1,j)} - \frac{f_1(i,j)}{g_1'(i,j)v(i,j)} \\ & + \frac{r_2(i,j)}{s_2'(i,j)u(i,j)}\frac{f_2(i,j+1)}{g_2'(i,j+1)v(i,j+1)} - \frac{f_2(i,j)}{g_2'(i,j)v(i,j)}. \end{aligned}$$

That is,

$$\begin{aligned} & s_1(i,j)s_2'(i,j)g_1(i,j)g_2'(i,j)g_1(i+1,j)g_2(i,j+1) \\ & \quad = f_1(i+1,j)r_1(i,j)s_2'(i,j)g_1(i,j)g_2'(i,j)g_2(i,j+1) \\ & \qquad - f_1(i,j)s_1(i,j)s_2'(i,j)g_2'(i,j)g_1(i+1,j)g_2(i,j+1) \\ & \qquad + f_2(i,j+1)r_2(i,j)s_1'(i,j)g_1(i,j)g_2'(i,j)g_1(i+1,j) \\ & \qquad - f_2(i,j)s_1(i,j)s_2'(i,j)g_1'(i,j)g_1(i+1,j)g_2(i,j+1). \end{aligned}$$

1. Suppose $p(i,j)$ is an irreducible factor of $v(i,j)$, and for some non-negative integer l, $p^l \mid v$. Note that $p(i+h_1, j+h_2)$ is also irreducible. Since

   $$\gcd(p(i+1,j), f_1(i+1,j)) = \gcd(p(i,j+1), f_2(i,j+1)) = 1,$$

   we have

   $$p^l(i+1,j) \mid r_1(i,j)s_2'(i,j)g_1(i,j)g_2'(i,j)g_2(i,j+1)$$

and

$$p^l(i,j+1) \,|\, r_2(i,j)s_1'(i,j)g_1(i,j)g_2'(i,j)g_1(i+1,j).$$

There are three cases:

- $p(i,j)$ is a polynomial depending only on i. Then $\gcd(p(i+1,j),g_1(i,j)) = 1$. Otherwise, by hypothesis $(H1)$ we have that $p(i+1,j) = p(i,j)$ is independent of i, which is a contradiction. Similarly, $\gcd(p(i+1,j),g_2(i,j)) = 1$. Since $p(i,j)$ is a polynomial depending only on i, we have

  $$\gcd(p(i+1,j),g_2(i,j+1)) = \gcd(p(i+1,j+1),g_2(i,j+1)) = 1.$$

  Hence,

  $$p^l(i+1,j) \,|\, r_1(i,j)s_2'(i,j).$$

  Let $v_1(i)$ denote the product of all irreducible factors of $v(i,j)$ that depend only on i. Then we have (2.5).

- $p(i,j)$ is a polynomial depending only on j. The same discussion leads to

  $$p^l(i,j+1) \,|\, r_2(i,j)s_1'(i,j).$$

  Let $v_2(j)$ denote the product of all irreducible factors of $v(i,j)$ that depend only on j. Then we have (2.6).

- $p(i,j)$ is a polynomial depending both on i and on j. Then either

  $$p(i+1,j) = p(i,j+1) \tag{2.10}$$

  or

  $$\gcd(p(i+1,j),p(i,j+1)) = 1. \tag{2.11}$$

  In the former case, $p(i,j)$ is a polynomial of $i+j$ (see [6, Lemma 3] or [15, Lemma 3.3]). For this case we do not have a bound. We denote by $v_3(i+j)$ the product of all irreducible factors $p(i,j)$ of $v(i,j)$ that satisfy (2.10). In the later case, by hypothesis $(H1)$, we have

  $$\gcd(p(i+1,j),g_1(i,j)g_2'(i,j)g_2(i,j+1)) = 1$$

  and

  $$\gcd(p(i,j+1),g_1(i,j)g_2'(i,j)g_1(i+1,j)) = 1.$$

Thus,

$$p^l(i,j) \mid \gcd\big(r_1(i-1,j)s_2'(i-1,j), r_2(i,j-1)s_1'(i,j-1)\big).$$

Let $v_4(i,j)$ denote the product of all irreducible factors $p(i,j)$ of $v(i,j)$ that satisfy (2.11). Then we have (2.7).

2. Suppose p is an irreducible factor of g_1' and $p^l|g_1'$ for some non-negative integer l. If $p(i,j) \mid v(i,j+1)$, then $p(i,j-1) \mid v(i,j)$. By hypothesis $(H1)$, $p(i,j-1) = p(i,j)$, which implies $p(i,j) \mid v(i,j)$, contradicting to hypothesis $(H2)$. Noting further that by hypothesis $(H3)$, for any $h_1, h_2 \in \{-1, 0, 1\}$,

$$\gcd(f_1(i,j), g_1(i,j)) = \gcd(g_1'(i,j), g_2'(i+h_1, j+h_2)) = 1,$$

we must have that

$$p^l(i,j) \mid s_1(i,j)s_2'(i,j)g_1(i+1,j).$$

If $p(i+1,j) \mid v(i,j+1)$, then by hypothesis $(H1)$, $p(i+1,j-1) = p(i,j)$, which implies $p(i,j) \mid v(i,j)$, contradicting to hypothesis $(H2)$. Therefore, by hypothesis $(H3)$,

$$p^l(i+1,j) \mid r_1(i,j)s_2'(i,j)g_1(i,j).$$

There are two cases:

- $p(i,j) = p(i+1,j)$. Then $p(i,j)$ is a polynomial depending only on j. For this case we also do not have a bound. We denote by $u_1(j)$ the product of all irreducible factors of $g_1'(i,j)$ that depend only on j.
- $\gcd(p(i,j), p(i+1,j)) = 1$. Then by hypothesis $(H1)$,

  $$\gcd(p(i,j), g_1(i+1,j)) = \gcd(p(i+1,j), g_1(i,j)) = 1,$$

  and hence,

  $$p^l(i,j) \mid \gcd\big(s_1(i,j)s_2'(i,j), r_1(i-1,j)s_2'(i-1,j)\big).$$

  Let $u_2(i,j)$ denote the product of all irreducible factors $p(i,j)$ of $g_1'(i,j)$ such that $\gcd(p(i,j), p(i+1,j)) = 1$. Then we have (2.8).

3. Similarly, suppose p is an irreducible factor of g'_2 and $p^l|g'_2$ for some non-negative integer l. Then either $p(i,j)$ is a polynomial depending only on i or

$$p^l(i,j) \mid \gcd\left(s_2(i,j)s'_1(i,j), r_2(i,j-1)s'_1(i,j-1)\right).$$

Let $w_1(i)$ denote product of irreducible factors of $g'_2(i,j)$ that depend only on i and $w_2(i,j)$ denote the product of the rest irreducible factors of $g'_2(i,j)$. Then we have (2.9). ∎

Note that $u_2(i,j)$ have no factors which are free of i and $w_2(i,j)$ have no factors which are free of j. We will need this property later for the algorithm EstDen.

3. Denominators in Our Telescoping Method

We are now ready to estimate the denominators of R_1 and R_2 in our telescoping method.

As in the case of single summations, the telescoping algorithm for double summations tries to find an operator

$$L = a_0(n) + a_1(n)N + \cdots a_r(n)N^r$$

and rational functions $R_1(n,i,j), R_2(n,i,j)$ such that

$$LF(n,i,j) = \Delta_i(R_1(n,i,j)F(n,i,j)) + \Delta_j(R_2(n,i,j)F(n,i,j)). \tag{3.1}$$

Let

$$\frac{F(n,i+1,j)}{F(n,i,j)} = \frac{r_1(n,i,j)}{s_1(n,i,j)}, \qquad \frac{F(n,i,j+1)}{F(n,i,j)} = \frac{r_2(n,i,j)}{s_2(n,i,j)}, \tag{3.2}$$

and $d(n,i,j)$ be the common denominator of

$$\frac{F(n+1,i,j)}{F(n,i,j)}, \quad \ldots, \quad \frac{F(n+r,i,j)}{F(n,i,j)}.$$

Then there exists a polynomial $c(n,i,j)$, not necessarily being coprime to d, such that

$$\frac{LF(n,i,j)}{F(n,i,j)} = \sum_{l=0}^{r} a_l(n)\frac{F(n+l,i,j)}{F(n,i,j)} = \frac{c(n,i,j)}{d(n,i,j)}. \tag{3.3}$$

Note that c is related to the polynomials $a_0, a_1, \ldots, a_r$ but d is independent of them.

Now, (3.1) can be written in the form of (2.1):

$$LF(n,i,j) = \Delta_i(R_1'(n,i,j)LF(n,i,j)) + \Delta_j(R_2'(n,i,j)LF(n,i,j)),$$

where

$$R_1'(n,i,j) = R_1(n,i,j)\frac{d(n,i,j)}{c(n,i,j)} \quad \text{and} \quad R_2'(n,i,j) = R_2(n,i,j)\frac{d(n,i,j)}{c(n,i,j)}.$$

This suggests us to assume

$$R_1(n,i,j) = \frac{1}{d(n,i,j)}\frac{f_1(n,i,j)}{g_1(n,i,j)} \quad \text{and} \quad R_2(n,i,j) = \frac{1}{d(n,i,j)}\frac{f_2(n,i,j)}{g_2(n,i,j)}, \tag{3.4}$$

where f_1, g_1 (f_2, g_2, respectively) are relatively prime polynomials.

Since the following discussion is independent of n, we omit the variable n for convenience. For example, we write $R_1(i,j)$ instead of $R_1(n,i,j)$. Using these notations, we have

Theorem 3.1 *Suppose the polynomials g_1, g_2 in (3.4) satisfy the hypotheses (H1)–(H3). Suppose further that for any $h_1, h_2 \in \{-1,0,1\}$,*

$$\gcd(g_1(i,j), d(i+h_1, j+h_2)) = \gcd(g_2(i,j), d(i+h_1, j+h_2)) = 1. \tag{3.5}$$

Then $g_1(i,j), g_2(i,j)$ can be factored into polynomials:

$$g_1(i,j) = v_1(i)v_2(j)v_3(i+j)v_4(i,j)u_1(j)u_2(i,j),$$
$$g_2(i,j) = v_1(i)v_2(j)v_3(i+j)v_4(i,j)w_1(i)w_2(i,j),$$

such that

$$v_1(i) \mid r_1(i-1,j)s_2'(i-1,j),$$
$$v_2(j) \mid r_2(i,j-1)s_1'(i,j-1),$$
$$v_4(i,j) \mid \gcd\big(r_1(i-1,j)s_2'(i-1,j), r_2(i,j-1)s_1'(i,j-1)\big),$$
$$u_2(i,j) \mid \gcd\big(s_1(i,j)s_2'(i,j), r_1(i-1,j)s_2'(i-1,j)\big),$$
$$w_2(i,j) \mid \gcd\big(s_2(i,j)s_1'(i,j), r_2(i,j-1)s_1'(i,j-1)\big),$$

where

$$s_1'(i,j) = s_1(i,j)/\gcd(s_1(i,j), s_2(i,j)),$$
$$s_2'(i,j) = s_2(i,j)/\gcd(s_1(i,j), s_2(i,j)). \tag{3.6}$$

Proof. Substituting (3.4) into (3.1) and dividing $F(i,j)$ on both sides, we obtain

$$\begin{aligned} \frac{c(i,j)}{d(i,j)} &= \frac{r_1(i,j)}{s_1(i,j)}\frac{f_1(i+1,j)}{d(i+1,j)g_1(i+1,j)} - \frac{f_1(i,j)}{d(i,j)g_1(i,j)} \\ &\quad + \frac{r_2(i,j)}{s_2(i,j)}\frac{f_2(i,j+1)}{d(i,j+1)g_2(i,j+1)} - \frac{f_2(i,j)}{d(i,j)g_2(i,j)}, \end{aligned} \tag{3.7}$$

i.e.,

$$\begin{aligned} c(i,j) &= \frac{r_1(i,j)d(i,j)}{s_1(i,j)d(i+1,j)}\frac{f_1(i+1,j)}{g_1(i+1,j)} - \frac{f_1(i,j)}{g_1(i,j)} \\ &\quad + \frac{r_2(i,j)d(i,j)}{s_2(i,j)d(i,j+1)}\frac{f_2(i,j+1)}{g_2(i,j+1)} - \frac{f_2(i,j)}{g_2(i,j)}. \end{aligned}$$

Let

$$\tilde r_1(i,j) = r_1(i,j)d(i,j), \quad \tilde s_1(i,j) = s_1(i,j)d(i+1,j),$$
$$\tilde r_2(i,j) = r_2(i,j)d(i,j), \quad \tilde s_2(i,j) = s_2(i,j)d(i,j+1).$$

All discussion in the proof of Theorem 2.1 still holds. Thus, we have

$$\begin{aligned} &v_1(i) \mid \tilde r_1(i-1,j)\tilde s_2'(i-1,j), \\ &v_2(j) \mid \tilde r_2(i,j-1)\tilde s_1'(i,j-1), \\ &v_4(i,j) \mid \gcd\left(\tilde r_1(i-1,j)\tilde s_2'(i-1,j), \tilde r_2(i,j-1)\tilde s_1'(i,j-1)\right), \\ &u_2(i,j) \mid \gcd\left(\tilde s_1(i,j)\tilde s_2'(i,j), \tilde r_1(i-1,j)\tilde s_2'(i-1,j)\right), \\ &w_2(i,j) \mid \gcd\left(\tilde s_2(i,j)\tilde s_1'(i,j), \tilde r_2(i,j-1)\tilde s_1'(i,j-1)\right), \end{aligned} \tag{3.8}$$

where

$$\tilde s_1'(i,j) = \tilde s_1(i,j)/\gcd(\tilde s_1(i,j), \tilde s_2(i,j)),$$
$$\tilde s_2'(i,j) = \tilde s_2(i,j)/\gcd(\tilde s_1(i,j), \tilde s_2(i,j)).$$

Since we have (3.5), we may replace $\tilde r_1, \tilde s_1, \tilde r_2, \tilde s_2$ by r_1, s_1, r_2, s_2 in (3.8), respectively. ∎

4. A Telescoping Method for Bivariate Hypergeometric Terms

Theorem 3.1 enables us to choose the denominators in the telescoping algorithm. Basically, we will use certain factors appearing in the bounds of the denominators as estimates of the denominators. In many cases, this approach seems to work quite efficiently although we are not able to give formula to bound the denominators because certain factors are not bounded in Theorem 3.1. Roughly speaking, the divisibility considerations in our method serve as a guide to guess the factors in the denominators. In fact, the estimated denominators are much smaller than the theoretical bounds given by Theorem 3.1. Only $u_2(i,j)$ and $w_2(i,j)$ are set to their theoretical bounds, while $v_2(j), v_3(i+j), v_4(i,j)$ are set to 1, $u_1(j)$ and $w_1(i)$ are set to factors of $s_1(i,j)s_2'(i,j)$, and $v_1(i)$ is set to a factor of its theoretical bound. See the following algorithm EstDen.

Algorithm EstDen
Input: A hypergeometric term $F(n,i,j)$.
Output: Estimated denominators $g_1(i,j)$ and $g_2(i,j)$ for bivariate Gosper's algorithm.

1. Calculate $r_1, r_2, s_1, s_2, s_1', s_2'$ defined by (3.2) and (3.6);

2. Set
   $v_1(i) :=$ the maximal factor of $r_1(i,j)s_2'(i,j)$ depending only on i;
   $v_2(j) :=$ the maximal factor of $r_2(i,j)s_1'(i,j)$ depending only on j;
   and
   $$v(i) := \gcd(v_1(i-1), v_2(i-1));$$

3. Set
   $u_1(j) :=$ the maximal factor of $s_1(i,j)s_2'(i,j)$ depending only on j;
   $w_1(i) :=$ the maximal factor of $s_1(i,j)s_2'(i,j)$ depending only on i;

4. Set $u_2(i,j)$ to be the maximal factor of
   $$\gcd(s_1(i,j)s_2'(i,j), r_1(i-1,j)s_2'(i-1,j))$$
   which depends on i;
   Set $w_2(i,j)$ to be the maximal factor of
   $$\gcd(s_1(i,j)s_2'(i,j), r_2(i,j-1)s_1'(i,j-1))$$

which depends on j.

5. Return $g_1(i,j) := v(i)u_1(j)u_2(i,j)$ and $g_2(i,j) := v(i)w_1(i)w_2(i,j)$.

Remark. Let $f(i,j)$ be a polynomial in i, j and a be a new variable. Then the maximal factor of $f(i,j)$ depending only on i can be obtained by

$$\gcd(f(i,j), f(i,j+a)),$$

and the maximal factor of $f(i,j)$ depending on i can be obtained by

$$f(i,j)/\gcd(f(i,j), f(i+a,j)).$$

We are now ready to describe our telescoping method for double summations:

Method BiZeil
Input: A hypergeometric term $F(n,i,j)$.
Output: An operator L and rational functions R_1 and R_2 such that (3.1) holds if succeed.

1. Using algorithm EstDen to obtain g_1 and g_2.

2. Set the order r of the linear difference operator L to be zero.

3. For the order r, calculate the common denominator $d(n,i,j)$ of

$$\frac{F(n+1,i,j)}{F(n,i,j)}, \quad \ldots, \quad \frac{F(n+r,i,j)}{F(n,i,j)}.$$

   (If $r = 0$, then take $d(n,i,j) = 1$.)

4. Set the degrees of f_1 and f_2 to be one more than those of $d \cdot g_1$ and $d \cdot g_2$, respectively.

5. Solve the equation (3.7) by undeterminate coefficients method to obtain $a_0, a_1, \ldots, a_r$ and f_1, f_2.

6. If $a_i \neq 0$ for some $i \in \{0, \ldots, r\}$, then return $L, f_1/(d \cdot g_1), f_2/(d \cdot g_2)$ and we are done.

   If $a_i = 0$ for all $i \in \{0, \ldots, r\}$, but $\deg f_1 - \deg(d \cdot g_1) \leq 2$, then increase the degrees of f_1 and f_2 by one and repeat step 5.

   Otherwise, set $r := r + 1$ and repeat the process from step 3.

Remarks.

1. In many cases, $g_1(i,j)$ and $g_2(i,j)$ can be further reduced by cancelling a factor of degree 1 and a factor of degree 2 from g_1 and g_2, respectively. In our implementation we first choose two arbitrary factors and use the reduced g_1 and g_2. When it fails, we then try the unreduced ones. This cancellation may reduce the time of calculation if we are lucky. For example, for the Andrews-Paule identity (see Example 1 in the following), the estimated denominators given by Theorem 3.1, by algorithm EstDen, and by reduction are, respectively,

   $$g_1(i,j) = (2n-2i+1)(n-i+1)(2n-2j+1)(n-j+1)(i+j)^2(j+1)^2,$$
   $$g_2(i,j) = (2n-2i+1)(n-i+1)(2n-2j+1)(n-j+1)(i+j)^2(i+1)^2;$$

   $$g_1(i,j) = (2n-2i+1)(n-i+1)(j+1)^2,$$
   $$g_2(i,j) = (2n-2i+1)(n-i+1)(i+1)^2;$$

   and

   $$g_1(i,j) = (2n-2i+1)(j+1)^2, \quad g_2(i,j) = (2n-2i+1)(n-i+1).$$

   The calculation times are 116 seconds, 5 seconds and 0.6 seconds, respectively. We should note that since our method is heuristic and it applies only to particular cases, we are more interested in the computation results which are verifiable. So we cannot claim the efficiency of the method or its applicability.

2. In all the following examples except Example 4, the degree of the numerator of R_1 (R_2) is one more than that of the denominator. While in Example 4, the difference is two.

   The degree bounds can be interpreted as follows. Let t_1, t_2, t_3, t_4 be the four terms of the right hand side of (3.7) after multiplying the common denominator. In most cases, the leading terms of t_1 and t_2 (t_3 and t_4, respectively) are cancelled.

3. There is a way to speed up the computation in Step 5. Given g_1 and g_2, we may derive part of the factors of f_1 and f_2 by divisibility. For example, suppose (3.7) becomes

   $$\frac{c(i,j)}{d(i,j)} = \frac{u_1(i,j)}{v_1(i,j)} f_1(i+1,j) - \frac{f_1(i,j)}{w_1(i,j)} + \frac{u_2(i,j)}{v_2(i,j)} f_2(i,j+1) - \frac{f_2(i,j)}{w_2(i,j)},$$

after substituting and simplification. Suppose further that $D(i,j)$ is the common denominator of the above equation. Then we immediately have that $f_1 \cdot D/w_1$ is divisible by $q_1 = \gcd(cD/d, u_1D/v_1, u_2D/v_2, D/w_2)$ and $f_1(i{+}1,j) \cdot u_1D/v_1$ is divisible by $q_2 = \gcd(cD/d, D/w_1, u_2D/v_2, D/w_2)$, and hence,

$$\frac{q_1}{\gcd(D/w_1, q_1)} \quad \text{and} \quad \frac{q_2}{\gcd(u_1D/v_1, q_2)}$$

are factors of $f_1(i,j)$ and $f_1(i+1,j)$, respectively.

5. Examples

In the following examples, let F denote the summand of the left hand side of the identity.

Example 1. The Andrews-Paule identity:

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{i}^2\binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2. \tag{5.1}$$

It was proved by G. Andrews and P. Paule [7, 8] by establishing a more general identity

$$\sum_{i=0}^{\lfloor \frac{m}{2}\rfloor}\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\binom{i+j}{i}^2\binom{m+n-2i-2j}{n-2i} = \frac{\lfloor\frac{m+n+1}{2}\rfloor!\lfloor\frac{m+n+2}{2}\rfloor!}{\lfloor\frac{m}{2}\rfloor!\lfloor\frac{m+1}{2}\rfloor!\lfloor\frac{n}{2}\rfloor!\lfloor\frac{n+1}{2}\rfloor!}.$$

Using the method BiZeil, we can deal with (5.1) directly. In fact, we have

$$g_1(i,j) = (2n{-}2i{+}1)(n{-}i{+}1)(j{+}1)^2, \quad g_2(i,j) = (2n{-}2i{+}1)(n{-}i{+}1)(i{+}1)^2.$$

Cancelling the factors $(n-i+1)$ and $(i+1)^2$ from $g_1(i,j)$ and $g_2(i,j)$, respectively, we obtain

$$\tilde{g}_1(i,j) = (2n-2i+1)(j+1)^2 \quad \text{and} \quad \tilde{g}_2(i,j) = (2n-2i+1)(n-i+1).$$

Finally, we get (in 1 second)

$$(2n+1)F(n,i,j) = \Delta_i R_1 F(n,i,j) + \Delta_j R_2 F(n,i,j),$$

where

$$R_1 = \frac{i^2(6n^2+5n+1+6jn^2+jn-j-in+2in^2-2i-4j^2n-2j^2-3ij-4ijn)}{(2n-2i+1)(1+j)^2},$$

$$R_2 = \frac{-2n^2+2jn^2+6in^2+9in+3jn-4ijn-4i^2n-n+j-3ij+2i-4i^2}{(2n-2i+1)},$$

which are the same as given in [20, p. 85]. Summing $i, j = 0, \ldots, n$, we get

$$\begin{aligned}(2n+1)&\sum_{i=0}^{n}\sum_{j=0}^{n} F(n,i,j)\\ =& \sum_{i=0}^{n}\big(R_2F(n,i,n+1)-R_2F(n,i,0)\big)+\sum_{j=0}^{n}\big(R_1F(n,n+1,j)-R_1F(n,0,j)\big)\end{aligned}$$

Note that there is only one nonzero term $R_1F(n,n+1,n)$ of the second summation. While applying Gosper's algorithm to the first summand, we obtain

$$\sum_{i=0}^{n}\big(R_2F(n,i,n+1)-R_2F(n,i,0)\big) = G(n+1)-G(0).$$

where

$$G(i) = \frac{(-2n+i-1)(-4n+2i-1)i}{-1+2i-2n}\binom{4n-2i}{2n-2i}.$$

Simplifying $G(n+1)-G(0)+R_1F(n,n+1,n)$, we finally get (5.1).

Example 2. Carlitz's identity [10] (see Also [21, Example 6.1.2]):

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{i}\binom{n-i}{j}\binom{n-j}{n-i-j} = \sum_{l=0}^{n}\binom{2l}{l}.$$

We have

$$g_1(i,j) = (j+1)^2(-n+j), \quad g_2(i,j) = (i+1)^2(-n+i).$$

Cancelling the factors $(-n+j)$ and $(i+1)(-n+i)$, we obtain

$$\tilde{g}_1(i,j) = (j+1)^2 \quad \text{and} \quad \tilde{g}_2(i,j) = i+1.$$

Notice that the common denominator of

$$\frac{F(n+1,i,j)}{F(n,i,j)} \quad \text{and} \quad \frac{F(n+2,i,j)}{F(n,i,j)}$$

is $(-n+i-1+j)^2(-n+i-2+j)^2$. We finally get (in 2 seconds)

$$L = (4n+6) - (8+5n)N + (n+2)N^2,$$

and

$$\begin{aligned} R_1 = \big(&-i^2(-n+i-1)(36-10ji^2n-13j^2ni+60j^2+60ji-2i^2-38j^2i \\ &-8ji^2+10i^3+36n^3-11in^3-14jn^3-2i^4-92jn^2+8i^2n-80in+5j^2n^2 \\ &+8j^2i^2+88jin+42j^2n-172jn+24jin^2+5i^2n^2+3i^3n-54in^2+88n^2+4j^3n \\ &-90j+6j^3-40i+5n^4+90n)\big) \Big/ \big((-n+i-1+j)^2(-n+i-2+j)^2(j+1)^2\big), \end{aligned}$$

$$\begin{aligned} R_2 = \big((&64-19ji^2n-6j^2ni+14j^2+74ji+54i^2-10j^2i-36ji^2+2i^3+39n^3 \\ &-16in^3-9jn^3-4i^4+6ji^3-53jn^2+50i^2n-176in+4j^2n^2+4j^2i^2+5n^4 \\ &+83jin+16j^2n-100jn+22jin^2+11i^2n^2+4i^3n-93in^2+112n^2-60j \\ &-108i+140n)(-n-1+j)\big) \Big/ \big((-n+i-2+j)^2(-n+i-1+j)^2\big), \end{aligned}$$

such that

$$LF(n,i,j) = \Delta_i R_1 F(n,i,j) + \Delta_j R_2 F(n,i,j), \tag{5.2}$$

By summing (5.2) over i, j from 0 to n, one derives that L annihilates the double sum on the left hand side. It is easily seen that the right hand side can be annihilated by $\big((n+2)N-(4n+6)\big)(N-1)$, which is exactly L. Then the identity follows from the initial values $n = 0, 1$.

The proofs of the following examples are similar to that of Example 2. We only need to give $\tilde{g}_1, \tilde{g}_2$ and L, R_1, R_2. Then these identities can be verified by checking the initial values.

Example 3. Carlitz's identity [9] (see also [21, Example 6.1.3]):

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{i+j}{i}\binom{m-i+j}{j}\binom{n-j+i}{i}\binom{m+n-i-j}{m-i} = \frac{(m+n+1)!}{m!n!}\sum_{k}\frac{1}{2k+1}\binom{m}{k}\binom{n}{k}.$$

By cancelling the factors $(1+j)$ and $(i+1)^2$, we obtain

$$\tilde{g}_1(i,j) = (n-j+i)(1+j) \quad \text{and} \quad \tilde{g}_2(i,j) = m-i+j$$

Notice that the common denominator of

$$\frac{F(n+1,i,j)}{F(n,i,j)} \quad \text{and} \quad \frac{F(n+2,i,j)}{F(n,i,j)}$$

is $(-n+j-1)^2(-n+j-2)^2$, which is denoted by $d(i,j)$. We finally get (in 37 seconds)

$$\begin{aligned} L = {} & 2(m+3+n)(2+m+n)^2 - \\ & (3m+2nm+4n^2+14+15n)(n+m+3)N + (2n+5)(n+2)^2N^2, \end{aligned}$$

and the denominators of R_1, R_2 are $d(i,j)\tilde{g}_1(i,j)$ and $d(i,j)\tilde{g}_2(i,j)$, respectively. The degrees of denominators and numerators of R_1, R_2 are both less than those given in [21].

Example 4. The Apéry-Schmidt-Strehl identity [19]:

$$\sum_i \sum_j \binom{n}{j}\binom{n+j}{j}\binom{j}{i}^3 = \sum_k \binom{n}{k}^2\binom{n+k}{k}^2.$$

By cancelling the factors $(-j-1+i)$ and $(i+1)^2$, we obtain

$$\tilde{g}_1(i,j) = (-j-1+i)^2 \quad \text{and} \quad \tilde{g}_2(i,j) = i+1$$

Notice that the common denominator of

$$\frac{F(n+1,i,j)}{F(n,i,j)} \quad \text{and} \quad \frac{F(n+2,i,j)}{F(n,i,j)}$$

is $(n+2-j)(n+1-j)$. We finally get (in 1 second)

$$L = (n+1)^3 - (3+2n)(17n^2+51n+39)N + (n+2)^3N^2,$$

and

$$\begin{aligned} R_1 = {} & \big(-2i^2(3+2n)(-10+30j^2-49n^2-j^3-4n^4-24n^3-2n^2i^2+n^2i-6ni^2 \\ & +3ni+3nji+n^2ji+3j^2i^2-3j^3i+3ji-4i^2-2j^2i-2ji^2+11n^2j^2+6n^2j \\ & +33nj^2+18nj-6j^4+2i+15j-39n)\big)\Big/\big((n+2-j)(n+1-j)(-j-1+i)^2\big), \end{aligned}$$

$$R_2 = \big(2(-j+i)(3+2n)(-8n^2i - 4n^2i^2 - 4n^2ji + 4n^2j + 4n^2j^2 + 12nj - 12nji - 24ni + 12nj^2 - 12ni^2 + 12j^2 - 4ji^2 + j^3 + 6j^2i^2 - 3j^4 + 8j + 5j^2i - 8i^2 + 3j^3i - 16i - 16ji)\big) \Big/ \big((n+2-j)(n+1-j)(i+1)\big).$$

The rational functions R_1, R_2 are simpler than those given in [19]. The operator L was used by Apéry in his proof of the irrationality of $\zeta(3)$ and Chyzak and Salvy obtained it using Ore algebras [12].

Example 5. The Strehl identity [19]:

$$\sum_i \sum_j \binom{n}{j}\binom{n+j}{j}\binom{j}{i}^2\binom{2i}{i}^2\binom{2i}{j-i} = \sum_k \binom{n}{k}^3\binom{n+k}{k}^3. \tag{5.3}$$

By cancelling the factor $(-3i-3+j)(-3i-2+j)$ from g_2, we obtain

$$\tilde{g}_1(i,j) = (j+1-i)^3 \quad \text{and} \quad \tilde{g}_2(i,j) = (-3i-1+j)(i+1)^3.$$

Notice that the common denominator of

$$\frac{F(n+1,i,j)}{F(n,i,j)}, \;\ldots,\; \frac{F(n+6,i,j)}{F(n,i,j)}$$

is $(n+1-j)(n+2-j)\cdots(n+6-j)$, which is denoted by $d(i,j)$. We finally get (in 2510 seconds) a linear difference operator L of order 6 and the denominators of R_1, R_2 are $d(i,j)\tilde{g}_1(i,j)$ and $d(i,j)\tilde{g}_2(i,j)$, respectively. The operator L is the same as the operator obtained by applying Zeilberger's algorithm to the right hand side of (5.3).

Example 6. The Graham-Knuth-Patashnik identity [14, p. 172]:

$$\sum_j \sum_k (-1)^{j+k}\binom{j+k}{k+l}\binom{r}{j}\binom{n}{k}\binom{s+n-j-k}{m-j} = (-1)^l\binom{n+r}{n+l}\binom{s-r}{m-n-l}. \tag{5.4}$$

By cancelling the factor $(j+1)(j+1-l)$ from g_2, we obtain

$$\tilde{g}_1(j,k) = (k+1)(k+l+1) \quad \text{and} \quad \tilde{g}_2(j,k) = 1.$$

Notice that the denominator of $\frac{F(r+1,j,k)}{F(r,j,k)}$ is $r-j+1$, which is denoted by $d(j,k)$. We finally get (in 8 seconds) a linear difference operator with respect to the variable r:

$$L = (r+n+1)(n+s+l-m-r) + (r-l+1)(r-s)R$$

and the denominators of R_1, R_2 are $d(j,k)\tilde{g}_1(j,k)$ and $d(j,k)\tilde{g}_2(j,k)$, respectively. Then (5.4) follows from the evaluation of the initial value ($r=0$) by Zeilberger's algorithm:

$$\sum_k (-1)^k \binom{k}{k+l}\binom{n}{k}\binom{s+n-k}{m} = (-1)^l \binom{n}{n+l}\binom{s}{m-n-l}.$$

Example 7. The Petkovšek-Wilf-Zeilberger identity [17, p. 33]:

$$\sum_r \sum_s (-1)^{n+r+s} \binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n} = \sum_k \binom{n}{k}^4. \tag{5.5}$$

By cancelling the factors $s+1$ and $(r+1)^2$, we obtain

$$\tilde{g}_1(r,s) = (n+r)(n+1-r)(s+1) \quad \text{and} \quad \tilde{g}_2(r,s) = (n+r)(n+1-r).$$

Notice that the common denominator of

$$\frac{F(n+1,r,s)}{F(n,r,s)} \quad \text{and} \quad \frac{F(n+2,r,s)}{F(n,r,s)}$$

is

$$(n+1)(n+2)(n+1-r)(n+2-r)(n+1-s)(n+2-s)(n-r-s+1)(n+2-r-s),$$

which is denoted by $d(r,s)$. We finally get (in 35 seconds)

$$L = 4(4n+5)(4n+3)(n+1) + 2(2n+3)(3n^2+9n+7)N - (n+2)^3 N^2$$

is a linear difference operator and the denominators of R_1, R_2 are $d(r,s)\tilde{g}_1(r,s)$ and $d(r,s)\tilde{g}_2(r,s)$, respectively. The recursion is the same as that obtained by applying Zeilberger's algorithm to the right hand side of (5.5).

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