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Non-Separating Paths in 4-Connected Graphs

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Abstract. In 1975, Lovász conjectured that for any positive integer k, there exists a minimum positive integer $f(k)$ such that, for any two vertices x, y in any $f(k)$-connected graph G, there is a path P from x to y in G such that $G - V(P)$ is k-connected. A result of Tutte implies $f(1) = 3$. Recently, $f(2) = 5$ was shown by Chen *et al.* and, independently, by Kriesell. In this paper, we show that $f(2) = 4$ except for double wheels.

Keywords: non-separating path, 4-connected graph, Lovász conjecture

1. Introduction

Throughout this paper, we consider simple graphs. A *plane graph* is a graph drawn in the plane with no pair of edges crossing. A graph is *planar* if it is isomorphic to a plane graph.

For a graph G, we use $V(G)$ and $E(G)$ to denote its *vertex set* and *edge set*, respectively. We use the shorthand notation xy (or yx) for an edge in $E(G)$ whose ends are x and y, and we say that x and y are *neighbors*. For two subgraphs G and H of a graph, we use $G \cup H$ and $G \cap H$ to denote their union and intersection, respectively. For convenience, we use $A := B$ to rename B as A or to define A as B.

Let G be a graph. Given $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and let $G - S := G[V(G) - S]$. We say that G is *k-connected* if $|V(G)| \geq$

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$k+1$ and, for any $S \subseteq V(G)$ with $|S| < k$, $G - S$ is connected. If G is connected and $x \in V(G) \cup E(G)$ for which $G - \{x\}$ is not connected, then x is called a *cut vertex* when $x \in V(G)$ and *cut edge* otherwise. For any $S \subseteq E(G)$, we let $G - S$ denote the graph with vertex set $V(G)$ and edge set $E(G) - S$. If $S = \{s\} \subseteq V(G) \cup E(G)$, we let $G - s := G - S$.

Again, let G be a graph. A subgraph H of G is an *induced* subgraph if $G[V(H)] = H$, and it is *non-separating* if $G - V(H)$ is connected. A *block* of G is a subgraph of G which is induced by a cut edge or is a maximal 2-connected subgraph. If a block is 2-connected, then we also say it is *non-trivial*.

Let P be a path between vertices u and v in a graph G; then P is called a u-v path, and u and v are called the *ends* of P. Given vertices x, y on P, we let $P[x, y]$ denote the path in P with ends x and y, and we define $P(x, y) = P[x, y] - \{x, y\}$. Two paths in a graph are said to be *internally disjoint* if no internal vertex of one path occurs in the other.

In 1975, Lovász [7] made the following.

Conjecture. 1.1. *For any positive integer k, there exists a minimum positive integer $f(k)$ such that, for any two vertices x, y in any $f(k)$-connected graph G, there is an x-y path P in G such that $G - V(P)$ is k-connected.*

It is not difficult to see $f(1) \leq 3$ using a theorem of Tutte [13] on non-separating cycles in 3-connected graphs. Also, $K_{2,3}$ shows that $f(1) \geq 3$. Hence $f(1) = 3$. Recently, $f(2) = 5$ was shown by Chen, Gould and Yu [1], and independently by Kriesell [6]. As far as we know, Conjecture 1.1 is open for $k \geq 3$.

An example for $f(2) = 5$ is the *double wheel*, which is the graph obtained from the union of a cycle C with two vertices u, v by adding all possible edges from $\{u, v\}$ to $V(C)$. The set $\{u, v\}$ is called the *center* of the double wheel and C is called the *ring* of the double wheel. Figure 1 shows an example of a double wheel. Note that a double wheel may have a representation with different centers and rings (for example, the square of the cycle of length six).

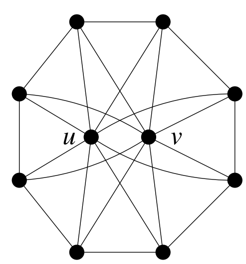

Figure 1: A double wheel with ring $\{u, v\}$.

Our main result says that $f(2) = 4$ except for double wheels.

Theorem 1.2. *Let G be a 4-connected graph and let $u, v \in V(G)$ be distinct vertices. Then exactly one of the following holds*:

(a) *There is a u-v path P in G such that $G - V(P)$ is 2-connected.*

(b) *G is a double wheel with center $\{u, v\}$.*

In [3], Curran and Yu proved that if G is 5-connected then G has an induced u-v path P such that $G - V(P)$ is 2-connected. If we require "induced" in Theorem 1.2, then the situation is different, as demonstrated by the "squares" of even cycles. For a graph H, the *square* of H is the graph obtained from H by adding all edges between vertices within distance two in H. See Figure 2 for an example. It would be interesting to obtain an "induced" version of Theorem 1.2.

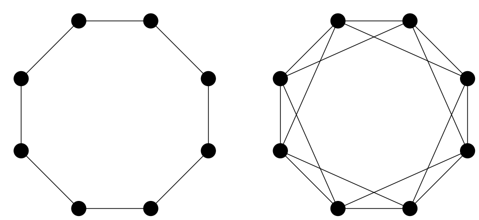

Figure 2: A cycle and its square.

The rest of the paper is organized into two sections. In Section 2, we will use contractible edges and contractible triangles to show the existence of certain non-separating paths. In the final section, we will use the result in Section 2 to complete the proof of 1.2.

2. Induced Paths

In this section we prove the existence of certain non-separating cycles in 4-connected graphs. Our approach is to find contractible subgraphs and apply induction. A connected subgraph H of a 4-connected G is *contractible* if the graph obtained from G by contracting H remains 4-connected. A *contractible edge* is a an edge whose induced graph is contractible. A 4-connected graph needs not to contain a contractible edge, for example, the square of a cycle of length at least 5.

Fontet [4] and Martinov [8] characterized those 4-connected graphs containing no contractible edges, which includes the line graphs of cyclically 4-edge-connected cubic graphs. To avoid the difficulty of dealing with such line graphs, we will use the following result proved by Kawarabayashi [5, Theorem 9].

Lemma 2.1. *Let G be a 4-connected graph with $|V(G)| \geq 7$ and assume that G is not the square of a cycle. Then G has a contractible edge or a contractible triangle.*

The following observation will be convenient.

Proposition 2.2. *Let G be a 4-connected graph with $|V(G)| = 6$. Then G contains the square of a cycle as a spanning subgraph.*

Proof. This is obvious if G is a complete graph. So assume G is not complete, and let x, y be two non-adjacent vertices of G. Since G is 4-connected, G contains four internally disjoint x-y paths. Since $|V(G)| = 6$, these four paths all have length 2. So let

xu_iy, $1 \le i \le 4$, denote these four paths. Again, since G is 4-connected, each u_i has at least two neighbors in $\{u_1, u_2, u_3, u_4\}$. So by a simple case checking, we can see that $G[\{u_1, u_2, u_3, u_4\}]$ contains a cycle of length 4. Hence, G contains the square of a cycle as a spanning subgraph. ∎

A result of Tutte [13] implies that for any 3-connected graph G and any distinct $u, v \in V(G)$, G contains a non-separating induced u-v path. Next we use 2.1 to prove a similar result for 4-connected graphs.

Theorem 2.3. *Let G be a 4-connected graph and let $u, v \in V(G)$ be distinct. Then exactly one of the following holds*:

(a) *G is a double wheel with center $\{u, v\}$.*
(b) *There is a non-separating induced u-v path P in G such that $G - V(P)$ has a non-trivial block.*

Proof. We will prove 2.3 by applying induction on $|V(G)|$. If $uv \in E(G)$, then since G is 4-connected, $G - \{u, v\}$ is 2-connected, and so, $P := G[\{u, v\}]$ gives the desired path for (b). So we may assume that

(1) $uv \notin E(G)$.

Suppose G is a double wheel. If $\{u, v\}$ is the center of the double wheel, then (a) holds. So assume that for any representation of G as a double wheel, $\{u, v\}$ is not the center. Therefore, it follows from (1) that both u and v lie on the ring of the double wheel. Moreover, $|V(G)| \ge 7$; for otherwise, G can be represented as a double wheel with center $\{u, v\}$. Let P denote a shortest path on the ring between u and v. Then we see that $G - V(P)$ is 2-connected. Hence, P gives the desired path for (b), and so, we may assume that

(2) G is not a double wheel.

Now suppose G is the square of a cycle C. Then by (2), $|V(G)| \ne 6$. Since G is 4-connected and by (1), $|V(G)| \ge 6$. Hence $|V(G)| \ge 7$. So let P be a shortest path on C between u and v. Then clearly, $G - V(P)$ is connected, and $G - V(P)$ contains a triangle. So P gives the desired path for (b). Hence, we may assume

(3) G is not the square of a cycle.

Suppose $|V(G)| = 6$. Then, since G is 4-connected and by (1), every vertex in $V(G) - \{u, v\}$ is adjacent to both u and v. By (3), $G - \{u, v\}$ must contain a triangle. Hence there is a u-v path P in G such that $G - V(P)$ is a triangle. Thus, P gives the desired path for (b). So we may assume

(4) $|V(G)| \ge 7$.

By (3) and (4), we may apply 2.1 to G and find a contractible edge or a contractible triangle in G. If G has a contractible edge then let R denote the set of vertices incident with that edge; and otherwise, let R be the vertex set of a contractible triangle in G. Let G' denote the graph resulted from G by contracting $G[R]$, let r denote the vertex of G' obtained from the contraction of $G[R]$, and for any $a \in V(G)$, let $a' = r$ if $a \in R$ and $a' = a$ if $a \notin R$. By (1), we have

(5) $\{u, v\} \not\subseteq R$.

Next, we distinguish two cases.

Case 1. G' cannot be represented as a double wheel with center $\{u', v'\}$.

Then by induction, G' has a non-separating induced u'-v' path P' such that $G' - V(P')$ has a non-trivial block.

If $r \notin V(P')$, then it is easy to see that $P := P'$ gives the desired path for (b). So assume $r \in V(P')$.

First, suppose $|R| = 3$. Let $R = \{x, y, z\}$, and let P denote a shortest induced u-v path in $G[(V(P') - \{r\}) \cup R]$. Then P contains at least one of $\{x, y, z\}$ and misses at least one of $\{x, y, z\}$. If P misses two of $\{x, y, z\}$, say y and z, then one of $\{y, z\}$ has a neighbor in $V(G') - V(P')$ (since G is 4-connected and P' is induced in G'), and so, P gives the desired path for (b). So assume that P misses exactly one of $\{x, y, z\}$, say z. Then it is easy to see that z has a neighbor in $V(G') - V(P')$ (since G is 4-connected and P was chosen to be shortest). Hence, P is the desired path for (b).

Now suppose $|R| = 2$. Let $R = \{x, y\}$ and let P be a shortest induced u-v path in $G[(V(P') - \{r\}) \cup R]$. If P contains both x and y, then $P := P'$ gives the desired path for (b). If P misses one of $\{x, y\}$, say y, then, since G is 4-connected and P' is induced in G', y has a neighbor in $V(G') - V(P')$. Hence P is the desired path for (b).

Case 2. G' is a double wheel with center $\{u', v'\}$.

Let C' denote the ring of G'. If $r \notin \{u', v'\}$, then $r \in V(C')$. Since G is not a double wheel with center $\{u, v\}$, $G - \{u, v\}$ contains a triangle T. So let P denote a shortest u-v path in $G - V(T)$ (P has length 2). We see that $G - V(P)$ is connected and has a non-trivial block.

Now assume that $r \in \{u', v'\}$. By symmetry, assume that $r = u'$.

Suppose $|R| = 3$, and let $R = \{x, y, z\}$ with $x = u$. Since G is 4-connected, there are distinct vertices u', y', z' on C' such that $uu', yy', zz' \in E(G)$. Let $P := uu'v$. Then $G - V(P)$ is connected, and the y'-z' path of $C' - u'$ forms a cycle with $y'yzz'$. So $G - V(P)$ has a non-trivial block, and P gives the desired path for (b).

Now assume $|R| = 2$ and let $R = \{x, y\}$ with $x = u$. Since G is 4-connected, there are distinct vertices u', y', y'' on C' such that $uu', yy', yy'' \in E(G)$. Let $P := uu'v$. Then $G - V(P)$ is connected, and the y'-y'' path of $C' - u'$ forms a cycle with $y'yy''$. So $G - V(P)$ has a non-trivial block, and P gives the desired path for (b). ∎

3. Non-Separating Paths

We begin with a result of Cheriyan and Maheshwari [2] which finds a second non-separating induced cycle in a 3-connected graph.

Lemma 3.1. *Let G be a 3-connected graph, let $uv \in E(G)$, and let D be a non-separating induced cycle in G through uv. Then G has a non-separating induced cycle C through e such that $V(C) \cap V(D) = \{u, v\}$.*

For notational convenience, we introduce the following definition. Let G be a graph with distinct vertices a, b, c, and d. We say that the ordered quintuple (G, a, b, c, d) is *planar* if G can be drawn in a closed disc in the plane with no pair of edges crossing such that a, b, c, d occur on the boundary of the disc in cyclic order. The following is proved in [3], which is an easy consequence of a result of Seymour [9] and Thomassen [10]

Lemma 3.2. *Let a, b, c, d be distinct vertices of a graph G. Suppose that for any $T \subseteq V(G)$ with $|T| \leq 3$, every component of $G - T$ contains at least one element of $\{a, b, c, d\}$. Then exactly one of the following is true*:

(a) *There are vertex disjoint paths joining a to b and c to d, respectively.*
(b) *(G, a, c, b, d) is planar.*

The following result from [3] is an easy consequence of a theorem of Thomassen [12].

Lemma 3.3. *Let (G, a, c, b, d) be planar and let $G - \{c, d\}$ contain an a-b path. Assume that, for any $T \subseteq V(G)$ with $|T| \leq 3$, every component of $G - T$ contains an element of $\{a, c, b, d\}$. Then $G - \{c, d\}$ contains a Hamiltonian a-b path.*

We also need the following lemma to prove our main result of this section.

Lemma 3.4. *Let G be a graph and $\{a, a', b, b'\} \subseteq V(G)$. Suppose*

(a) *for any $T \subseteq V(G)$ with $|T| \leq 2$, every component of $G - T$ contains an element of $\{a, a', b, a'\}$, and*
(b) *G contains disjoint paths from a, b to a', b', respectively.*

Then there exists a non-separating induced a-a′ path P in G such that $V(P) \cap \{b, b'\} = \emptyset$.

Proof. Suppose 3.4 is not true. Let $\mathcal{P}$ denote the set of all induced a-a' paths P in G for which $G - V(P)$ is not connected and $\{b, b'\}$ is contained in a component of $G - V(P)$. By (b), $\mathcal{P} \neq \emptyset$. For each $P \in \mathcal{P}$, let U_P denote the component of $G - V(P)$ containing $\{b, b'\}$, and let W_P denote a component of $G - V(P)$ such that $W_P \neq U_P$ and $|V(W_P)|$ is minimum.

We choose $P \in \mathcal{P}$ so that $|V(W_P)|$ is minimum. Let $x, y \in V(P)$ be the neighbors of W_P such that $P[x, y]$ is maximal. By (a) and since P is induced in G, $P(x, y)$ contains a neighbor of some component of $G - V(P)$. Let R denote an induced x-y path in $G[V(W_P) \cup \{x, y\}]$, and let $Q := (P - V(P(x, y))) \cup R$. Then Q is an induced a-a' path in G, and B_P is contained in a component of $G - V(Q)$. Hence $Q \in \mathcal{P}$. It is easy to see that W_Q is properly contained in W_P, contradicting the choice of P. ∎

Before we prove our next result, we introduce the concept of a bridge. Let G be a graph, and $S \subseteq V(G)$. An *S-bridge* of G is a subgraph of G which is either induced by an edge of G with both ends in S or is induced by the edges in a component of $G - S$ and all edges from that component to S.

Theorem 3.5. *Let G be a 4-connected graph and let $a, b \in V(G)$ be distinct. Suppose G contains a non-separating induced a-b path P such that $G - V(P)$ contains a non-trivial block. Then G has an a-b path Q such that $G - V(Q)$ is 2-connected.*

Proof. Let $\mathcal{P}$ denote the set of those non-separating induced a-b paths P in G for which $G - V(P)$ contains a non-trivial block. By hypothesis, $\mathcal{P} \neq \emptyset$. For any $P \in \mathcal{P}$, let B_P denote a non-trivial block of $G - V(P)$ with maximum number of vertices. Choose $P \in \mathcal{P}$ so that

(1) $|V(B_P)|$ is maximum.

For convenience, let $H := G - V(P)$. If H is 2-connected, then $Q := P$ is the desired path. So assume that H is not 2-connected. Let $X := \{v_1, v_2, \ldots, v_n\}$ be the set of cut vertices of H which are contained in B_P. Let $B_i^1, B_i^2, \ldots, B_i^{n_i}$ denote the v_i-bridges of H which do not contain B_P. Then $n_i \geq 1$, because v_i is a cut vertex of H. Let $\mathcal{B} := \{B_i^j : 1 \leq i \leq n, 1 \leq j \leq n_i\}$. See Figure 3 for an illustration.

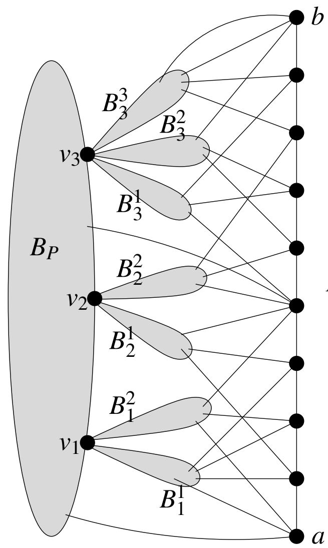

Figure 3: Example of a graph G, a path P and the v_i-bridges of $G - V(P)$.

Because G is 4-connected, $B_i^j - v_i$ has at least three neighbors on P. Let a_i^j, b_i^j be the neighbors of $B_i^j - v_i$ on P such that $P[a_i^j, b_i^j]$ is maximal and a, a_i^j, b_i^j, b occur on P in order. Next, we prove the following claim.

(2) For each $B_i^j \in \mathcal{B}$, there is some $u_i^j \in V(B_P) - \{v_i\}$ such that all paths in G from $P(a_i^j, b_i^j)$ to B_P internally disjoint from $B_p \cup P \cup B_i^j$ end at u_i^j.

Since G is 4-connected and P is induced in G, there must be a path from $P(a_i^j, b_i^j)$ to $B_P - v_i$ internally disjoint from $B_P \cup P \cup B_i^j$. Suppose (2) does not hold. Then there are paths P_1, P_2 from $r_1, r_2 \in V(P(a_i^j, b_i^j))$ to $s_1, s_2 \in V(B_P)$, respectively, such that $s_1 \neq s_2$ and P_1 and P_2 are internally disjoint from $B_p \cup P \cup B_i^j$. We will show that there is a path $R \in \mathcal{P}$ contradicting the choice of P.

Let $v_i^1, \ldots, v_i^k$ be those neighbors of $B_i^j - v_i$ on P, occurring in order on P with $v_i^1 = a_i^j$ and $v_i^k = b_i^j$. Let A_i^j denote the graph obtained from $G[V(B_i^j \cup P[a_i^j, b_i^j]) - \{v_i\}]$ by (for all $1 \leq s \leq k-1$) deleting $P(v_i^s, v_i^{s+1})$ and adding edges $v_i^1 v_i^k$ and $v_i^s v_i^{s+1}$. Let D_i^j denote the cycle $v_i^1 \cdots v_i^k v_i^1$ in A_i^j. Clearly, D_i^j is a non-separating induced cycle in A_i^j.

We claim that A_i^j is 3-connected. For otherwise, let $T \subseteq V(A_i^j)$ with $|T| \leq 2$ such that $A_i^j - T$ is not connected. Note that $T \nsubseteq V(D_i^j)$ because $B_i^j - v_i$ is connected and every vertex of $D_i^j - T$ has a neighbor in $V(B_i^j - v_i)$. Hence, $D_i^j - T$ is contained in a single component of $A_i^j - T$. Therefore, there is a component D of $A_i^j - T$ such that $D \subseteq B_i^j - v_i$. Then D is also a component of $G - (T \cup \{v_i\})$. But $|T \cup \{v_i\}| \leq 3$, contradicting the assumption that G is 4-connected.

By applying 3.1 to $A_i^j, D_i^j, v_i^1 v_i^k$, we find a non-separating induced cycle C in A_i^j such that $v_i^1 v_i^k \in E(C)$ and $V(C) \cap V(D_i^j) = \{v_i^1, v_i^k\}$. Now let $R_i^j := C - v_i^1 v_i^k$. Then $R_i^j - \{v_i^1, v_i^k\} \subseteq B_i^j - v_i$. Let $R := (P - V(P(a_i^j, b_i^j))) \cup R_i^j$. It is easy to see that R is an induced path in G and $B_P \cup P(a_i^j, b_i^j) \cup P_1 \cup P_2 \subseteq G - V(R)$. Since $A_i^j - V(C)$ is connected and $V(C) \cap V(D_i^j) = \{v_i^1, v_i^k\}$, we see that $G - V(R)$ is connected. So $R \in \mathcal{P}$. Since $B_P \cup P(a_i^j, b_i^j) \cup P_1 \cup P_2 \subseteq G - V(R)$, we see that $B_P \subseteq B_R$ and $B_P \neq B_R$. This contradicts (1), completing the proof of (2).

Next we show that all B_i^j's are associated with 4-cuts of G. For ease of presentation, we define a new graph $\mathcal{G}$ whose vertices are B_i^j's, and B_i^j is adjacent to B_k^l in $\mathcal{G}$ if $E(P[a_i^j, b_i^j]) \cap E(P[a_k^l, b_k^l]) \neq \emptyset$. Let $\mathcal{G}_i^j$ denote the component of $\mathcal{G}$ containing B_i^j. Let c_i^j, d_i^j be the vertices on P such that c_i^j and d_i^j are neighbors in G of members of $\mathcal{G}_i^j$ and, subject to this, $P[c_i^j, d_i^j]$ is maximal. See Figure 4 for an illustration of these definitions, using the graph in Figure 3. Observe that

(3) all $P[c_i^j, d_i^j]$ are edge disjoint.

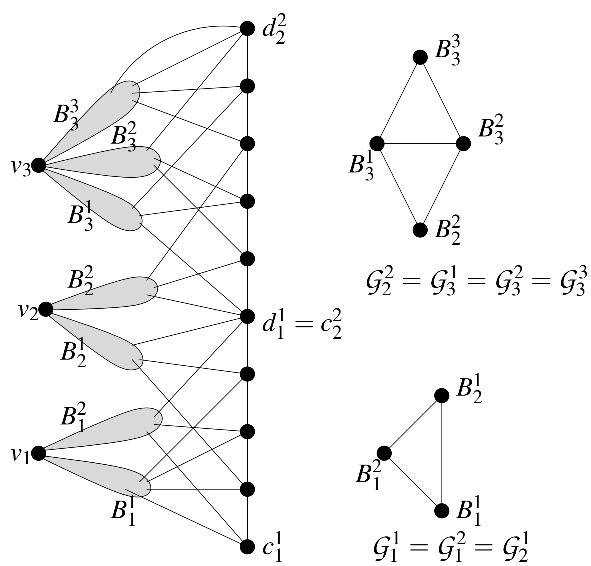

Figure 4: Components of the graph $\mathcal{G}$.

Let G_i^j denote the union of $P[c_i^j, d_i^j]$, those B_i^j's with a neighbor in $P(c_i^j, d_i^j)$, and those edges of G from B_P to $P(c_i^j, d_i^j)$. We claim that

(4) $\{u_i^j, v_i, c_i^j, d_i^j\}$ is a 4-cut of G, and G_i^j is a $\{u_i^j, v_i, c_i^j, d_i^j\}$-bridge of G.

It suffices to show that, for each $B_k^l \in V(\mathcal{G}_i^j)$ with $B_k^l \neq B_i^j$, we have $\{v_k, u_k^l\} = \{v_i, u_i^j\}$. Since $\mathcal{G}_i^j$ is connected, we only need to show $\{v_k, u_k^l\} = \{v_i, u_i^j\}$ for those B_k^l which are adjacent to B_i^j in $\mathcal{G}_i^j$. Since $B_k^l - v_k$ and $B_i^j - v_i$ each has at least three neighbors on P, we see that $P(a_i^j, b_i^j)$ contains a neighbor of $B_k^l - v_k$ or $P(a_k^l, b_k^l)$ contains a neighbor of $B_i^j - v_i$. By symmetry, we may assume that $P(a_i^j, b_i^j)$ contains a neighbor of $B_k^l - v_k$. Then by (2), $u_i^j = v_k$ and $v_k \neq v_i$. If $P(a_k^l, b_k^l)$ contains a neighbor of $B_i^j - v_i$, then it follows from (2) that $u_k^l = v_i$, and hence $\{v_k, u_k^l\} = \{v_i, u_i^j\}$. So we may assume that $P(a_k^l, b_k^l)$ contains no neighbor of $B_i^j - v_i$. Then $P(a_k^l, b_k^l) \subseteq P(a_i^j, b_i^j)$. Hence by (2), $u_k^l = u_i^j$ and $u_k^l \neq v_k$. This contradicts the earlier conclusion that $u_i^j = v_k$.

Hence, $\{u_i^j, v_i, c_i^j, d_i^j\}$ is a 4-cut of G. It is easy to see that G_i^j is a $\{u_i^j, v_i, c_i^j, d_i^j\}$-bridge of G.

We further claim that

(5) $(G_i^j, c_i^j, u_i^j, d_i^j, v_i)$ is planar.

Now suppose that $(G_i^j, c_i^j, u_i^j, d_i^j, v_i)$ is not planar. Then by 3.2, G_i^j contains disjoint paths from c_i^j to d_i^j and from u_i^j to v_i. So by 3.4, we can find a non-separating induced c_i^j-d_i^j path R_i^j in $G_i^j - \{u_i^j, v_i\}$. Now let $R := (P - V(P(c_i^j, d_i^j))) \cup R_i^j$. Then R is a non-separating induced a-b path in G. Hence $R \in \mathcal{P}$. But B_R contains B_P and a u_i^j-v_i path in $G_i^j - V(R_i^j)$, contradicting (1).

By (5), we may apply 3.3 to $(G_i^j, c_i^j, u_i^j, d_i^j, v_i)$ and find a Hamiltonian c_i^j-d_i^j path Q_i^j in $G_i^j - \{u_i^j, v_i\}$. Let $Q := (P - \bigcup V(P(c_i^j, d_i^j))) \cup (\bigcup Q_i^j)$. Then Q is an a-b path, and $G - V(Q) = B_P$ is 2-connected. ∎

It is now easy to see that our main result 1.2 follows from 2.3 and 3.5.

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