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Covering planar graphs with forests

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Abstract

We study the problem of covering graphs with trees and a graph of bounded maximum degree. By a classical theorem of Nash-Williams, every planar graph can be covered by three trees. We show that every planar graph can be covered by two trees and a forest, and the maximum degree of the forest is at most 8. Stronger results are obtained for some special classes of planar graphs.
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1. Introduction

For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. For two subgraphs H and K of a graph, we use $H \cup K$ to denote the union

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of H and K. We say that a graph G can be *covered* by subgraphs $G_1, \ldots, G_k$ of G if $\bigcup_{i=1}^{k} G_i = G$.

A well-known theorem of Nash-Williams [5] (based on a result proved independently in [4,7]) states that the edges of a graph G can be covered by t trees if, and only if, for every $A \subseteq V(G)$, $e(A) \leqslant (|A| - 1)t$, where $e(A)$ denotes the number of edges of G with both ends in A. One way to extend this result is to cover graphs with trees (or forests) and a graph with bounded degree. We say that a graph is (t, D)-*coverable* if it can be covered by at most t forests and a graph of maximum degree D.

It is easy to check that if a graph G is (t, D)-coverable, then, for any two disjoint subsets A, B of $V(G)$, $f_t(A) + e(A, B) \leqslant D \cdot |A| + t(|A| + |B| - 1)$, where $e(A, B)$ denotes the number of edges of G with one endpoint in A and the other in B, $f_t(A) = e(A)$ if $e(A) \leqslant t(|A| - 1)$, and $f_t(A) = 2e(A) - t(|A| - 1)$ otherwise. Unfortunately, this condition is not sufficient. For example, by deleting one edge from the Petersen graph, we obtain a graph that satisfies the above inequality with $t = D = 1$, but is not $(1, 1)$-coverable.

It is interesting to know what can be said about planar graphs. The aforementioned theorem of Nash-Williams implies that every planar graph is $(3, 0)$-coverable. As pointed out by Lovász [3] there are infinitely many planar graphs which are not (2,3)-coverable: take a triangle, put a vertex inside and connect it to the vertices of the triangle, and repeat this operation for each new triangle. After repeating this process for a while, we get a graph on n vertices with roughly $2n/3$ vertices of degree 3. This graph does not satisfy the above inequality about $f_t(A)$ (with $t = 2$, $D = 3$, B the set of vertices of degree 3, and A the set of vertices of degree at least 4), and so, it is not (2,3)-coverable. The double wheel on $2D + 4$ vertices (that is, a cycle of length $2D + 2$ plus two vertices and all edges from these two vertices to the cycle) shows that planar graphs need not be $(1, D)$-coverable. However, we believe the following is correct.

Conjecture 1. *Every simple planar graph is* $(2, 4)$-*coverable.*

As evidence for this conjecture, we shall prove that every simple planar graph is $(2, 8)$-coverable. This will be done in Section 3, with the help of a result from Section 2. In Section 4, we shall show that every simple outerplanar graph is $(1, 3)$-coverable, and as a consequence, every 4-connected planar graph is $(2, 6)$-coverable. We shall also consider graphs which are series-parallel or contain no $K_{3,2}$-subdivision. We conclude this section with some notation.

Throughout the remainder of this paper, we shall consider only simple graphs. Let G be a graph. An edge of G with endpoints x and y will be denoted by xy or yx. Paths and cycles in G will be denoted by sequences of vertices of G. For any $x \in V(G)$, let $N_G(x) := \{y \in V(G) : xy \in E(G)\}$, and let $d_G(x) := |N_G(x)|$, the degree of x. When G is known from the context, we shall simply write $N(x)$ and $d(x)$. Let $\Delta(G) := \max\{d(x) : x \in V(G)\}$. For any $S \subseteq V(G)$, we use $G - S$ to denote the graph with vertex set $V(G) - S$ and edge set $\{uv \in E(G) : \{u, v\} \subseteq V(G) - S\}$. For any $S \subseteq E(G)$, we use $G - S$ to denote the graph with vertex set $V(G)$ and edge set $E(G) - S$. When $S = \{s\}$, we shall simply write $G - s$. Let H be a subgraph of G and let $S \subseteq V(G) \cup E(G)$ such that every edge of G in S has both endpoints in $V(H) \cup (S \cap V(G))$, then we use $H + S$ to denote the graph with vertex set $V(H) \cup (S \cap V(G))$ and edge set $E(H) \cup (S \cap E(G))$.

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Recall that a plane graph is a graph drawn in the plane with no pairs of edges crossing. A *facial cycle* of a plane graph G is a cycle that bounds a face of G. A *planar triangulation* is a plane graph in which every face is bounded by a triangle.

2. High vertices

In this section, we shall prove the following result about planar graphs. This result will be used in the next section to prove that all planar graphs are (2, 8)-coverable. Let G be a graph and $x \in V(G)$. Then x is said to be *high* if $d(x) \geqslant 11$, and *low* otherwise.

Theorem 2. *Every planar graph contains a vertex of degree at most* 5 *which is adjacent to at most two high vertices.*

Proof. Suppose the statement is not true. Then there is a planar triangulation G such that every vertex of degree at most 5 is adjacent to at least three high vertices. Therefore, all vertices of G have degree at least 3.

Let $v \in V(G)$ with $d(v) = 4$. We say that v is *4-independent* if, for any $u \in N(v)$, $d(u) \neq 4$; otherwise, we say that v is *4-dependent*. Let u_1, u_2 be two adjacent 4-dependent vertices. Then $G - \{u_1, u_2\}$ has a facial cycle $v_1v_2v_3v_4v_1$, and v_1, v_2, v_3, v_4 are all high vertices of G. Furthermore, the notation can be chosen so that v_1, v_3 are adjacent to both u_1 and u_2, and v_2 (respectively, v_4) is adjacent with u_1 (respectively, u_2). In this case we say that v_1, v_3 are u_1*-weak* and v_2 is u_1*-strong*, and v_1, v_3 are u_2*-weak* and v_4 is u_2*-strong*.

Next, we define a weight function $\omega : V(G) \to \mathbb{R}$ by making changes to the degree function $d : V(G) \to \mathbb{R}$. For each high vertex v of G, we make changes to $d(v)$ and $d(u)$ for all $u \in N(v)$ with $d(u) \leqslant 5$, according to the following rules:

(R1) If $u \in N(v)$ and $d(u) = 3$, then subtract 1 from $d(v)$ and add 1 to $d(u)$.
(R2) If $u \in N(v)$ and $d(u) = 5$, then subtract $\frac{1}{3}$ from $d(v)$ and add $\frac{1}{3}$ to $d(u)$.
(R3) If $u \in N(v)$ and u is 4-independent, then subtract $\frac{2}{3}$ from $d(v)$ and add $\frac{2}{3}$ to $d(u)$.
(R4) If $u \in N(v)$, u is 4-dependent, and v is u-strong, then subtract 1 from $d(v)$ and add 1 to $d(u)$.
(R5) If $u \in N(v)$, u is 4-dependent, and v is u-weak, then subtract $\frac{1}{2}$ from $d(v)$ and add $\frac{1}{2}$ to $d(u)$.

Let $\omega : V(G) \to \mathbb{R}$ denote the resulting function. For convenience, when we subtract a quantity α from $d(v)$ and add a quantity α to $d(u)$, we will simply say that v sends *charge* α to u or u receives *charge* α from v.

Clearly,

$$\sum_{x \in V(G)} d(x) = \sum_{x \in V(G)} \omega(x).$$

Since G has $3|V(G)| - 6$ edges, $\sum_{x \in V(G)} d(x) < 6|V(G)|$. Hence there exists a vertex x of G such that $\omega(x) < 6$. We shall derive a contradiction by showing that $\omega(x) \geqslant 6$ for all $x \in V(G)$. Let $x \in V(G)$. We distinguish two cases.

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Case 1: x is low.

If $d(x) = 3$ then, since all its neighbors are high, $\omega(x) = d(x) + 3 = 3 + 3 = 6$ by (R1).

If $d(x) = 5$ then, since x has $k \geqslant 3$ high neighbors, $\omega(x) = d(x) + k/3 = 5 + k/3 \geqslant 6$ by (R2).

Now assume $d(x) = 4$. If x is 4-independent then, since x has $k \geqslant 3$ high neighbors, $\omega(x) = d(x) + 2k/3 = 4 + 2k/3 \geqslant 6$ by (R3). If x is 4-dependent then, since x has three high neighbors (two are x-weak and one is x-strong), $\omega(x) = 4 + \frac{1}{2} + \frac{1}{2} + 1 = 6$ by (R4) and (R5).

If $6 \leqslant d(x) \leqslant 10$, then $\omega(x) = d(x) \geqslant 6$.

Case 2: x is high.

Let $d(x) = k$. Then $k \geqslant 11$. Since G is a planar triangulation, $G - x$ has a facial cycle C_k such that $V(C_k) = N(x)$. We partition $V(C_k)$ into the following five sets. Let $A := \{u \in N(x) : d(u) = 3$, or u is 4-dependent and x is u-strong$\}$. Let $B := \{u \in N(x) : u$ is 4-dependent and x is u-weak$\}$. Let $C := \{u \in N(x) : u$ is 4-independent$\}$. Let $D := \{u \in N(x) : d(u) = 5\}$. Finally, let $S := \{u \in N(x) : d(u) \geqslant 6\}$. Because every vertex of degree at most 5 has at least 3 high neighbors, one can easily check that the following statements hold:

(1) if $u \in A$, then u has two neighbors in S, and u receives charge 1 from x (by (R1) and (R4)).
(2) if $u \in B$, then (by planarity) u has one neighbor in B and one neighbor in S, and u receives charge $\frac{1}{2}$ from x (by (R5)).
(3) if $u \in C$, then u has at least one neighbor in S and at most one neighbor in D, and u receives charge $\frac{2}{3}$ from x (by (R3)).
(4) if $u \in D$, then u can have neighbors in $C \cup D \cup S$, and u receives charge $\frac{1}{3}$ from x (by (R2)).
(5) if $u \in S$, then u receives no charge from x.

Therefore, if $S = \emptyset$, then $A = B = C = \emptyset$, and hence, $D = V(C_k)$ and, by (4), $\omega(x) = k - (k/3) \geqslant \frac{22}{3} > 6$.

So assume $S \neq \emptyset$. Let $S = \{s_1, \ldots, s_m\}$ such that $s_1, \ldots, s_m$ occur on C_k in that clockwise order. If $m = 1$, let $S_1 = C_k$ and $s_2 = s_1$. If $m \geqslant 2$, the vertices in S divide C_k into k internally disjoint paths: for $1 \leqslant i \leqslant k$, let S_i denote the clockwise subpath of C_k from s_i to s_{i+1}, where $s_{m+1} = s_1$. Let $S'_i := S_i - \{s_i, s_{i+1}\}$.

We claim that, for each $1 \leqslant i \leqslant m$, one of the following holds:

(a) $|V(S'_i)| \leqslant 1$.
(b) $|V(S'_i)| = 2$ and $V(S'_i) \subseteq B$.
(c) $|V(S'_i)| = 2$, $V(S'_i) \subseteq C \cup D$ and $V(S'_i) \cap D \neq \emptyset$.
(d) $|V(S'_i)| \geqslant 3$, $V(S'_i) \subseteq C \cup D$ and all internal vertices of S'_i are contained in D.

To prove this claim, assume that $|V(S'_i)| \geqslant 2$ (that is, not (a)) and let $S_i = x_0x_1, \ldots, x_nx_{n+1}$, where $x_0 = s_i$ and $x_{n+1} = s_{i+1}$. Thus, $x_0, x_{n+1} \in S$, $n \geqslant 2$, and $x_1, \ldots, x_n \notin S$. Recall that we allow $x_0 = x_{n+1}$, which occurs when $m = 1$. Then, for each $1 \leqslant j \leqslant n$, $x_j \notin A$; for otherwise, by (1), $\{x_{j-1}, x_{j+1}\} \subseteq S$, contradicting the fact that $x_1, \ldots, x_n \notin S$.

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Now assume that there is some $x_j \in B$. Since x_j has at least three high neighbors, one element of $\{x_{j-1}, x_{j+1}\}$ is high. By symmetry we may assume that x_{j-1} is high. Then $x_{j-1} \in S$. Since $x_j \in B$, x is x_j-weak. So $x_{j+1} \in B$, x_{j+2} is high, and $x_{j+2} \in S$. Hence, $x_{j-1} = x_0$ and $x_{j+2} = x_{n+1}$, $n = 2$, and $\{x_1, x_2\} \subseteq B$. That is, $V(S_i')$ consists of exactly two vertices which are in B, and (b) holds. So we may assume that $\{x_1, \ldots, x_n\} \subseteq C \cup D$, that is, $V(S_i') \subseteq C \cup D$. Then, since each x_j has at least three high neighbors, $x_2, \ldots, x_{n-1} \in D$ and, if $n = 2$ then $x_1 \in D$, or $x_n \in D$. So we have (c) and (d).

Now, let us calculate $\omega(x)$ by finding out how much charge x sends to vertices of S_i'. Suppose (a) holds for S_i'. If $|V(S_i')| = 1$ then the charge that x sends to S_i' is at most $1 = \lfloor \frac{|V(S_i')|+1}{2} \rfloor$. If $|V(S_i')| = 0$ then the charge that x sends to S_i' is $0 = \lfloor \frac{|V(S_i')|+1}{2} \rfloor$. If (b) holds for S_i', then by (2), the charge that x sends to vertices of S_i' is $\frac{1}{2} + \frac{1}{2} = 1 = \lfloor |V(S_i')|/2 \rfloor$. Now assume (c) holds for S_i'. If $|V(S_i')| = 2$ then by (c) at least one vertex of S_i' is in D, and by (3) and (4), the charge that x sends to vertices of S_i' is at most $\frac{2}{3} + \frac{1}{3} = 1 = \lfloor |V(S_i')|/2 \rfloor$. If $|V(S_i')| \geqslant 3$, then by (d), all internal vertices of S_i' are in D, and by (3) and (4), the charge that x sends to S_i' is at most $(n-2)/3 + \frac{2}{3} + \frac{2}{3} = (n+2)/3 \leqslant \lfloor (n+1)/2 \rfloor = \lfloor (|V(S_i')|+1)/2 \rfloor$ (because $n = |V(S_i')| \geqslant 3$). By (5), x sends no charge to vertices in S. Hence, the total charge that x sends to its neighbors is at most

$$\sum_{i=1}^{m} \left\lfloor \frac{|V(S_i')|+1}{2} \right\rfloor \leqslant \left\lfloor \frac{(\sum_{i=1}^{m} |V(S_i')|) + m}{2} \right\rfloor = \lfloor d(x)/2 \rfloor.$$

So $\omega(x) \geqslant d(x) - \lfloor d(x)/2 \rfloor$. Since $d(x) \geqslant 11$, $\omega(x) \geqslant 6$. □

Theorem 2 no longer holds if we define high vertices as those of degree 10 or more. Consider a planar triangulation with vertices of degrees 6 and 5. Put into each triangle a vertex and join it to all vertices of the triangle. We get a planar triangulation with vertices of degrees 3, 10, 12, and each vertex has at least 3 neighbors of degree at least 10.

3. Covering with forests

In this section we prove that every planar graph is (2,8)-coverable. In fact, we prove the following stronger result.

Theorem 3. *For each planar graph G, there exist forests T_1, T_2, and T_3 such that $G = T_1 \cup T_2 \cup T_3$ and $\Delta(T_3) \leqslant 8$.*

The proof is by way of contradiction. Suppose Theorem 3 is not true. Let G be a counter example with $|V(G)|$ minimum. Without loss of generality, we may assume that G is a planar triangulation. Hence the minimum degree of G is at least 3. We shall derive a contradiction to Theorem 2 by showing that every vertex of G with degree at most 5 has at least three high neighbors.

Lemma 4. *If $x \in V(G)$ and $d(x) = 3$, then all three neighbors of x are high.*

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Proof. Consider the graph $G' := G - x$. By the choice of G, G' can be covered by three forests T_1', T_2', and T_3' such that $\Delta(T_3') \leqslant 8$. Without loss of generality, we may further assume that T_1', T_2', T_3' are edge disjoint, and subject to this, $|E(T_3')|$ is minimum. Therefore, for any $u \in V(T_3')$, $d_{T_i'}(u) \geqslant 1$ for $i = 1, 2$. Hence, $d_{T_3'}(v) \leqslant d_{G'}(v) - 2$ for every vertex v of G'.

Suppose some neighbor of x is not high, say y. Then $d_G(y) \leqslant 10$. So $d_{G'}(y) \leqslant 9$, and $d_{T_3'}(y) \leqslant d_{G'}(y) - 2 \leqslant 7$. Let v, w be the other two neighbors of x. Let $T_1 := T_1' + \{x, xv\}$, $T_2 := T_2' + \{x, xw\}$, and let $T_3' := T_3 + \{x, xy\}$. It is easy to check that T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(y) = d_{T_3'}(y) + 1 \leqslant 8$ and, for any $u \in V(T_3) - \{y\}$, $d_{T_3}(u) = d_{T_3'}(u) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence, the existence of T_1, T_2, T_3 contradicts the choice of G. So all neighbors of x are high. $\square$

Lemma 5. *If $x \in V(G)$ and $d(x) = 4$, then at least three neighbors of x are high.*

Proof. Let u, y, v and z denote the neighbors of x, occurring in that clockwise order around x. Since G is planar, $uv \notin E(G)$ or $yz \notin E(G)$. Without loss of generality we may assume that $yz \notin E(G)$. Then $G' := (G - x) + yz$ is a planar triangulation. By the choice of G, G' can be covered by three forests T_1', T_2', T_3' such that $\Delta(T_3') \leqslant 8$. We may further assume that T_1', T_2', T_3' are edge disjoint, and subject to this, $|E(T_3')|$ is minimum. Therefore, $d_{T_3'}(v) \leqslant d_{G'}(v) - 2$ for every vertex v of G'.

If $yz \in E(T_3')$, we let $T_1 := T_1' + \{x, ux\}$, $T_2 := T_2' + \{x, vx\}$ and $T_3 := (T_3' - yz) + \{x, yx, xz\}$. It is easy to see that T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 2$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence, the existence of T_1, T_2, T_3 contradicts the choice of G.

So $yz \notin E(T_3')$. Then $yz \in E(T_1') \cup E(T_2')$. By symmetry, we may assume that $yz \in E(T_1')$.

We claim that u must be high. For, suppose u is low. Then $d_{G'}(u) = d_G(u) - 1 \leqslant 9$ and $d_{T_3'}(u) \leqslant d_{G'}(u) - 2 \leqslant 7$. Let $T_1 := (T_1' - yz) + \{x, xy, xz\}$, $T_2 := T_2' + \{x, xv\}$, and $T_3 := T_3' + \{x, xu\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 1$ and $d_{T_3}(u) = d_{T_3'}(u) + 1 \leqslant 8$, and for any $w \in V(T_3) - \{u, x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

By a symmetric argument, we can show that v is also high.

Next we show that y is high or z is high. Suppose both y and z are low. Since T_1' is a forest and $yz \in E(T_1')$, $T_1' - yz$ does not contain both a y–v path and a z–v path. By symmetry, we may assume that $T_1' - yz$ contain no y–v path. Let $T_1 := (T_1' - yz) + \{x, v, yx, xv\}$, $T_2 := T_2' + \{x, ux\}$ and $T_3 := T_3' + \{x, xz\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 1$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3') \leqslant 8$. Hence, the existence of T_1, T_2, T_3 contradicts the choice of G.

Therefore, at least three neighbors of x are high. $\square$

Lemma 6. *Let $x \in V(G)$ with $d(x) = 5$, and let x_0, x_1, x_2, x_3 and x_4 denote the neighbors of x which occur around x in that clockwise order. For any $0 \leqslant i \leqslant 4$, if $x_i x_{i+2} \notin E(G)$ and $x_i x_{i-2} \notin E(G)$, then both x_{i-1} and x_{i+1} are high.* (*Subscripts are taken modulo* 5.)

Proof. Since G is a planar triangulation, $x_0x_1x_2x_3x_4x_0$ is a facial cycle of $G - x$. Suppose $0 \leqslant i \leqslant 4$, $x_i x_{i+2} \notin E(G)$, and $x_i x_{i-2} \notin E(G)$. Then by the choice of G, $G' = (G - x) +$

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$\{x_ix_{i+2}, x_ix_{i-2}\}$ can be covered by three forests T_1', T_2', T_3', with $\Delta(T_3') \leqslant 8$. We may further assume that T_1', T_2', T_3' are edge disjoint, and subject to this, $|E(T_3')|$ is minimum. Therefore, $d_{T_3'}(v) \leqslant d_{G'}(v) - 2$ for every vertex v of G'.

Case 1: $\{x_ix_{i+2}, x_ix_{i-2}\} \subseteq E(T_3')$.

Let $T_1 := T_1' + \{x, xx_{i+1}\}$, $T_2 := T_2' + \{x, xx_{i-1}\}$ and $T_3 := (T_3' - \{x_ix_{i+2}, x_ix_{i-2}\}) + \{x, xx_{i+2}, xx_{i-2}, xx_i\}$. Clearly, T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 3$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Case 2: $\{x_ix_{i+2}, x_ix_{i-2}\} \subseteq E(T_1')$ or $\{x_ix_{i+2}, x_ix_{i-2}\} \subseteq E(T_2')$.

By symmetry, we may assume that $\{x_ix_{i+2}, x_ix_{i-2}\} \subseteq E(T_1')$. We show that both x_{i+1} and x_{i-1} are high. For, assume by symmetry that x_{i-1} is low. Then $d_{G'}(x_{i-1}) = d_G(x_{i-1}) - 1 \leqslant 9$ and $d_{T_3'}(x_{i-1}) \leqslant d_{G'}(x_{i-1}) - 2 \leqslant 7$. Let $T_1 := (T_1' - \{x_ix_{i+2}, x_ix_{i-2}\}) + \{x, xx_{i+2}, xx_{i-2}, xx_i\}$, $T_2 := T_2' + \{x, x_{i+1}x\}$, $T_3 := T_3' + \{x, xx_{i-1}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 1$, $d_{T_3}(x_{i-1}) \leqslant 8$ and, for any $w \in V(T_3) - \{x, x_{i-1}\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Case 3: One element of $\{x_ix_{i+2}, x_ix_{i-2}\}$ is in $E(T_3')$ and the other is in $E(T_1') \cup E(T_2')$.

By symmetry, we may assume that $x_ix_{i+2} \in E(T_1')$ and $x_ix_{i-2} \in E(T_3')$. We consider five subcases.

Subcase 3.1: $T_1' - x_ix_{i+2}$ contains an x_i-x_{i+1} path. Then $T_1' - x_ix_{i+2}$ contains no x_{i+1}-x_{i+2} path. In this case, let $T_1 := (T_1' - x_ix_{i+2}) + \{x, xx_{i+2}, xx_{i+1}\}$, $T_2 := T_2' + \{x, xx_{i-1}\}$ and $T_3 := (T_3' - x_ix_{i-2}) + \{x, xx_i, xx_{i-2}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 2$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 3.2: $T_1' - x_ix_{i+2}$ contains an x_i-x_{i-1} path. Then $T_1' - x_ix_{i+2}$ contains no x_{i-1}-x_{i+2} path. In this case, let $T_1 := (T_1' - x_ix_{i+2}) + \{x, xx_{i+2}, xx_{i-1}\}$, $T_2 := T_2' + \{x, xx_{i+1}\}$ and $T_3 := (T_3' - x_ix_{i-2}) + \{x, xx_i, xx_{i-2}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 2$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 3.3: $T_1' - x_ix_{i+2}$ contains neither an x_i-x_{i+1} path nor an x_i-x_{i-1} path, and $x_ix_{i-1} \in E(T_3')$. Let $T_1 := (T_1' - x_ix_{i+2}) + \{x, x_{i-1}, x_ix_{i-1}, xx_{i+2}\}$, $T_2 := T_2' + \{x, xx_{i+1}\}$, and $T_3 := (T_3' - \{x_ix_{i-2}, x_ix_{i-1}\}) + \{x, xx_i, xx_{i-1}, xx_{i-2}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 3$, $d_{T_3}(x_i) = d_{T_3'}(x_i) - 1$, and for any $w \in V(T_3') - \{x, x_i\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 3.4: $T_1' - x_ix_{i+2}$ contains neither an x_i-x_{i+1} path nor an x_i-x_{i-1} path, and $x_ix_{i+1} \in E(T_3')$. Let $T_1 := (T_1' - x_ix_{i+2}) + \{x, x_{i+1}, x_ix_{i+1}, xx_{i+2}\}$, $T_2 := T_2' + \{x, xx_{i-1}\}$, and $T_3 := (T_3' - \{x_ix_{i-2}, x_ix_{i+1}\}) + \{x, xx_i, xx_{i+1}, xx_{i-2}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 3$, $d_{T_3}(x_i) = d_{T_3'}(x_i) - 1$, and for any $w \in V(T_3') - \{x, x_i\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 3.5: $T_1' - x_ix_{i+2}$ contains neither an x_i-x_{i+1} path nor an x_i-x_{i-1} path, and $x_ix_{i-1}, x_ix_{i+1} \notin E(T_3')$. Then $x_ix_{i-1}, x_ix_{i+1} \in E(T_2')$. Let $T_1 := (T_1' - x_ix_{i+2}) + \{x, x_{i+1}, x_ix_{i+1}, xx_{i+2}\}$, $T_2 := (T_2' - x_ix_{i+1}) + \{x, xx_{i-1}, xx_{i+1}\}$, and $T_3 := (T_3' - x_ix_{i-2}) + \{x, xx_i, xx_{i-2}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 2$ and, for any

$w \in V(T_3') - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Case 4: One element of $\{x_i x_{i+2}, x_i x_{i-2}\}$ is in $E(T_1')$ and the other is in $E(T_2')$.

Without loss of generality, we may assume that $x_i x_{i+2} \in E(T_1')$ and $x_i x_{i-2} \in E(T_2')$. Then, up to symmetry, it suffices to check the following six subcases.

Subcase 4.1: $T_1' - x_i x_{i+2}$ contains neither an x_i-x_{i-1} path nor an x_i-x_{i+1} path, and $T_2' - x_i x_{i-2}$ contains neither an x_i-x_{i-1} path nor an x_i-x_{i+1} path.

Then $\{x_i x_{i+1}, x_i x_{i-1}\} \subseteq E(T_3')$. Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, x_{i+1}, x_i x_{i+1}, x x_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, x_{i-1}, x_i x_{i-1}, x x_{i-2}\}$, and $T_3 := (T_3' - \{x_i x_{i+1}, x_i x_{i-1}\}) + \{x, x x_{i+1}, x x_i, x x_{i-1}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 3$, $d_{T_3}(x_i) = d_{T_3'}(x_i) - 1$ and, for any $w \in V(T_3) - \{x, x_i\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 4.2: $T_1' - x_i x_{i+2}$ contains both an x_i-x_{i-1} path and an x_i-x_{i+1} path, or $T_2' - x_i x_{i-2}$ contains both an x_i-x_{i-1} path and an x_i-x_{i+1} path.

By symmetry, we may assume that $T_1' - x_i x_{i+2}$ contains an x_i-x_{i-1} path and an x_i-x_{i+1} path. Then $T_1' - x_i x_{i+2}$ contains no x_{i+1}-x_{i+2} path. Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, x x_{i+1}, x x_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, x x_i, x x_{i-2}\}$, and $T_3 := T_3' + \{x, x x_{i-1}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 1$ and, for any $w \in V(T_3) - \{x, x_{i-1}\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. If x_{i-1} is low, then $d_{T_3'}(x_{i-1}) \leqslant d_{G'}(x_{i-1}) - 2 = d_G(x_{i-1}) - 3 \leqslant 7$, and so, $d_{T_3}(x_{i-1}) \leqslant 8$ and $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G. So x_{i-1} must be high.

Similarly, the forests $T_1 := (T_1' - x_i x_{i+2}) + \{x, x x_{i-1}, x x_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, x x_i, x x_{i-2}\}$, and $T_3 := T_3' + \{x, x x_{i+1}\}$ allow us to conclude that x_{i+1} must be high.

Subcase 4.3: There is an x_i-x_{i+1} path in $T_1' - x_i x_{i+2}$, and there are no x_i-x_{i-1} paths in $T_1' - x_i x_{i+2}$ and $T_2' - x_i x_{i-2}$.

Then $x_i x_{i-1} \in E(T_3')$ and $T_1' - x_i x_{i+2}$ contains no x_{i+1}-x_{i+2} path. Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, x x_{i+1}, x x_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, x_i x_{i-1}, x x_{i-2}\}$, and $T_3 := (T_3' - x_i x_{i-1}) + \{x, x x_{i-1}, x x_i\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 2$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 4.4: There is a x_i-x_{i-1} path in $T_1' - x_i x_{i+2}$, and there are no x_i-x_{i+1} paths in $T_1' - x_i x_{i+2}$ and $T_2' - x_i x_{i-2}$.

Then $x_i x_{i+1} \in E(T_3')$ and $T_1' - x_i x_{i+2}$ contains no x_{i-1}-x_{i+2} path. Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, x x_{i-1}, x x_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, x_i x_{i+1}, x x_{i-2}\}$, and $T_3 := (T_3' - x_i x_{i+1}) + \{x, x x_{i+1}, x x_i\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 2$ and, for any $w \in V(T_3) - \{x\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. So $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G.

Subcase 4.5: There is an x_i-x_{i+1} path in $T_1' - x_i x_{i+2}$, there is no x_i-x_{i-1} path in $T_1' - x_i x_{i+2}$, there is an x_i-x_{i-1} path in $T_2' - x_i x_{i-2}$, and there is no x_i-x_{i+1} path in $T_2' - x_i x_{i-2}$.

Then $T_1' - x_i x_{i+2}$ contains no x_{i+1}-x_{i+2} path, and $T_2' - x_i x_{i-2}$ contains no x_{i-1}-x_{i-2} path.

Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, x x_{i+1}, x x_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, x x_i, x x_{i-2}\}$, and $T_3 := T_3' + \{x, x x_{i-1}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 1$ and, for any $w \in V(T_3) - \{x, x_{i-1}\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. If x_{i-1} is low, then $d_{T_3'}(x_{i-1}) \leqslant d_{G'}(x_{i-1})$

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$-2 = d_G(x_{i-1}) - 3 \leqslant 7$, and so, $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G. So x_{i-1} must be high.

Similarly, the forests $T_1 := (T_1' - x_i x_{i+2}) + \{x, xx_i, xx_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, xx_{i-1}, xx_{i-2}\}$, and $T_3 := T_3' + \{x, xx_{i+1}\}$ allow us to conclude that x_{i+1} must be high.

Subcase 4.6: There is an x_i-x_{i-1} path in $T_1' - x_i x_{i+2}$, there is no x_i-x_{i+1} path in $T_1' - x_i x_{i+2}$, there is an x_i-x_{i+1} path in $T_2' - x_i x_{i-2}$, and there is no x_i-x_{i-1} path in $T_2' - x_i x_{i-2}$.

Then $T_1' - x_i x_{i+2}$ contains no x_{i-1}-x_{i+2} path, and $T_2' - x_i x_{i-2}$ contains no x_{i+1}-x_{i-2} path.

Let $T_1 := (T_1' - x_i x_{i+2}) + \{x, xx_{i-1}, xx_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, xx_i, xx_{i-2}\}$, and $T_3 := T_3' + \{x, xx_{i+1}\}$. Then T_1, T_2, T_3 are forests and cover G. Note that $d_{T_3}(x) = 1$ and, for any $w \in V(T_3) - \{x, x_{i+1}\}$, $d_{T_3}(w) = d_{T_3'}(w) \leqslant 8$. If x_{i+1} is low, then $d_{T_3'}(x_{i+1}) \leqslant d_{G'}(x_{i+1}) - 2 = d_G(x_{i+1}) - 3 \leqslant 7$, and so, $\Delta(T_3) \leqslant 8$. Hence the existence of T_1, T_2, T_3 contradicts the choice of G. So x_{i+1} must be high.

Similarly, the forests $T_1 := (T_1' - x_i x_{i+2}) + \{x, xx_i, xx_{i+2}\}$, $T_2 := (T_2' - x_i x_{i-2}) + \{x, xx_{i+1}, xx_{i-2}\}$, and $T_3 := T_3' + \{x, xx_{i-1}\}$ allow us to conclude that x_{i-1} must be high.

Therefore x_{i-1} and x_{i+1} are high. $\square$

We can now complete the proof of Theorem 3 as follows.

Proof. By Theorem 2, there is a vertex x of G such that $d(x) \leqslant 5$ and x has at most two high neighbors. By Lemmas 4 and 5, we see that $d(x) = 5$. Let $x_0, x_1, \ldots, x_4$ denote the neighbors of x such that $x_0 x_1 \ldots x_4 x_0$ is a facial cycle of $G - x$. By planarity, there exist $0 \leqslant i \neq j \leqslant 4$ such that $x_i x_{i-2}, x_i x_{i+2}, x_j x_{j-2}, x_j x_{j+2} \notin E(G)$. So by Lemma 6, $x_{i-1}, x_{i+1}, x_{j-1}, x_{j+1}$ are high vertices. Since $x_i \neq x_j$ and $x_0 x_1 x_2 x_3 x_4 x_0$ is a cycle, $|\{x_{i-1}, x_{i+1}, x_{j-1}, x_{j+1}\}| \geqslant 3$. But this means that x has at least three high neighbors, a contradiction.

It is not hard to see that we may further require T_1, T_2 be trees. $\square$

4. Special planar graphs

In this section, we shall see that Theorem 3 can be improved for some special classes of planar graphs, thereby providing further evidence for Conjecture 1. Recall that a graph is outerplanar if it can be embedded in the plane such that all vertices are incident with its infinite face.

Theorem 7. *Let G be a 2-connected outerplanar graph and let C be the cycle of an outerplanar embedding of G bounding the infinite face. Let $y \in V(C)$ and let $yx, yz \in E(C)$. Then there is a forest T in G such that $d_{G-E(T)}(y) = 0$, $d_{G-E(T)}(x) \leqslant 1$, $d_{G-E(T)}(z) \leqslant 2$, $\Delta(G - E(T)) \leqslant 3$, and $G - E(T)$ is a forest.*

Proof. We apply induction on $|V(G)|$. It is easy to see that the theorem holds when $|V(G)| = 3$. So assume that $|V(G)| \geqslant 4$. Without loss of generality, we may assume that x, y, z occur on C in the clockwise order listed.

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First, we consider the case when $d(y) = 2$. Let $H := (G - y) + xz$ and $D := (C - y) + xz$. Then H can be embedded in the plane so that H is an outerplanar graph with D bounding its infinite face. Let $xx' \in E(D)$ with $x' \neq z$ (because $|V(G)| \geqslant 4$). We apply induction to H, D, z, x, x' (as G, C, x, y, z, respectively). There is a forest S in H such that $d_{H-E(S)}(x) = 0$, $d_{H-E(S)}(z) \leqslant 1$, $d_{H-E(S)}(x') \leqslant 2$, $\Delta(H - E(S)) \leqslant 3$, and $H - E(S)$ is a forest. Now let T be the forest in G obtained from S by replacing the edge xz of S with the path xyz in G. It is easy to see that $d_{G-E(T)}(y) = 0$. Because $d_{H-E(S)}(x) = 0$, $d_{G-E(T)}(x) \leqslant 1$. Because $d_{H-E(S)}(z) \leqslant 1$, $d_{G-E(T)}(z) \leqslant 2$. The possible increase of 1 in the degrees comes from the edge xz. Therefore, because $\Delta(H - E(S)) \leqslant 3$, we have $\Delta(G - E(T)) \leqslant 3$. Since $G - E(T) = (H - E(S)) + xz$ and $d_{H-E(S)}(x) = 0$, we see that $G - E(T)$ is also a forest.

So we may assume that $d(y) \geqslant 3$. We label the neighbors of y as $y_1, \ldots, y_{k+1}$ in counterclockwise order on C. Then $k \geqslant 2$. Without loss of generality, assume that $y_1 = x$ and $y_{k+1} = z$. For $i = 1, \ldots, k$, let C_i denote the cycle which is the union of $y_{i+1}yy_i$ and the counterclockwise subpath of C from y_i to y_{i+1}, and let H_i denote the subgraph of G contained in the closed disc bounded by C_i. Then H_i is an outerplanar graph and C_i bounds its infinite face. For each $1 \leqslant i \leqslant k$, we apply induction to $H_i, C_i, y_i, y, y_{i+1}$ (as G, C, x, y, z, respectively). Therefore, for each $1 \leqslant i \leqslant k$, H_i has a forest T_i such that $d_{H_i - E(T_i)}(y) = 0$, $d_{H_i - E(T_i)}(y_i) \leqslant 1$, $d_{H_i - E(T_i)}(y_{i+1}) \leqslant 2$, $\Delta(H_i - E(T_i)) \leqslant 3$, and $H_i - E(T_i)$ is a forest. Let $T := \bigcup_{i=1}^{k} T_i$. Then T is a forest in G. It is easy to see that $d_{G-E(T)}(y) = 0$, $d_{G-E(T)}(x) \leqslant 1$, and $d_{G-E(T)}(z) \leqslant 2$. Note that for $i = 1, \ldots, k$, $d_{H_i - E(T_i)}(y_i) \leqslant 1$ and $d_{H_i - E(T_i)}(y_{i+1}) \leqslant 2$. Hence, $d_{G-E(T)}(y_i) \leqslant 3$ for $i = 2, \ldots, k-1$. Thus, $\Delta(G - E(T)) \leqslant 3$. It is also easy to see that $G - E(T) = \bigcup_{i=1}^{k}(H_i - E(T_i))$. Since $d_{G-E(T)}(y) = 0$, $G - E(T)$ is a forest. □

The following example gives a family of outerplanar graphs which are not $(1, 2)$-coverable. Take a long cycle $C = v_0v_1 \ldots v_{2n+1}v_0$ and add the following edges: v_0v_{2i+1} for $i = 1, \ldots, n-1$ and $v_{2i-1}v_{2i+1}$ for $i = 1, \ldots, n$.

Next, we show that all 4-connected planar graphs are $(2, 6)$-coverable. But first, we consider Hamiltonian planar graphs.

Corollary 8. *If G is a Hamiltonian planar graph, then it is* $(2, 6)$*-coverable.*

Proof. Take a plane embedding of G and let C be a Hamiltonian cycle in G. Let G_1 (respectively, G_2) denote the subgraph of G inside (respectively, outside) the closed disc bounded by C. Then G_1 and G_2 are outer planar graphs (with C as the boundary cycle). Pick a vertex $y \in V(C)$, and apply Theorem 7 to G_i, $i = 1, 2$, we find a forest T_i in G_i such that $d_{G_i - E(T_i)}(y) = 0$ and $\Delta(G_i - E(T_i)) \leqslant 3$. It is easy to verify that $\Delta(G - E(T_1 \cup T_2)) \leqslant 6$. □

Tutte [6] proved that every 4-connected planar graph contains a Hamilton cycle. Thus, by Corollary 8, we have the following result.

Corollary 9. *If G is a* 4*-connected planar graph, then it is* $(2, 6)$*-coverable.*

It is well known that a graph is outerplanar if and only if it contains no K_4-subdivision or $K_{3,2}$-subdivision [1, Proposition 7.3.1]. In view of Theorem 7, it is natural to consider

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the class of graphs containing no K_4-subdivisions and the class of graphs containing no $K_{3,2}$-subdivisions.

The graphs containing no K_4-subdivisions are also called *series-parallel* graphs. It is known that any simple series-parallel graph has a vertex of degree at most two (see [2]). Therefore, by applying induction on the number of vertices, we can show that any simple series-parallel graph is $(2, 0)$-coverable.

On the other hand, the graph $K_{n,2}$ is series-parallel, but is not $(1, \lfloor \frac{n}{2} - 2 \rfloor)$-coverable. So it is natural to consider graphs containing no $K_{n,2}$-subdivisions. An easier question is to determine the smallest t and D so that every simple graph with no $K_{n,2}$-minors is (t, D)-coverable, for $n \geqslant 2$. To this end, we consider the cases $n = 2, 3$. We note that when $n = 2, 3$, a graph contains a $K_{n,2}$-minor if, and only if, it contains a $K_{n,2}$-subdivision.

Note that if G is a simple graph containing no $K_{2,2}$-minor, then every block of G is either a triangle or induced by an edge. So it is easy to see that any simple graph containing no $K_{2,2}$-minor is $(1, 1)$-coverable.

For graphs with no $K_{3,2}$-minor, we have the following result.

Proposition 10. *If G is a simple graph containing no $K_{3,2}$-subdivision, then G is both $(1, 3)$-coverable and $(2, 0)$-coverable.*

Proof. First we shall prove the existence of a $(1, 3)$-cover. To do this, we prove the following stronger result.

(1) For any vertex v of G there is a forest T in G such that $d_{G-E(T)}(v) = 0$ and $\Delta(G - E(T)) \leqslant 3$.

We use induction on the number of K_4-subdivisions contained in G. If G contains no K_4-subdivision, then it is outerplanar, and (1) follows from Theorem 7. So assume that G contains a K_4-subdivision. In fact, every K_4-subdivision in G must be isomorphic to K_4, since any K_4-subdivision not isomorphic to K_4 is also a $K_{3,2}$-subdivision.

Let $\{v_1, v_2, v_3, v_4\} \subseteq V(G)$ induce a K_4 in G. Since G has no $K_{3,2}$-subdivision, $G - \{v_i v_j : 1 \leqslant i \neq j \leqslant 4\}$ has exactly four components C_i with $v_i \in V(C_i)$, $i = 1, 2, 3, 4$. Without loss of generality, we may assume that $v \in V(C_1)$. By applying induction to C_1, we conclude that C_1 contains a forest T_1 such that $d_{C_1-E(T_1)}(v) = 0$ and $\Delta(C_1 - E(T_1)) \leqslant 3$. Similarly, by applying induction to C_i, $i = 2, 3, 4$, C_i contains a forest T_i such that $d_{C_i-E(T_i)}(v_i) = 0$ and $\Delta(C_i - E(T_i)) \leqslant 3$. Let $T := (\bigcup_{i=1}^{4} T_i) + \{v_1 v_2, v_1 v_3, v_1 v_4\}$. It is easy to check that T is a forest, $d_{G-E(T)}(v) = 0$, and $\Delta(G - E(T)) \leqslant 3$.

To prove that G is $(2, 0)$-coverable, it suffices to prove the following result (by using Nash-Williams' theorem).

(2) If G is a graph containing no $K_{3,2}$-subdivision, then G contains at most $2|V(G)| - 2$ edges.

It is easy to check that (2) holds when $|V(G)| \leqslant 4$. So assume that $|V(G)| \geqslant 5$. Then G is not a complete graph. Further, G is not 3-connected. For otherwise, there are three internally disjoint paths in G between two non-adjacent vertices, and they would form a $K_{3,2}$-subdivision in G.

So let $\{u, v\}$ be a 2-cut of G and let C be a component of $G - \{u, v\}$. We choose $\{u, v\}$ and C so that $|V(C)|$ is minimum (among all choices of 2-cuts of G). Assume for the moment that $|V(C)| = 1$. Let $V(C) = \{x\}$. Then $d_G(x) = 2$. By applying induction to $G - x$, we

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see that $|E(G-x)| \leqslant 2|V(G-x)|-2$. Thus, $|E(G)| \leqslant 2|V(G)|-2$. Hence we may assume $|V(C)| \geqslant 2$. Let S denote the set of edges of G with one endpoint in $\{u, v\}$ and one endpoint in $V(C)$, and let $C^* := C + (\{u, v, uv\} \cup S)$. By the choice of $\{u, v\}$ and C, we can prove that C^* is 3-connected. Therefore, $C^* - uv$ contains two internally disjoint paths P, Q between u and v. On the other hand, $G - V(C)$ contains a path R from u to v and containing at least three vertices. Now $P \cup Q \cup R$ gives a $K_{3,2}$-subdivision in G, a contradiction. □

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