# A $q$-Analog of Dual Sequences with Applications

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**Abstract.** In the present paper combinatorial identities involving $q$-dual sequences or polynomials with coefficients that are $q$-dual sequences are derived. Further, combinatorial identities for $q$-binomial coefficients(Gaussian coefficients), $q$-Stirling numbers and $q$-Bernoulli numbers and polynomials are deduced.

*Keywords*: $q$-dual sequence, $q$-binomial coefficients, $q$-Stirling numbers, $q$-Bernoulli numbers, $q$-Bernoulli polynomials

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## 1 Introduction

Given a sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ of elements of a commutative ring $R$ (for example, the complex numbers, polynomials or rational functions), one usually describes as Euler-Seidel matrix associated with $(a_n)$ the double sequence $(a_n^k)$ $(n \geq 0, k \geq 0)$ given by the recurrence [7]:

$$a_n^0 = a_n, \quad a_n^k = a_n^{k-1} + a_{n+1}^{k-1} \quad (k \geq 1, n \geq 0).$$

The sequence $(a_n^0)$ of the first row of the matrix is the *initial sequence*. The sequence $(a_0^n)$ of the first column of the matrix is the *final sequence*. Such a matrix is equivalent to the table obtained by computing the finite difference of consecutive terms of $(a_0^n)$ and iterating the procedure. One passes from the initial sequence to the last one and conversely through

$$a_0^n = \sum_{i=0}^{n} \binom{n}{i} a_i^0 \Longleftrightarrow a_n^0 = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a_0^i. \tag{1}$$

If one sets $a_n = (-1)^n a_n^0$ and $a_n^* = (-1)^n a_0^n$, then the above relations can be written as

$$a_n^* = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i \Longleftrightarrow a_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i^*. \tag{2}$$

In [12] the sequence $(a_n^*)$ is called the *dual sequence* of $(a_n)$. It is well-known that if $a_n = (-1)^n B_n$, where $(B_n) = (1, -1/2, 1/6, 0, -1/30, \ldots)$ is the sequence of Bernoulli numbers, then $a_n^* = a_n$, that is $((-1)^n B_n)$ is *self-dual*. Generalizing the results of Kaneko [10] and Momiyama [11] on Bernoulli numbers, Sun [12] has recently proved some remarkable identities on dual sequences. Other generalizations of Kaneko's identity have been obtained by Gessel [9] using umbral calculus.

The aim of this paper is to give a $q$-version of Sun's results in [12]. In the last two decades there has been an increasing interest in generalizing the classical results with a generic parameter $q$, which is the so-called phenomenon of "$q$-disease". As regards Euler-Seidel matrix Clarke et al. [6] have given a $q$-analog of (1) with application to $q$-enumeration of derangements.

We shall need some standard $q$-notation, which can be found in Gasper and Rahman's book [8]. The $q$-shifted factorial $(a;q)_n$ is defined by $(a;q)_0 = 1$ and

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

if $n$ is a positive integer. For $k \in \mathbb{Z}$ the $k$-integer $[k]_q$ is defined by $[k]_q = \frac{1-q^k}{1-q}$, so $[-k]_q = -q^{-k}[k]_q$. For integer $k$, the $q$-binomial coefficient $\begin{bmatrix}\alpha\\k\end{bmatrix}$ is defined by $\begin{bmatrix}\alpha\\k\end{bmatrix} = 0$ if $k < 0$ and

$$\begin{bmatrix}\alpha\\k\end{bmatrix} = \frac{(1-q^\alpha)(1-q^{\alpha-1})\cdots(1-q^{\alpha-k+1})}{(q;q)_k}$$

if $k$ is a positive integer. Let $(a_n)$ be a sequence of a commutative ring. We call the sequence $(a_n^*)$ given by

$$a_n^* = \sum_{i=0}^{n} \begin{bmatrix}n\\i\end{bmatrix} (-1)^i a_i q^{\binom{i}{2}} \tag{3}$$

the $q$-*dual sequence* of $(a_n)$. By Gauss inversion [1, p. 96] we get

$$a_n = \sum_{r=0}^{n} \begin{bmatrix}n\\r\end{bmatrix} (-1)^r a_r^* q^{\binom{r+1}{2}-nr}. \tag{4}$$

We will need the following $q$-analog of binomial formula [3, p. 36]:

$$(z;q)_n = \sum_{j=0}^{n} \begin{bmatrix}n\\j\end{bmatrix} (-1)^j z^j q^{\binom{j}{2}}, \tag{5}$$

and the $q$-Chu-Vandermonde formula [8, p.354]:

$${}_2\Phi_1\begin{bmatrix}q^{-n}, a\\ c\end{bmatrix}; q, q\Bigg] := \sum_{k=0}^{\infty} \frac{(q^{-n};q)_k (a;q)_k}{(c;q)_k} \frac{z^k}{(q;q)_k} = \frac{(c/a;q)_n}{(c;q)_n} a^n. \tag{6}$$

The following is our basic theorem.

**Theorem 1.** *For $k, l \in \mathbb{N}$ the following identities hold true:*

$$\sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} \frac{(-1)^j a^*_{k+j+1}}{[k+j+1]_q} q^{\binom{j+1}{2}-l(k+j+1)} + \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(-1)^j a_{l+j+1}}{[l+j+1]_q} q^{\binom{j+1}{2}} = \frac{a_0}{[k+l+1]_q \begin{bmatrix} k+l \\ k \end{bmatrix}}, \tag{7}$$

$$\sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} (-1)^j a^*_{k+j} q^{\binom{j+1}{2}-l(k+j)} = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j a_{l+j} q^{\binom{j}{2}}, \tag{8}$$

$$\sum_{j=0}^{l+1} \begin{bmatrix} l+1 \\ j \end{bmatrix} (-1)^{j+1} [k+j+1]_q a^*_{k+j} q^{\binom{j}{2}-l(k+j)-k} = \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} (-1)^j [l+j+1]_q a_{l+j} q^{\binom{j-1}{2}}. \tag{9}$$

The above theorem is a $q$-analog of Theorem 2.1 in Sun [12]. Note also that (8) was also a $q$-analog of Theorem 7.4 in Gessel [9].

The rest of this paper will be organized as follows: we prove Theorem 1 in Section 2 and present a $q$-analog of Sun's main theorem in Section 3. In Section 4, we present some interesting examples as applications of our Theorems 1 and 2.

## 2 Proof of Theorem 1

Plugging (3) into the first sum of the left-hand side of (7), we have

$$LHS = a_0 B + \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j \frac{a_{l+j+1}}{[l+j+1]_q} q^{\binom{j+1}{2}} + C, \tag{10}$$

where

$$B = \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} (-1)^j \frac{q^{\binom{j+1}{2}-l(k+j+1)}}{[k+j+1]_q},$$

and

$$C = \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} (-1)^j \frac{q^{\binom{j+1}{2}-l(k+j+1)}}{[k+j+1]_q} \sum_{i=1}^{k+j+1} \begin{bmatrix} k+j+1 \\ i \end{bmatrix} (-1)^i a_i q^{\binom{i}{2}}.$$

It is known (see [13] for further applications) that

$$\frac{1}{(x+a_0)(x+a_1)\cdots(x+a_l)} = \sum_{j=0}^{l} \frac{\prod_{\substack{i=0\\ i\neq j}}^{l} (a_i - a_j)^{-1}}{x+a_j}. \tag{11}$$

Setting $x = -q^{-k-1}$ and $a_i = q^i$ $(0 \le i \le l)$ in (11) we obtain

$$\sum_{j=0}^{l} (-1)^j \frac{q^{\binom{j+1}{2}-l(k+j+1)}}{(q;q)_j (q;q)_{l-j} (1-q^{k+j+1})} = \frac{1}{(q^{k+1};q)_{l+1}}. \tag{12}$$

It follows that

$$B = \frac{(1-q)(q;q)_l}{(q^{k+1};q)_{l+1}} = \frac{1}{[k+l+1]_q \left[{k+l \atop k}\right]}.$$

Exchanging the order of summation we can rewrite $C$ as follows:

$$\begin{aligned} C &= \sum_{i=1}^{k+l+1} (-1)^i \frac{a_i}{[i]_q} q^{\binom{i}{2}} \sum_{j=i-k-1}^{l} (-1)^j \begin{bmatrix} l \\ j \end{bmatrix} \begin{bmatrix} k+j \\ i-1 \end{bmatrix} q^{\binom{j+1}{2}-lj-(k+1)l} \\ &= \sum_{i=1}^{k} (-1)^i \frac{a_i}{[i]_q} q^{\binom{i}{2}} \begin{bmatrix} k \\ i-1 \end{bmatrix} {}_2\Phi_1 \begin{bmatrix} q^{-l}, q^{k+1} \\ q^{k-i+2} \end{bmatrix} ; q, q \end{bmatrix} q^{-(k+1)l}. \end{aligned}$$

Applying the $q$-Chu-Vandermonde formula (6) we obtain

$$\begin{aligned} C &= \sum_{i=1}^{k} (-1)^i \frac{a_i}{[i]_q} q^{\binom{i}{2}} \begin{bmatrix} k \\ i-1 \end{bmatrix} \frac{(q^{-i+1};q)_l}{(q^{k-i+2};q)_l} \\ &= \sum_{i=1}^{k} (-1)^{i+l} \begin{bmatrix} k \\ i-l-1 \end{bmatrix} \frac{a_i}{[i]_q} q^{\binom{i}{2}+\binom{l+1}{2}-il} \\ &= \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^{j+1} \frac{a_{l+j+1}}{[l+j+1]_q} q^{\binom{j+1}{2}}. \end{aligned}$$

Substituting the values of $B$ and $C$ into (10) yields (7).

To derive (8) and (9) from (7) we define the linear operator $\delta_q$ by

$$\delta_q(a_n) = -q^{1-n}[n]_q a_{n-1} \qquad \text{for} \quad n \geq 0.$$

Then $\delta_q(a_n^*) = [n]_q a_{n-1}^*$. Indeed,

$$\begin{aligned} \delta_q(a_n^*) &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^i \delta_q(a_i) q^{\binom{i}{2}} = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^{i+1} q^{1-i} [i]_q a_{i-1} q^{\binom{i}{2}} \\ &= [n]_q \sum_{i=0}^{n} (-1)^i \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} (-1)^{i-1} a_{i-1} q^{\binom{i-1}{2}} = [n]_q a_{n-1}^*. \end{aligned}$$

Now, applying $\delta_q$ to (7) yields (8). Furthermore, replacing $k$ by $k+1$ and $l$ by $l+1$ in (8) then applying $\delta_q$ on both sides yields (9).

**Remark**: We can also prove (8) and (9) directly by using the $q$-Chu-Vandermonde formula.

## 3 A $q$-analog of Sun's main theorem

In this section, we assume that $x$, $y$ and $z$ are commuting indeterminates. Define $[x,y]^n$ by $[x,y]^0 = 1$ and

$$[x,y]^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} x^i y^{n-i}$$

for positive integer $n$. So $[x,y]^n=(x+y)^n$ when $q=1$. Similarly

$$[x,y,z]^n=[x,[y,z]]^n=\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}x^i[y,z]^{n-i}=\sum_{\substack{i,j,k\geq 0\\ i+j+k=n}}\frac{[n]_q!}{[i]_q![j]_q![k]_q!}x^iy^jz^k,$$

and hence $[x,y,z]^n$ is a symmetric polynomial of $x,y,z$ and $[x,y,z]^n=(x+y+z)^n$ when $q=1$.

Like the definition of Bernoulli polynomials, we introduce

$$A_n(x)=\sum_{i=0}^{n}(-1)^i\begin{bmatrix}n\\ i\end{bmatrix}a_iq^{\binom{i}{2}}x^{n-i}\quad\text{and}\quad A_n^*(x)=\sum_{i=0}^{n}(-1)^i\begin{bmatrix}n\\ i\end{bmatrix}a_i^*x^{n-i}.$$

The following is our $q$-analog of the main theorem of Sun [12, Th. 1.1].

**Theorem 2.** *Let $k,l\in\mathbb{N}$, then*

$$(-1)^l\sum_{j=0}^{l}\begin{bmatrix}l\\ j\end{bmatrix}x^{l-j}\frac{A^*_{k+j+1}(z)}{[k+j+1]_q}q^{-kj-\binom{k+1}{2}}$$
$$+(-1)^k\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}x^{k-j}\frac{A_{l+j+1}([1,-z,-x])}{[l+j+1]_q}q^{\binom{j+1}{2}-k(l+j+1)}=\frac{a_0(-x)^{k+l+1}}{[k+l+1]\begin{bmatrix}k+l\\ k\end{bmatrix}}. \quad (13)$$

$$(-1)^l\sum_{j=0}^{l}\begin{bmatrix}l\\ j\end{bmatrix}x^{l-j}A^*_{k+j}(z)q^{k(l-j)}=(-1)^k\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}x^{k-j}A_{l+j}([1,-z,-x])q^{\binom{k-j}{2}}. \quad (14)$$

$$(-1)^{l+1}\sum_{j=0}^{l+1}\begin{bmatrix}l+1\\ j\end{bmatrix}x^{l+1-j}[k+j+1]_qA^*_{k+j}(z)q^{(k+1)(l-j)+1}$$
$$=(-1)^k\sum_{j=0}^{k+1}\begin{bmatrix}k+1\\ j\end{bmatrix}x^{k+1-j}[l+j+1]_qA_{l+j}([1,-z,-x])q^{\binom{k-j}{2}-j}. \quad (15)$$

*Proof.* We derive from (4) and (5) that

$$\begin{aligned}A_n([1,-x]) &= \sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}(-1)^ia_i[1,-x]^{n-i}q^{\binom{i}{2}}\\ &= \sum_{i,j,s\geq 0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}\begin{bmatrix}i\\ j\end{bmatrix}(-1)^{i-j}a_j^*\begin{bmatrix}n-i\\ s\end{bmatrix}(-x)^sq^{\binom{i-j}{2}}\\ &= \sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}a_j^*\sum_{i,s\geq 0}\begin{bmatrix}n-j\\ s\end{bmatrix}(-1)^sx^s\begin{bmatrix}n-j-s\\ i-j\end{bmatrix}(-1)^{i-j}q^{\binom{i-j}{2}}\\ &= (-1)^n\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}(-1)^ja_j^*x^{n-j}\\ &= (-1)^nA_n^*(x), \end{aligned} \quad (16)$$

and

$$\begin{aligned}
A_n([1,-z,-x]) &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^i a_i [1,-z,-x]^{n-i} q^{\binom{i}{2}} \\
&= \sum_{i,j\geq 0} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n-i \\ j \end{bmatrix} (-1)^{i+j} x^j a_i [1,-z]^{n-i-j} q^{\binom{i}{2}} \\
&= \sum_{i,j\geq 0} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j x^j \begin{bmatrix} n-j \\ i \end{bmatrix} (-1)^i a_i [1,-z]^{n-i-j} q^{\binom{i}{2}} \\
&= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j x^j A_{n-j}([1,-z]) \\
&= (-1)^n \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} x^j A^*_{n-j}(z). \qquad (17)
\end{aligned}$$

Denote the first sum of the left-hand side in (13) by $\mathcal{C}$. Applying (16) and (17), the left-hand side of (13) is equal to

$$\begin{aligned}
&(-1)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{x^{k-j}}{[l+j+1]_q} q^{\binom{j+1}{2} - k(l+j+1)} \sum_{i=0}^{l+j+1} \begin{bmatrix} l+j+1 \\ i \end{bmatrix} A_i([1,-z])(-x)^{l+j+1-i} + \mathcal{C} \\
= \;& a_0 (-x)^{k+l+1} \mathcal{B} + S + \mathcal{C}, \qquad (18)
\end{aligned}$$

where

$$\mathcal{B} = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j \frac{q^{\binom{j+1}{2} - k(l+j+1)}}{[l+j+1]_q},$$

and

$$S = (-1)^{k+l+1} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2} - k(l+j+1)} \sum_{i=1}^{l+j} \begin{bmatrix} l+j \\ i-1 \end{bmatrix} x^{k+l+1-i} \frac{A^*_i(z)}{[i]_q}.$$

Exchanging $k$ and $l$ in (12) yields

$$\mathcal{B} = \frac{1}{[k+l+1]_q \begin{bmatrix} k+l \\ k \end{bmatrix}}.$$

Now, we show that $S = -\mathcal{C}$. Exchanging the order of summation we have

$$\begin{aligned}
S &= (-1)^{k+l+1} \sum_{i=1}^{l} \begin{bmatrix} l \\ i-1 \end{bmatrix} \frac{A^*_i(z)}{[i]_q} x^{k+1+l-i} {}_2\Phi_1 \left[ \begin{matrix} q^{-k}, q^{l+1} \\ q^{l-i+2} \end{matrix} ; q, q \right] q^{-k(l+1)} \\
&= (-1)^{k+l+1} \sum_{i=1}^{l} \begin{bmatrix} l \\ i-1 \end{bmatrix} \frac{A^*_i(z)}{[i]_q} x^{k+l-i} \frac{(q^{-i+1};q)_k}{(q^{l-i+2};q)_k} \quad \text{(by $q$-Chu-Vandemonde)} \\
&= (-1)^{l+1} \sum_{i=1}^{l} \begin{bmatrix} l \\ i-k-1 \end{bmatrix} \frac{A^*_i(z)}{[i]_q} x^{k+1+l-i} q^{-ik + \binom{k+1}{2}} \\
&= (-1)^{l+1} \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} \frac{A^*_{k+j+1}(z)}{[k+j+1]_q} x^{l-j} q^{-jk - \binom{k+1}{2}} = -\mathcal{C}.
\end{aligned}$$

Next, the right-hand side of (14) is equal to

$$
\begin{aligned}
& (-1)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} x^{k-j} \sum_{i=0}^{l+j} \begin{bmatrix} l+j \\ i \end{bmatrix} (-x)^{l+j-i} A_i([1,-z]) q^{\binom{k-j}{2}} \\
= \quad & (-1)^{k+l} \sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix} x^{k+l-i} A_i^*(z) \; {}_2\Phi_1 \left[ \begin{matrix} q^{-k}, q^{l+1} \\ q^{l-i+1} \end{matrix} ; q, q \right] q^{\binom{k}{2}} \\
= \quad & (-1)^{k+l} \sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix} x^{k+l-i} A_i^*(z) \frac{(q^{-i};q)_k}{(q^{l-i+1};q)_k} q^{\binom{k}{2}+k(l+1)} \\
= \quad & (-1)^{l} \sum_{i=0}^{l} \begin{bmatrix} l \\ i-k \end{bmatrix} x^{k+l-i} A_i^*(z) q^{-ik+k^2+kl} \\
= \quad & (-1)^{l} \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} x^{l-j} A_{k+j}^*(z) q^{k(l-j)},
\end{aligned}
$$

which is exactly the left-hand side of (14).

Finally, exchanging the order of summation, the right-hand side of (15) can be written as

$$
\begin{aligned}
R \quad = \quad & (-1)^k \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} x^{k+1-j} [l+j+1]_q \sum_{i=0}^{l+j} \begin{bmatrix} l+j \\ i \end{bmatrix} (-x)^{l+j-i} A_i([1,-z]) q^{\binom{k-j}{2}-j} \\
= \quad & (-1)^{k+l} \sum_{i,j\geq 0} \begin{bmatrix} k+1 \\ j \end{bmatrix} x^{k+l+1-i} [i+1]_q (-1)^j \begin{bmatrix} l+j+1 \\ i+1 \end{bmatrix} A_i^*(z) q^{\binom{k-j}{2}-j} \\
= \quad & (-1)^{k+l} \sum_{i=0}^{l+1} \begin{bmatrix} l+1 \\ i \end{bmatrix} x^{k+l+1-i} [i+1]_q A_i^*(z) \; {}_2\Phi_1 \left[ \begin{matrix} q^{-k-1}, q^{l+2} \\ q^{l-i+1} \end{matrix} ; q, q \right] q^{\binom{k}{2}}.
\end{aligned}
$$

By $q$-Chu-Vandermonde formula we have

$$
\begin{aligned}
R \quad = \quad & (-1)^{k+l} \sum_{i=0}^{l+1} \begin{bmatrix} l+1 \\ i \end{bmatrix} x^{k+l+1-i} [i+1]_q A_i^*(z) \frac{(q^{-i-1};q)_{k+1}}{(q^{l-i+1};q)_{k+1}} q^{\binom{k}{2}+(k+1)(l+2)} \\
= \quad & (-1)^{l+1} \sum_{i=0}^{l+1} \begin{bmatrix} l+1 \\ i-k \end{bmatrix} x^{k+l+1-i} [i+1]_q A_i^*(z) q^{(k+1)(l+2-i)+k^2-k-1} \\
= \quad & (-1)^{l+1} \sum_{j=0}^{l+1} \begin{bmatrix} l+1 \\ j \end{bmatrix} x^{l+1-j} [k+j+1]_q A_{k+j}^*(z) q^{(k+1)(l-j)+1},
\end{aligned}
$$

which is exactly the left-hand side of (15). □

**Remark.** When $q = 1$, Theorems 1 and 2, which correspond to Theorems 2.2 and 1.1 of Sun[12], are actually equivalent. Indeed, in such case, we have

$$(-1)^n A_n^*(1-x) = A_n(x), \tag{19}$$

which can be verified as follows:

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}a_i^*(x-1)^{n-i} &= \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\sum_{j=0}^{i}\binom{i}{j}(-1)^j a_j(1-x)^{n-i}\\ &= \sum_{j=0}^{n}\binom{n}{j}(-1)^j a_j\sum_{i=j}^{n}\binom{n-j}{i-j}(x-1)^{n-i}\\ &= \sum_{j=0}^{n}\binom{n}{j}(-1)^j a_j x^{n-j}.\end{aligned}$$

Now, taking $a_n=(-1)^{l+k+n}x^{k+l-n}A_n(y)$ with $q=1$,

$$\begin{aligned}a_n^* &= \sum_{i=0}^{n}\binom{n}{i}(-1)^{l+k}x^{k+l-i}A_i(y)\\ &= \sum_{i=0}^{n}\binom{n}{i}(-1)^{l+k}x^{k+l-i}\sum_{j=0}^{i}\binom{i}{j}(-1)^j a_j y^{i-j}\\ &= (-1)^{l+k+n}x^{k+l-n}\sum_{j=0}^{n}\binom{n}{j}a_j(-1)^{n-j}\sum_{i=j}^{n}\binom{n-j}{i-j}x^{n-i}y^{i-j}\\ &= (-1)^{l+k}x^{k+l-n}A_n(x+y).\end{aligned}$$

It follows from (19) that

$$a_n^*=(-1)^{k+l+n}x^{k+l-n}A_n^*(1-x-y).$$

Substituting the above values of $a_n$ and $a_n^*$ in Theorem 1 we obtain Theorem 2. Conversely, it is easy to see that Theorem 1 is a special case of Theorem 2 because

$$A_n(0)=(-1)^n a_n,\qquad A_n(1)=a_n^*.$$

Hence we have proved that Theorems 1 and 2 are actually equivalent when $q=1$.

## 4 Some applications

In this section we derive some examples from our main theorem, most of them are $q$-analogs of results in Sun [12].

### Example 1

For any fixed integer $i\geq 0$ let $a_n=(-1)^n\begin{bmatrix}n\\i\end{bmatrix}t^{n-i}q^{\binom{i}{2}}$, then it follows from (3) and (5) that

$$\begin{aligned}a_n^* &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}k\\i\end{bmatrix}t^{k-i}q^{\binom{i}{2}+\binom{k}{2}}\\ &= \begin{bmatrix}n\\i\end{bmatrix}q^{i^2-i}\sum_{k=i}^{n}\begin{bmatrix}n-i\\k-i\end{bmatrix}(tq^i)^{k-i}q^{\binom{k-i}{2}}\\ &= \begin{bmatrix}n\\i\end{bmatrix}(-tq^i;q)_{n-i}q^{i^2-i}.\end{aligned}$$

Substituting the above values in (8) of Theorem 1 yields

$$\sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} \begin{bmatrix} k+j \\ i \end{bmatrix} (-1)^{l-j} (-q^i t; q)_{k+j-i} q^{j(j+1)/2 - lj + \binom{i}{2}} = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} l+j \\ i \end{bmatrix} t^{l+j-i} q^{kl+j(j-1)/2}. \quad (20)$$

For variations of methods, we will give two more proofs of (20). Note that when $q = 1$ Eq. (20) reduces to a crucial result of Sun [12, Lemma 3.1], which was proved by using derivative operator.

We first $q$-generalize Sun's proof by using $q$-derivative operator. For any polynomial $f(t)$ in $t$, let $D_q$ be the $q$-derivative operator with respect to $t$:

$$D_q f(t) = \frac{f(tq) - f(t)}{(q-1)t}.$$

Clearly we have

$$D_q t^n = \frac{q^n - 1}{q - 1} t^{n-1}, \quad D_q((-t;q)_n) = [n]_q (-qt; q)_{n-1}.$$

For integer $i \geq 0$ define $[i]_q! = \prod_{j=0}^{i} [j]_q$, then

$$D_q^i(t^n) = [i]_q! \begin{bmatrix} n \\ i \end{bmatrix} t^{n-i}, \quad (21)$$

$$D_q^i((-t;q)_n) = q^{i(i-1)/2} [i]_q! \begin{bmatrix} n \\ i \end{bmatrix} (-q^i t; q)_{n-i}. \quad (22)$$

By Gauss inversion, the $q$-binomial formula (5) is equivalent to

$$z^n = \sum_{j=0}^{n} (-1)^j q^{\binom{j+1}{2} - nj} \begin{bmatrix} n \\ j \end{bmatrix} (z; q)_j.$$

Substituting $z$ by $-tq^k$ we get

$$(tq^k)^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^{n-j} q^{j(j+1)/2 - nj} (-tq^k; q)_j. \quad (23)$$

Now, using the $q$-derivative operator and (21)-(23), we can write the difference of the two sides of (20) as follows:

$$\frac{1}{[i]_q!} D_q^i \left( (-t;q)_k \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} (-1)^{l-j} q^{j(j+1)/2 - lj} (-tq^k; q)_j - (tq^k)^l \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{j(j-1)/2} t^j \right)$$

$$= \frac{1}{[i]_q!} D_q^i \left( (-t;q)_k (tq^k)^l - (tq^k)^l (-t;q)_k \right),$$

which is clearly equal to 0.

Our second proof of (20) uses the machinery of basic hypergeometric functions. Rewriting (20) in terms of basic hypergeometric functions, we have

$$\begin{bmatrix} k \\ i \end{bmatrix} (-1)^l (-q^i t; q)_{k-i} q^{\binom{i}{2}} {}_3\phi_2 \left[ \begin{matrix} q^{-l}, \ q^{k+1}, \ -tq^k \\ q^{k-i+1}, \ 0 \end{matrix} ; q, q \right]$$

$$= \begin{bmatrix} l \\ i \end{bmatrix} t^{l-i} q^{kl} {}_2\phi_1 \left[ \begin{matrix} q^{-k}, \ q^{l+1} \\ q^{l-i+1} \end{matrix} ; q, -tq^k \right]. \quad (24)$$

A standard proof of (24) goes then as follows:

$$\begin{bmatrix} k \\ i \end{bmatrix}(-1)^l(-tq^i;q)_{k-i}q^{\binom{i}{2}}(-tq^{k-i})^l{}_3\phi_2\left[\begin{matrix} q^{-l}, q^{-i}, (-tq^{i-1})^{-1} \\ q^{k-i+1}, 0 \end{matrix}; q, q\right]$$

(by [8, p.241(III.11)])

$$= \begin{bmatrix} k \\ i \end{bmatrix} t^l q^{(k+i)l} q^{\binom{i}{2}}(-tq^i;q)_{k-i}\frac{(q^{l-i+1};q)_i}{(q^{-k};q)_i}(tq^{k+l})^{-i}{}_2\phi_1\left[\begin{matrix} q^{-i}, q^{k+l+1-i} \\ q^{l-i+1} \end{matrix}; q, -tq^i\right]$$

(by [8, p.241(III.6)])

$$= \begin{bmatrix} l \\ i \end{bmatrix} q^{kl}t^{l-i}(-tq^i;q)_{k-i}\frac{(-tq^k;q)_\infty}{(-tq^i;q)_\infty}{}_2\phi_1\left[\begin{matrix} q^{-k}, q^{l-1} \\ q^{l-i+1} \end{matrix}; q, -tq^k\right]$$

(by [8, p.241(III.3)])

which is equal to the right-hand side of (24).

## Example 2

Let $a_n = \left[\begin{smallmatrix} x+n \\ m \end{smallmatrix}\right] q^{-mn}$ for $n \in \mathbb{N}$. By the notation (3), we have

$$a_n^* = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}(-1)^i \begin{bmatrix} x+i \\ m \end{bmatrix} q^{\binom{i}{2}-im} = \begin{cases} (-1)^n \left[\begin{smallmatrix} x \\ m-n \end{smallmatrix}\right] q^{-mn+\binom{n}{2}} & \text{if } m \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1 implies that

$$\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(-1)^j}{[l+j+1]_q} \begin{bmatrix} x+l+j+1 \\ m \end{bmatrix} q^{-m(l+j+1)+\binom{j}{2}}$$

$$= (-1)^k \sum_{k\leq j\leq m} \frac{1}{[j]_q}\begin{bmatrix} l \\ j-k-1 \end{bmatrix}\begin{bmatrix} x \\ m-j \end{bmatrix} q^{\binom{j-k}{2}+\binom{j}{2}-l(j-1)-mj} + \frac{\left[\begin{smallmatrix} x \\ m \end{smallmatrix}\right]}{[k+l+1]_q \left[\begin{smallmatrix} k+l \\ k \end{smallmatrix}\right]}.$$

## Example 3

Let $c_n = \left[\begin{smallmatrix} y \\ n \end{smallmatrix}\right] / \left[\begin{smallmatrix} x \\ n \end{smallmatrix}\right]$ for $n \in \mathbb{N}$. Then $c_n^* = \left[\begin{smallmatrix} x-y \\ n \end{smallmatrix}\right] q^{ny} / \left[\begin{smallmatrix} x \\ n \end{smallmatrix}\right]$. In fact,

$$\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^k \begin{bmatrix} -n+k-1 \\ k \end{bmatrix} q^{nk-\binom{k}{2}}.$$

$$\begin{aligned} \begin{bmatrix} x \\ n \end{bmatrix} c_n^* &= \sum_{i=0}^{n} \begin{bmatrix} x-i \\ n-i \end{bmatrix}(-1)^i \begin{bmatrix} y \\ i \end{bmatrix} q^{\binom{i}{2}} = (-1)^n \sum_{i=0}^{n} \begin{bmatrix} x-n+1 \\ n-i \end{bmatrix}\begin{bmatrix} y \\ i \end{bmatrix} q^{(x-i)(n-i)-\binom{n-i}{2}+\binom{i}{2}} \\ &= (-1)^n q^{xn-\binom{n}{2}} \begin{bmatrix} n-x-1+y \\ n \end{bmatrix} = \begin{bmatrix} x-y \\ n \end{bmatrix} q^{ny}. \end{aligned}$$

By the identities in Theorem 1, we obtain

$$\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(-1)^j \left[\begin{smallmatrix} y \\ l+j+1 \end{smallmatrix}\right]}{\left[\begin{smallmatrix} x-1 \\ l+j \end{smallmatrix}\right]} q^{\binom{j+1}{2}} + \sum_{j=0}^{l} \frac{(-1)^j \left[\begin{smallmatrix} x-y \\ k+j+1 \end{smallmatrix}\right]}{\left[\begin{smallmatrix} x-1 \\ k+j \end{smallmatrix}\right]} q^{(k+j+1)y+\binom{j+1}{2}-l(k+j+1)} = \frac{[x]_q}{[k+l+1]\left[\begin{smallmatrix} k+l \\ k \end{smallmatrix}\right]}$$

and

$$\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}(-1)^j\frac{\begin{bmatrix}y\\l+j\end{bmatrix}}{\begin{bmatrix}x\\l+j\end{bmatrix}}q^{\binom{j}{2}}=\sum_{j=0}^{l}(-1)^j\frac{\begin{bmatrix}x-y\\k+j\end{bmatrix}}{\begin{bmatrix}x\\k+j\end{bmatrix}}q^{(k+j)y+\binom{j+1}{2}-l(k+j)}.$$

### Example 4

Carlitz [4, (3.1)] defined the *q-Stirling numbers of the second kind* $\left\{ {m \atop n} \right\}_q$ by

$$[n]_q^m=\sum_{i=0}^{m}\left\{ {m \atop i} \right\}_q [i]_q!\begin{bmatrix}n\\i\end{bmatrix}q^{\binom{i}{2}}.$$

By Gauss inversion we get

$$\begin{aligned}\left\{ {m \atop n} \right\}_q &= \frac{q^{-\binom{n}{2}}}{[n]_q!}\sum_{i=0}^{n}(-1)^i q^{\binom{i}{2}}\begin{bmatrix}n\\i\end{bmatrix}[n-i]_q^m\\ &= \frac{1}{[n]_q!}\sum_{i=0}^{n}(-1)^{n-i}q^{\binom{i+1}{2}-ni}\begin{bmatrix}n\\i\end{bmatrix}[i]_q^m.\end{aligned}$$

So we have the following $q$-dual sequences:

$$a_n=(-1)^n[n]_q!\left\{ {m \atop n} \right\}_q,\quad a_n^*=[n]_q^m.$$

Substituting these values in Theorem 1 yields corresponding identities. For example, applying (8) we obtain

$$\frac{1}{[l]_q!}\sum_{j=0}^{l}(-1)^{l-j}q^{\binom{j+1}{2}-l(k+j)}\begin{bmatrix}l\\j\end{bmatrix}[k+j]_q^m=\sum_{j=0}^{k}q^{\binom{j}{2}}\begin{bmatrix}l+j\\j\end{bmatrix}\frac{[k]_q!}{[k-j]_q!}\left\{ {m \atop l+j} \right\}_q.$$

The left-hand side of the above identity is called a *non-central q-Stirling number of the second kind*, with non-centrality parameter $k$, by Charalambides [5]. This number was first discussed by Carlitz [4, (3.8)] and recently by Charalambides [5, (3.5)]. Note that for $k=0$ these numbers reduce to the usual $q$-Stirling numbers of the second kind, while for $k\neq 0$ the above identity connects the non-central to the usual $q$-Stirling numbers of the second kind.

### Example 5

Taking $e(t)=\sum_{n=0}^{\infty}\frac{x^n}{[n]_q!}$ as a q-analog of the exponential function $e^x$, Al-Salam [2, 2.1] defined a $q$-analog of Bernoulli numbers $B_n$ by

$$\frac{1}{e(t)-1}=\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}B_n.$$

These $q$-Bernoulli numbers $B_n$ satisfy the following recurrence relation (see [2, 4.3]):

$$[1,B]^n = \begin{cases} B_n & n>1, \\ 1+B_1 & n=1. \end{cases}$$

Now, if $a_0^* = B_0 = 1$ and for $n \geq 1$, $a_n^* = B_n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} B_i$, then $a_0 = 1$ and for $n \geq 1$

$$\begin{aligned} a_n &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^i q^{\binom{i+1}{2}-in} \sum_{j=0}^{i} \begin{bmatrix} i \\ j \end{bmatrix} B_j \\ &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} B_j \sum_{i \geq j} \begin{bmatrix} n-j \\ i-j \end{bmatrix} (-1)^i q^{\binom{i+1}{2}-ni} \\ &= (-1)^n B_n q^{-\binom{n}{2}}. \end{aligned}$$

Theorem 1 infers then the following identities, which are $q$-analogs of the identities of Kaneko [10] and Momiyama [11] on Bernoulli numbers.

**Proposition 1.** *For $k, l \in \mathbb{N}$,*

$$\sum_{j=0}^{k} \begin{bmatrix} l \\ j \end{bmatrix} \frac{(-1)^j B_{k+j+1}}{[k+j+1]_q} q^{\binom{j+1}{2}-l(k+j+1)} + \sum_{j=0}^{l} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(-1)^{l+1} B_{l+j+1}}{[l+j+1]_q} q^{-\binom{l+1}{2}-lj} = \frac{1}{[k+l+1]_q \begin{bmatrix} k+l \\ k \end{bmatrix}},$$

$$\sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} (-1)^j B_{k+j} q^{\binom{l-j}{2}} = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^l B_{l+j} q^{l(k-j)},$$

$$\sum_{j=0}^{l+1} \begin{bmatrix} l+1 \\ j \end{bmatrix} (-1)^{j+1} [k+j+1]_q B_{k+j} q^{\binom{l-j}{2}-j} = \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} (-1)^l [l+j+1]_q B_{l+j} q^{(k-j)(l+1)+1}.$$

## Example 6

Al-Salam [2] also defined the $q$-Bernoulli polynomials $B_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} B_k x^{n-k}$. By Example 5, if $a_n = (-1)^n B_n q^{-\binom{n}{2}}$ then $a_n^* = B_n$. Therefore

$$A_n(x) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} B_i x^{n-i} = B_n(x) \quad \text{and} \quad A_n^*(x) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^i B_i x^{n-i} = B_n^*(x).$$

If we replace $A_n(x)$ and $A_n^*(x)$ by $B_n(x)$ and $B_n^*(x)$, respectively, we get the following result.

**Proposition 2.** *For $k, l \in \mathbb{N}$,*

$$(-1)^l \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} x^{l-j} \frac{B^*_{k+j+1}(z)}{[k+j+1]_q} q^{-kj-\binom{k+1}{2}}$$
$$+(-1)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} x^{k-j} \frac{B_{l+j+1}([1,-x,-z])}{[l+j+1]_q} q^{\binom{j+1}{2}-k(l+j+1)} = \frac{a_0(-x)^{k+l+1}}{[k+l+1]\begin{bmatrix} k+l \\ k \end{bmatrix}},$$

$$(-1)^l \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} x^{l-j} B^*_{k+j}(z) q^{k(l-j)} = (-1)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} x^{k-j} B_{l+j}([1,-x,-z]) q^{\binom{k-j}{2}},$$

$$(-1)^{l+1} \sum_{j=0}^{l+1} \begin{bmatrix} l+1 \\ j \end{bmatrix} x^{l+1-j} [k+j+1]_q B^*_{k+j}(z) q^{(k+1)(l-j)+1}$$
$$= (-1)^k \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} x^{k+1-j} [l+j+1]_q B_{l+j}([1,-x,-z]) q^{\binom{k-j}{2}-j}.$$

**Remark:** It is easy to see that $B_n(0) = B_n$ and $B_n(0)^* = (-1)^n B_n$. Hence Proposition 1 can be derived from Proposition 2 by taking $x = 1$ and $z = 0$.

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## References

[1] M. Aigner, *Comninatorial Theory*, Grundlehren der mathematischen Wissenschaften 234, Springer-Verlag, 1979.

[2] W. A. Al-Salam, *q-Bernoulli numbers and polynomials*, Math. Nachr., **17**(1959), 239-260.

[3] G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, 1984.

[4] L. Carlitz, *q-Bernoulli numbers and polynomials*, Duke Math., **15**(1948), 987-1000.

[5] Ch. A. Charalambides, *Non-central generlized q-factorial coefficients and q-Stirling numbers*, Discrete Math., **275**(2004), 67-85.

[6] R.J. Clarke, G.N. Han and J. Zeng, *A combinatorial interpretation of Seidel generation of q-derangement numbers*, Annals of Combinatorics, **1**(1997), 313-327.

[7] D. Dumont, *Matrices d'Euler-Seidel*, Séminaire Lotharingien de Combinatoire, B05c (1981).

[8] G. Gasper & M. Rahman, *Basic hypergeometric series* (second ed.), Encyclopedia of Math. and its Applications, **96**(2004).

[9] I. Gessel, *Applications of classical umbral calculus*, Algebra Universalis, **49**(2003), 397-434.

[10] M. Kaneko, *A recurrence formula for the Bernoulli numbers*, Proc. Japan Acad. Ser. A. Math. Sci, **71**(1995), 192-193.

[11] H. Momiyama, *A new recurrence formula for Bernoulli numbers*, Fibonacci Quart.**39**(2001) 324-333.

[12] Z. W. Sun, *Combinatorial identities in dual sequences*, European J. Combin., **24**(2003), no.6, 709-718.

[13] J. Zeng, *On some q-identities related to divisor functions*, Adv. Appl. Math., 34(2005) 313-315.