Smallest number of vertices in a 2-arc-strong digraph without good pairs

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March 19, 2022

Abstract

Branchings play an important role in digraph theory and algorithms. In particular, a chapter in the monograph of Bang-Jensen and Gutin, Digraphs: Theory, Algorithms and Application, Ed. 2, 2009 is wholly devoted to branchings. The well-known Edmonds Branching Theorem provides a characterization for the existence of k arc-disjoint out-branchings rooted at the same vertex. A short proof of the theorem by Lovász (1976) leads to a polynomial-time algorithm for finding such out-branchings. A natural related problem is to characterize digraphs having an out-branching and an in-branching which are arc-disjoint. Such a pair of branchings is called a good pair.

Bang-Jensen, Bessy, Havet and Yeo (2020) pointed out that it is NP-complete to decide if a given digraph has a good pair. They also showed that every digraph of independence number at most 2 and arc-connectivity at least 2 has a good pair, which settled a conjecture of Thomassen for digraphs of independence number 2. Then they asked for the smallest number n_{ngp} of vertices in a 2-arc-strong digraph which has no good pair. They proved that $7 \leq n_{ngp} \leq 10$. In this paper, we prove that $n_{ngp} = 10$, which solves the open problem.

Keywords: Arc-disjoint branchings; out-branching; in-branching; arc-connectivity

1 Introduction

Let $D = (V, A)$ be a digraph. For a non-empty subset $X \subset V$, the *in-degree* (resp. *out-degree*) of the set X, denoted by $d_D^-(X)$ (resp. $d_D^+(X)$), is the number of arcs with head (resp. tail) in X and tail (resp. head) in $V \setminus X$. The *arc-connectivity* of D, denoted by $\lambda(D)$, is the minimum out-degree of a proper subset of vertices. A digraph is *k-arc-strongly connected* (or, just *k-arc-strong*) if $\lambda(D) \geq k$. In particular, a digraph is *strongly connected* (or, just *strong*) if $\lambda(D) \geq 1$.

An *out-branching* (*in-branching*) of a digraph $D = (V, A)$ is a spanning tree in the underlying graph of D whose edges are oriented in D such that only one vertex, called the *root*, has in-degree (out-degree) zero and others have one. Branchings play an important role in digraph theory and algorithms. In particular, Chapter 9 in the monograph [5] is wholly devoted to branchings. The well-known Edmonds Branching Theorem (see e.g. [5]) provides a characterization for the existence of k arc-disjoint out-branchings rooted at the same vertex. A short proof of the theorem by Lovász [11] leads to a polynomial-time algorithm for finding such out-branchings. A natural related problem is to characterize digraphs having an out-branching and an in-branching which are arc-disjoint. Such a pair of branchings is called a *good pair*.

Thomassen [12] conjectured the following:

Conjecture 1. *There is a constant c, such that every digraph with arc-connectivity at least c has an out-branching and an in-branching which are arc-disjoint.*

He also proved that it is NP-complete to decide whether a given digraph D has an out-branching and an in-branching both rooted at the same vertex such that these are arc-disjoint. This implies that it is NP-complete to decide if a given digraph has a good pair [2]. Conjecture 1 has been verified for semicomplete digraphs [1] and their genearlizations: locally semicomplete digraphs [7] and semicomplete compositions [6] (it follows from the main result in [6]).

An out-branching and an in-branching of D are *k-distinct* if each of them has at least k arcs, which are absent in the other. Bang-Jensen et al. [8] proved that the problem of deciding whether a strongly connected digraph D has k-distinct out-branching and in-branching is fixed-parameter tractable when parameterized by k. Settling an open problem in [8], Gutin et al. [10] extended this result to arbitrary digraphs.

In [2], Bang-Jensen et al. showed that every digraph of independence number at most 2 and arc-connectivity at least 2 has a good pair, which settles the conjecture for digraphs of independence number 2.

Theorem 2. *If D is a digraph with $\alpha(D) \leq 2 \leq \lambda(D)$, then D has a good pair.*

Moreover, they also proved that every digraph on at most 6 vertices and arc-connectivity at least 2 has a good pair and gave an example of a 2-arc-strong digraph D on 10 vertices with independence number 4 that has no good pair. Then they asked for the smallest number n_{ngp} of vertices in a 2-arc-strong digraph which has no good pair and proved that $7 \leq n_{ngp} \leq 10$.

The following is the first problem in Section 8 of [2].

Problem 3. *What is the smallest number n of vertices in a 2-arc-strong digraph which has no good pair?*

In this paper, we prove that every digraph on at most 9 vertices and arc-connectivity at least 2 has a good pair, which answers this problem. The main results of the paper are shown below.

Theorem 4. *Every 2-arc-strong digraph on 7 vertices has a good pair.*

Theorem 5. *Every 2-arc-strong digraph on 8 vertices has a good pair.*

Theorem 6. *Every 2-arc-strong digraph on 9 vertices has a good pair.*

This paper is organised as follows. In the rest of this section, we provide further terminology and notation on digraphs, and undefined terms can be found in [4, 5]. In next section, we outline the proofs of Theorem 4, 5 and 6 and state some auxiliary lemmas which we use in their proofs. Section 3 contains a number of technical lemmas which will be used in proofs of our main results. Then we respectively devote one section for proofs of each theorem and its relevant auxiliary lemmas. Some proofs will not be given in this paper, but interested readers may refer to reference [9].

Additional Terminology and Notation. For a positive integer n, $[n]$ denotes the set $\{1, 2, \ldots, n\}$. Throughout this paper, we will only consider digraphs without loops and multiple arcs. Let $D = (V, A)$ be a digraph. We denote by uv the arc whose *tail* is u and whose *head* is v. Two vertices u, v are *adjacent* if at least one of uv and vu belongs to A. If u and v are adjacent, then we also say that u is a *neighbour* of v and vice versa. If $uv \in A$, then v is called an *out-neighbour* of u and u is called an *in-neighbour* of v. Moreover, we say uv is an *out-arc* of u and an *in-arc* of v and that u *dominates* v. The *order* $|D|$ of D is $|V|$.

In this paper, we will extensively use *digraph duality*, which is as follows. Let D be a digraph and let $D^{\rm rev}$ be the *reverse* of D, i.e., the digraph obtained from D by reversing every arc xy to yx. Clearly, D contains a subdigraph H if and only if $D^{\rm rev}$ contains $H^{\rm rev}$. In particular, D contains a good pair if and only if $D^{\rm rev}$ contains a good pair.

Let $N_D^-(X) = \{y : yx \in A, x \in A\}$ and $N_D^+(X) = \{y : xy \in A, x \in A\}$. Note that X may be just a vertex. For two non-empty disjoint subsets $X, Y \subset V$, we use $N_Y^-(X)$ to denote $N_D^-(X) \cap Y$ and $d_Y^-(X) = |N_Y^-(X)|$. Analogously, we can define $N_Y^+(X)$ and $d_Y^+(X)$. For two non-empty subsets $X_1, X_2 \subset V$, define $(X_1, X_2)_D = \{(v_1, v_2) \in A\colon v_1 \in X_1 \text{ and } v_2 \in X_2\}$ and $[X_1, X_2]_D = (X_1, X_2)_D \cup (X_2, X_1)_D$. We will drop the subscript when the digraph is clear from the context.

We write $D[X]$ to denote the subdigraph of D induced by X. A *clique* in D is an induced subdigraph $D[X]$ such that any two vertices of X are adjacent. We say that D contains K_p if it has a clique on p vertices. A vertex set X of D is *independent* if no pair of vertices in X are adjacent. A dipath (dicycle) of D with t vertices is denoted by P_t (C_t). We drop the subscript when the order is not specified. A dipath P from v_1 to v_2, denoted by $P_{(v_1,v_2)}$, is often called a (v_1, v_2)-*dipath*. A dipath P is a *Hamilton* dipath if $V(P) = V(D)$. We call C_2 a *digon*. A digraph without digons is called an *oriented graph*. If two digons have exactly one common vertex, then we call this structure a *bidigon*. A *semicomplete* digraph is a digraph D that each pair of vertices has an arc between them. A *tournament* is a semicomplete oriented graph.

In- and out-branchings were defined above. An *out-tree* (*in-tree*) is an out-branching (in-branching) of a subdigraph of D. We use B_s^+ (B_t^-) to denote an out-branching rooted at s (an in-branching rooted at t). The root s (t) is called *out-generator* (*in-generator*) of D. We denote by $\mathrm{Out}(D)$ ($\mathrm{In}(D)$) the set of out-generators (in-denerators) of D. If the root is not specified, then we drop the subscripts of B_s^+ and B_t^-. We also use O_D (I_D) to denote an out-branching (in-branching) of a digraph D. If O_D and I_D are arc-disjoint, then we write (O_D, I_D) to denote a good pair in D.

2 Proofs Outline

In this section, we outline constructions for proof of our main results and give statements of some auxiliary lemmas. Since some of their proofs are too complicated, we give only a simple version in this paper and a more specific one can be found in [9] for interested readers. We prove each of our main results by contradiction, and assume that $|D_1| = 7$, $|D_2| = 8$ and $|D_3| = 9$ for simplicity in this section.

2.1 Theorem 4

First we show that the largest clique in D_1 is a tournament by Lemma 17, and next we prove that D_1 is an oriented graph in Claim 4.1 by Lemma 18. We give these two lemmas in next section. Then we use Proposition 25 to show that D_1 has a Hamilton dipath in Section 4.

Proposition 25 A 2-arc-strong oriented graph D on n vertices has a P_7, where $7 \leq n \leq 9$.

After that, we prove that D_1 has a good pair by Propositon 23, which is shown in Section 3.

2.2 Theorem 5

Our proof will follow three steps.

Firstly, we prove that the largest clique R in D_2 has 3 vertices by Lemma 17. Then we show that R is a tournament through Claim 5.1, which is proved by Lemmas 17 and 18.

Our second step is to prove that D_2 is an oriented graph in Claim 5.2 by Lemmas 19, 20 and 21, which are given in next section.

In the last step, we proceed as follows in Section 5. We use Proposition 28 to show that D_2 has a Hamilton dipath.

Proposition 28 Let D be a 2-arc-strong digraph on n vertices without a good pair, where $8 \leq n \leq 9$. If D is an oriented graph without K_4 as a subdigraph, then D has a P_8.

To prove it, we show Proposition 27 first.

Proposition 27 Let D be a 2-arc-strong oriented graph on n vertices without K_4 as a subdigraph, where $8 \leq n \leq 9$. If D has two disjoint cycles C^1 and C^2 which cover 7 vertices, then D contains a P_8.

After that, we prove that D_2 has a good pair by Propositon 23.

2.3 Theorem 6

Our proof will follow four steps.

Firstly, we show that the largest clique R in D_3 has 3 vertices by Claim 6.1, which is proved using Proposition 7 given in next section, and Lemmas 17 and 18.

Next we show that R has no digons by Claim 6.2, which is proved analogously to Claim 5.1 using Lemmas 18, 19, 20 and 21.

Our third step is to show that D_3 is an oriented graph in Claim 6.3. To do this we need Lemmas 30 and 32 given in Section 6.

Lemma 30 Let D be a 2-arc-strong digraph on 9 vertices that contains a digon Q. Assume that D has no subdigraph with a good pair on 3 or 4 vertices. Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$. If $|X| = 3$ and $|Y| = 2$, then D has a good pair.

Lemma 32 Let D be a 2-arc-strong digraph on 9 vertices that contains a digon Q. Assume that D has no subdigraph with a good pair on 3 or 4 vertices. Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$. If $|X| = 2$ and $|Y| = 2$, then D has a good pair.

For the first lemma, we give a generalization of Proposition 8 as Proposition 29, and for the second one, we prove Lemma 31 first.

Lemma 31 Let $D = (V, A)$ be a 2-arc-strong digraph on 9 vertices that contains a digon Q. Assume that D has no subdigraph with a good pair on at least 3 vertices. Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$ and $W = V - V(Q) - X - Y$. Assume that $|X| = |Y| = 2$ and there is an arc $e = st \in A$ such that $s \in Y$ and $t \in W$ (resp. $s \in W$ and $t \in X$). If there are at least three arcs in $D[Y \cup \{t\}]$ (resp. $D[X \cup \{s\}]$), then D has a good pair.

Then we use Lemma 34 to show that D_3 has a Hamilton dipath in Section 6.

Lemma 34 Let D be a 2-arc-strong digraph on 9 vertices without good pair. If D is an oriented graph without K_4 as a subdigraph, then D has a Hamilton dipath.

To prove it, we show Proposition 33 first.

Proposition 33 Let D be a 2-arc-strong oriented graph on 9 vertices without K_4 as a subdigraph. If D has two cycles C^1 and C^2 with $C^1 \cap C^2 = \emptyset$ which cover 8 vertices, then D contains a Hamilton dipath.

After that, we prove that D_3 has a good pair by Proposition 23.

3 Preliminaries and useful lemmas

Proposition 7. *Let D be a digraph with $\lambda(D) \geq 2$ and with a good pair (B_s^+, B_s^-). If there exists a vertex t in D such that $D[\{s, t\}]$ is a digon, then D has a good pair (B_t^+, B_t^-).*

Proof. Let $B_t^+ = ts + B_s^+ - e_1$ and $B_t^- = B_s^- + st - e_2$, where e_1 (e_2) is the only in-arc (out-arc) of t in B_s^+ (B_s^-). Observe that B_t^+ (B_t^-) is an out-branching (in-branching) rooted at t in D. Since the root of any out-branching has in-degree zero, if $ts \in B_s^+ \cup B_s^-$, then ts must be in B_s^- and moreover ts is the only out-arc e_2 of t in B_s^-. Similarly, if $st \in B_s^+ \cup B_s^-$, then st must be in B_s^+ and moreover st is the only in-arc e_1 of t in B_s^+. Thus, B_t^+ and B_t^- are arc-disjoint and so (B_t^+, B_t^-) is a good pair of D. □

Proposition 8. *Let D be a digraph with a subdigraph Q that has a good pair (O_Q, I_Q). Let $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$ and $X \cup Y = V - V(Q)$. Let X_i (Y_j) be the initial (terminal) strong components in $D[X]$ ($D[Y]$), $i \in [a]$ ($j \in [b]$). If one of the following holds, then D has a good pair. Meanwhile, we can always get two arc-disjoint $\mathcal{P}_X, \mathcal{P}_Y$ and respectively an out- and an in-forest T_X and T_Y in D.*

1. $d_Y^-(X_1) \geq 1$, $d_Y^-(X_i) \geq 2$, $i \in \{2, \dots, a\}$ *and* $d_X^+(Y_j) \geq 2$, $j \in [b]$.
2. $d_X^+(Y_1) \geq 1$, $d_X^+(Y_j) \geq 2$, $j \in \{2, \dots, b\}$ *and* $d_Y^-(X_i) \geq 2$, $i \in [a]$.

Proof. Let B^+ be an out-tree containing O_Q and an in-arc of any vertex in Y from Q. Let B^- be an in-tree containing I_Q and an out-arc of any vertex in X to Q. Set $\mathcal{X} = \{X_i,\ i \in [a]\}$ and $\mathcal{Y} = \{Y_j,\ j \in [b]\}$. By the digraph duality, it suffices to prove that condition 1 implies that D has a good pair.

Now assume that $d_Y^-(X_1) \geq 1$, $d_Y^-(X_i) \geq 2$, $i \in \{2, \dots, a\}$, and $d_X^+(Y_j) \geq 2$, $j \in [b]$. Then there are at least two arcs from Y_j (for each $j \in [b]$) to X, at least two arcs from Y to X_i (for each $i \in \{2, \dots, a\}$) and at least one arc from Y to X_1. Set $X_1' = X_1$. If there is an arc y^1x_1 from Y to X_1' with y^1 in some Y_j, $j \in [b]$, then we choose such an arc and let $Y_1' = Y_j$, otherwise we choose an arbitrary arc y^1x_1 from Y to X_1' and let Y_1' be an arbitrary strong component in $\mathcal{Y}$. Let $\mathcal{P}_X = \{y^1x_1\}$. There now exists an arc, y_1x^1, out of Y_1' ($x^1 \in X$) which is different from y^1x_1 (as Y_1' has at least two arcs out of it). If there is such an arc y_1x^1 with x^1 in some X_i, $i \in \{2, \dots, a\}$, then we choose one of these arcs and let $X_2' = X_i$, otherwise we choose such an arbitrary arc y_1x^1 out of Y_1' ($x^1 \in X$) and let X_2' be an arbitrary strong component in $\mathcal{X} - X_1'$. Let $\mathcal{P}_Y = \{y_1x^1\}$. Likewise, for $t \geq 2$, we get an arc y^tx_t into X_t' ($y^t \in Y$) which is different from $y_{t-1}x^{t-1}$ in $\mathcal{P}_Y$. If there is such an arc y^tx_t with y^t in some $Y_j \in \mathcal{Y} - \{Y_1', \dots, Y_{t-1}'\}$, then choose one of these arcs and let $Y_t' = Y_j$, otherwise we choose such an arbitrary arc y^tx_t and let Y_t' be an arbitrary strong component in $\mathcal{Y} - \{Y_1', \dots, Y_{t-1}'\}$. Add y^tx_t to $\mathcal{P}_X$. For $s \geq 2$, we get an arc y_sx^s out of Y_s' ($x^s \in X$) which is different from y^sx_s in $\mathcal{P}_X$. If there is such an arc y_sx^s with x^s in some $X_i \in \mathcal{X} - \{X_1', \dots, X_{s-1}'\}$, then we choose one of these arcs and let $X_s' = X_i$, otherwise we choose such an arbitrary arc y_sx^s and let X_s' be an arbitrary strong component in $\mathcal{X} - \{X_1', \dots, X_{s-1}'\}$. Add y_sx^s to $\mathcal{P}_Y$. Hence we get two arc sets $\mathcal{P}_X$ and $\mathcal{P}_Y$ with $\mathcal{P}_X \cap \mathcal{P}_Y = \emptyset$.

We will now show that D has a good pair. Let D_X be the digraph obtained from $D[X]$ by adding one new vertex y^* and arcs from y^* to x_i for $i \in [a]$. Analogously let D_Y be the digraph obtained from $D[Y]$ by adding one new vertex x^* and arcs from y_j to x^* for $j \in [b]$. Since Out(D_X)$= \{y^*\}$ and In(D_Y)$= \{x^*\}$, there exists an out-branching $B_{y^*}^+$ in D_X and an in-branching $B_{x^*}^-$ in D_Y. Set $T_X = B_{y^*}^+ - y^*$ and $T_Y = B_{x^*}^- - x^*$.

By construction, (O_D, I_D) is a good pair of D with $O_D = B^+ + \mathcal{P}_X + T_X$ and $I_D = B^- + \mathcal{P}_Y + T_Y$. □

Corollary 9. *Let D be a digraph with $\lambda(D) \geq 2$ that contains a subdigraph Q with a good pair. Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$. If $X \cap Y = \emptyset$ and $X \cup Y = V - V(Q)$, then D has a good pair.*

Proof. Let X_i be the initial strong components in $D[X]$ and Y_j be the terminal strong components in $D[Y]$, $i \in [a]$ and $j \in [b]$. Since $\lambda(D) \geq 2$, $d_Y^-(X_i) \geq 2$ and $d_X^+(Y_j) \geq 2$, for any $i \in [a]$ and $j \in [b]$, which implies that D has a good pair by Proposition 8. □

Lemma 10 ([2])**.** *Let D be a digraph and $X \subset V(D)$ be a set such that every vertex of X has both an in-neighbour and an out-neighbour in $V - X$. If $D - X$ has a good pair, then D has a good pair.*

By Lemma 10, in this paper we will often use the fact that if Q is a maximal subdigraph of D with a good pair and $X = N_D^-(Q), Y = N_D^+(Q)$, then $X \cap Y = \emptyset$.

Lemma 11. *Let D be a 2-arc-strong digraph containing a subdigraph Q with a good pair, $X = N_D^-(Q)$ and $Y = N_D^+(Q)$. If $X \cap Y = \emptyset$ and $X \cup Y = V - V(Q) - \{w\}$, where $w \in V - V(Q)$, then D has a good pair.*

Proof. Assume that Q has a good pair (O_Q, I_Q). Let B^+ be an out-tree containing O_Q with an in-arc of any vertex in Y from Q, while B^- be an in-tree containing I_Q with an out-arc of any vertex in X to Q.

First assume that either $(Y, w)_D \neq \emptyset$ or $(w, X)_D \neq \emptyset$. By the digraph duality, we may assume that $(Y, w)_D \neq \emptyset$, i.e., there exists an arc e from Y to w in D. Let $D' = D - e$. Set $X' = N^-_{D'}(Q) = X$ and $Y' = N^+_{D'}(Q) \cup \{w\} = Y \cup \{w\}$. Let X'_i be the initial strong components in $D'[X']$ and Y'_j be the terminal strong components in $D'[Y']$, $i \in [a]$ and $j \in [b]$. If w has an in-neighbour v in Y with v in some Y'_j, $j \in [b]$, then let $e = vw$ and $Y^*_1 = Y'_j$, otherwise we choose an arbitrary in-neighbour v of w in Y and let $e = vw$ and Y^*_1 be an arbitrary terminal strong component of $D'[Y']$. Since $\lambda(D) \geq 2$, $d^+_{X'}(Y^*_1) \geq 1$, $d^+_{X'}(Y'_j) \geq 2$ and $d^-_{Y'}(X'_i) \geq 2$, for any $Y'_j \neq Y^*_1$, $j \in [b]$ and $i \in [a]$, which implies that we get arc sets $\mathcal{P}_{X'}$ and $\mathcal{P}_{Y'}$ with $\mathcal{P}_{X'} \cap \mathcal{P}_{Y'} = \emptyset$, and digraphs $T_{X'}$ and $T_{Y'}$ by Proposition 8. By construction, D has a good pair $(B^+ + \mathcal{P}_{X'} + T_{X'} + e, B^- + \mathcal{P}_{Y'} + T_{Y'})$.

Now assume that $(Y, w)_D = \emptyset$ and $(w, X)_D = \emptyset$, which implies that $d^-_X(w) \geq 2$ and $d^+_Y(w) \geq 2$. Let X_i be the initial strong components in $D[X]$ and Y_j be the terminal strong components in $D[Y]$, $i \in [a]$ and $j \in [b]$. Since $\lambda(D) \geq 2$ and $(w, X)_D = (Y, w)_D = \emptyset$, $d^-_Y(X_i) \geq 2$ and $d^+_X(Y_j) \geq 2$ for any $i \in [a]$ and $j \in [b]$. By Proposition 8, we get $\mathcal{P}_X, T_X$ and $\mathcal{P}_Y, T_Y$ with $\mathcal{P}_X \cap \mathcal{P}_Y = \emptyset$. It follows that $(B^+ + \mathcal{P}_X + T_X + w^-w, B^- + \mathcal{P}_Y + T_Y + ww^+)$ is a good pair of D, where $w^- \in X$ and $w^+ \in Y$. ☐

Proposition 12 ([2])**.** *Every digraph on 3 vertices with at least 4 arcs has a good pair.*

Following [4], we shall use $\delta_0(D)$ to denote the *minimum semi-degree* of D, which is the minimum over all in- and out-degrees of vertices of D.

Proposition 13 ([2])**.** *Let D be a digraph on 4 vertices with at least 6 arcs except E_4 (see Figure 1). If $\delta^0(D) \geq 1$ or D is a semicomplete digraph, then D has a good pair.*

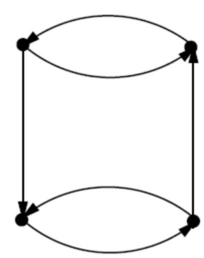

Figure 1: E_4.

Lemma 14 ([5], p.354)**.** *Let $D = (V, A)$ be a digraph. Then D is k-arc-strong if and only if it contains k arc-disjoint (s, t)-paths for every choice of distinct vertices $s, t \in V$.*

Lemma 15 (Edmonds' branching theorem [4])**.** *A directed multigraph $D = (V, A)$ with a special vertex z has k arc-disjoint out-branchings rooted at z if and only if $d^-(X) \geq k$ for all $\emptyset \neq X \subseteq V - z$.*

Lemma 16 ([2])**.** *If D is a 2-arc-strong digraph on n vertices that contains a subdigraph on $n - 3$ vertices with a good pair, then D has a good pair.*

Lemma 17. *If D is a 2-arc-strong digraph on n vertices that contains a subdigraph Q on $n - 4$ vertices with a good pair, then D has a good pair.*

Proof. Let (O_Q, I_Q) be a good pair of Q and set $X = N^-_D(Q)$ and $Y = N^+_D(Q)$. Since $|Q| = n-4$, $|D-Q| = 4$. If $X \cap Y \neq \emptyset$, then there is a vertex v in $D - Q$ which has both an in-neighbour and an out-neighbour in Q. By Lemma 10, $D[V(Q) \cup \{v\}]$ has a good pair. Moreover, since $|Q \cup \{v\}| = n - 3$, D has a good pair by Lemma 16. Henceforth, we may assume that $X \cap Y = \emptyset$.

If $X \cup Y = V - V(Q)$, namely $|X \cup Y| = 4$, then D has a good pair by Corollary 9. If $|X \cup Y| = 3$, then D has a good pair by Lemma 11. Therefore, we may assume that $|X \cup Y| = 2$, i.e., $|X| = |Y| = 1$.

Set $Y = \{y\}$, $X = \{x\}$ and $V - V(Q) - X - Y = \{w_1, w_2\}$, and let e_y (resp. e_x) be an arc from Q to y (resp. from x to Q). Since $\lambda(D) \geq 2$, D contains 2 arc-disjoint (y, x)-paths $P^1_{(y,x)}$ and $P^2_{(y,x)}$ by Lemma 14. Obviously, $(P^1_{(y,x)} \cup P^2_{(y,x)}) \cap V(Q) = \emptyset$ and $(P^1_{(y,x)} \cup P^2_{(y,x)}) \cap \{w_1, w_2\} \neq \emptyset$ (since there are no multiple arcs). W.l.o.g., assume that $P^1_{(y,x)} \cap \{w_1, w_2\} \neq \emptyset$.

Case 1: $P^2_{(y,x)} \cap \{w_1, w_2\} = \emptyset$.

That is $P^2_{(y,x)} = yx$. Since $P^1_{(y,x)} \cup P^2_{(y,x)}$ uses at most one arc from $\{y, x\}$ to $\{w_1, w_2\}$, there is one arc $a_1 \notin P^1_{(y,x)} \cup P^2_{(y,x)}$ from $\{y, x\}$ to $\{w_1, w_2\}$. W.l.o.g., assume that the head of a_1 is w_1. Moreover, since $d^-_D(w_2) \geq 2$, there is one arc $a_2 \notin P^1_{(y,x)} \cup P^2_{(y,x)}$ with head w_2. Obviously, we can make $\{a_1\} \cup \{a_2\} \neq C_2$. Set $O_D = O_Q + e_y + P^2_{(y,x)} + a_1 + a_2$.

Now, we want to find I_D. If $P^1_{(y,x)} \supseteq \{w_1, w_2\}$, then let $I_D = P^1_{(y,x)} + e_x + I_Q$. If $P^1_{(y,x)} \cap \{w_1, w_2\} = w_1$, then there is one arc a_3 with tail w_2. Note that $a_3 \neq a_1$ (since the tail of a_1 is in $\{y, x\}$), $a_3 \neq a_2$ (since the head of a_2 is w_2) and obviously $a_3 \notin P^1_{(y,x)} \cup P^2_{(y,x)}$. Set $I_D = P^1_{(y,x)} + a_3 + e_x + I_Q$. If $P^1_{(y,x)} \cap \{w_1, w_2\} = w_2$, then there is one arc $a_4 \neq a_2$ with tail w_1 since $d^+_D(w_1) \geq 2$. Similarly, $a_4 \neq a_1$ and $a_4 \notin P^1_{(y,x)} \cup P^2_{(y,x)}$. Set $I_D = P^1_{(y,x)} + a_4 + e_x + I_Q$. Therefore, (O_D, I_D) is a good pair of D.

We omit the proof of the analogous case when "$P^2_{(y,x)} \cap \{w_1, w_2\} \neq \emptyset$" here.[1] □

Lemma 18. *Let D be a 2-arc-strong digraph on n vertices that contains a subdigraph Q on $n - 5$ vertices with a good pair, $X = N^-_D(Q)$, $Y = N^+_D(Q)$ and $X \cap Y = \emptyset$. If $|X| \geq 2$ or $|Y| \geq 2$, then D has a good pair.*

Proof. Let (O_Q, I_Q) be a good pair of Q. If $X \cup Y = V - V(Q)$, namely $|X \cup Y| = 5$, then D has a good pair by Corollary 9. If $|X \cup Y| = 4$, then D has a good pair by Lemma 11. Therefore, assume that $|X \cup Y| \leq 3$. Moreover, from the assumption that $|X| \geq 2$ or $|Y| \geq 2$, and the fact that $|X| \geq 1$ and $|Y| \geq 1$, we have $|X \cup Y| = 3$. W.l.o.g., assume that $|X| = 1$ and $|Y| = 2$.

Set $X = \{x\}$, $Y = \{y_1, y_2\}$ and $V - V(Q) - X - Y = \{w_1, w_2\}$, and let e_x (resp. e_{y_1}, e_{y_2}) be an arc from x to Q (resp. from Q to y_1, y_2). Let $B^+ = O_Q + e_{y_1} + e_{y_2}$ and $B^- = I_Q + e_x$. Since $\lambda(D) \geq 2$, by the definition of arc-connectivity and Lemma 15, D contains two arc-disjoint in-branchings rooted at x, denoted by B^-_1 and B^-_2. Since $X = N^-_D(Q)$ and $Y = N^+_D(Q)$, clearly the restriction of B^-_1, B^-_2 to $V - V(Q)$ are still two arc-disjoint in-branchings rooted at x. We denote them by $\hat{B}^-_1$ and $\hat{B}^-_2$.

Now, in $\hat{B}^-_i$ ($i = 1$ or 2), if there exists a (y_1, x)-path or a (y_2, x)-path containing w_j ($j = 1$ or 2), then we let $f_i(w_j) = 1$; otherwise, let $f_i(w_j) = 0$. Note that, in any in-branching rooted at x, there is a unique (s, x)-path for any vertex s. Hence, for $\hat{B}^-_i$, if $f_i(w_1) = 1$ and $f_i(w_2) = 0$, then it is impossible that w_2 is the out-neighbor of y_1, y_2 or w_1. So w_2 must be a leaf in $\hat{B}^-_i$.

Next, we consider the following cases:

Case 1: $f_i(w_1) + f_i(w_2) = 2$ for some $i \in [2]$.

W.l.o.g., assume that $i = 1$, i.e., $f_1(w_1) = f_1(w_2) = 1$. Then, $\hat{B}^-_1$ contains a $P_{(y_i,x)}$ containing w_1 and a $P_{(y_j,x)}$ containing w_2, where $i, j \in \{1, 2\}$. Since $P_{(y_i,x)} \neq P_{(y_j,x)}$, $i \neq j$. Obviously, $B^+ + P_{(y_i,x)} + P_{(y_j,x)}$ contains an out-branching of D, denoted by O_D. Now, set $I_D = \hat{B}^-_2 + B^-$. Since $P_{(y_i,x)} \cup P_{(y_j,x)} \subseteq \hat{B}^-_1$ and $\hat{B}^-_1$ and $\hat{B}^-_2$ are arc-disjoint, (O_D, I_D) is a good pair.

Case 2: $f_i(w_1) + f_i(w_2) = 1$ for any $i \in [2]$.

Subcase 2.1: $f_1(w_i) = f_2(w_i) = 1$ and $f_1(w_{3-i}) = f_2(w_{3-i}) = 0$, where $i \in [2]$.

W.l.o.g., assume that $i = 1$. Since $f_1(w_1) = 1$ and $f_1(w_2) = 0$, w_2 is a leaf of $\hat{B}^-_1$. Similarly, w_2 is also a leaf of $\hat{B}^-_2$. Thus, $\hat{B}^-_1 \cup \hat{B}^-_2$ does not use any in-arc to w_2. Let a be an arc with head w_2 and then $a \notin \hat{B}^-_1 \cup \hat{B}^-_2$. Since $f_1(w_1) = 1$, $\hat{B}^-_1$ contains a $P_{(y_j,x)}$ containing w_1 ($j = 1$ or 2). Now, D has a good pair (O_D, I_D) with $O_D = B^+ + P_{(y_j,x)} + a$ and $I_D = \hat{B}^-_2 + B^-$.

Other cases can be derived analogously, and we just omit them here. □

[1]The corresponding content of omitted proofs in this paper can be found in [9] for interested readers.

Lemma 19. *Let D be a 2-arc-strong digraph on n vertices that contains a subdigraph Q on $n-6$ vertices with a good pair. Let $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$. If $|X| = |Y| = 2$ and at most one of X and Y is an independent set, then D has a good pair.*

Proof. Let $D = (V, A)$ and $W = V - X - Y - V(Q) = \{w_1, w_2\}$. Set $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. By contradiction, suppose that D has no good pair. Assume that (B_Q^+, B_Q^-) is a good pair of Q. Let B^+ be an out-tree containing B_Q^+ and an in-arc of y_j from Q for any $j \in [2]$. Let B^- be an in-tree contraining B_Q^- and an out-arc of x_i into Q for any $i \in [2]$. By construction, B^+ and B^- are arc-disjoint.

Claim 19.1 *None of the following holds:*

1. $|(Y, X)_D| = 4$;
2. *X is not independent and there exists a vertex in Y which dominates each vertex in X. Analogously Y is not independent and there exists a vertex in X which is dominated by each vertex in Y;*
3. *Both X and Y are not independent, say $x_i x_{3-i}, y_j y_{3-j} \in A$, and $y_j x_i, y_{3-j} x_{3-i} \in A$, $i, j \in [2]$.*

Proof. We will show that for each of the three cases D has a good pair.

1. Since $|(Y, X)_D| = 4$, for each $y \in Y$, y dominates each vertex in X. Thus, $(B^+ + y_1x_1 + y_2x_2, B^- + y_1x_2 + y_2x_1)$ is a good pair of $D \setminus W$, which implies that D has a good pair by Lemma 10.
2. By the digraph duality, it suffices to prove the first part. Assume that $x_1x_2 \in A$ and y_1 dominates each vertex in X. Then $(B^+ + y_1x_1x_2, B^- + y_1x_2)$ is a good pair of $D \setminus \{W \cup \{y_2\}\}$, which implies that D has a good pair by Lemma 16.
3. W.l.o.g., assume that $x_1x_2, y_1y_2, y_1x_1, y_2x_2 \in A$. Then $(B^+ + y_1x_1x_2, B^- + y_1y_2x_2)$ is a good pair of $D \setminus W$, which implies that D has a good pair by Lemma 10. ◊

Since at most one of X and Y is independent and $|X| = |Y| = 2$, it suffices to consider the case when X is not independent by the digraph duality. W.l.o.g., assume $x_1x_2 \in A$. We now prove the following claims.

Claim 19.2 *No vertex in W dominates both vertices of X. Analogously, if Y is not an independent set, then no vertex in W is dominated by both vertices in Y.*

Proof. By the digraph duality, it suffices to prove the first part. Suppose that w_1 dominates both vertices in X. Let $e_1 = w_1x_2$.

First assume that there exists an arc from Y to w_2, say e_2. Set $D' = D - \{e_1, e_2\}$, $X' = X \cup \{w_1\}$ and $Y' = Y \cup \{w_2\}$. There is only one initial strong component in $D'[X']$, say X_1'. Note that $d_{Y'}^-(X_1') \geq 2$. Let Y_j' be the terminal strong components in $D'[Y']$, $j \in [a]$. Note that $a \leq 3$. Then $d_{X'}^+(Y_j') \geq 2$ for all $j \in [a]$ except at most one, say Y_1', has $d_{X'}^+(Y_1') = 1$. Note that $e_2 \in (Y_1', w_2)_D$. By Proposition 8, we get arc-disjoint $\mathcal{P}_{X'}$ and $\mathcal{P}_{Y'}$ and $T_{X'}$, $T_{Y'}$. Then D has a good pair $(B^+ + e_2 + \mathcal{P}_{X'} + T_{X'}, B^- + e_1 + \mathcal{P}_{Y'} + T_{Y'})$, a contradiction.

Henceforth we may assume that $(Y, w_2)_D = \emptyset$. Now $d_{X \cup \{w_1\}}^-(w_2) \geq 2$. Let e_2 be an arbitrary out-arc of w_2. Set $D' = D - \{e_1, e_2\}$ and $X' = X \cup W$. There is only one initial strong component in $D'[X']$, say X_1'. Note that $d_Y^-(X_1') \geq 2$. For any terminal strong component Y_j in $D'[Y]$, $d_{X'}^+(Y_j) \geq 2$, where $j \in [a]$ and $a \leq 2$. By Proposition 8, we get arc-disjoint $\mathcal{P}_{X'}$ and $\mathcal{P}_Y$, and $T_{X'}$, T_Y. Then D has a good pair $(B^+ + \mathcal{P}_{X'} + T_{X'}, B^- + e_1 + e_2 + \mathcal{P}_Y + T_Y)$, a contradiction. ◊

Claim 19.3 *If Y is not independent, then at least one of $(w_k, X)_D$ and $(Y, w_{3-k})_D$ is empty, for any $k \in [2]$.*

Proof. W.l.o.g., assume that $y_1y_2 \in A$ and $k = 1$. Suppose that neither $(w_1, X)_D$ nor $(Y, w_2)_D$ is empty.

Case 1: $y_2w_2 \in A$ ($w_1x_1 \in A$).

By the digraph duality, it suffices to prove the case of $y_2w_2 \in A$. We distinguish several subcases as follows.

Subcase 1.1: $(Y, x_1)_D \neq \emptyset$.

Assume $y_jx_1 \in (Y, x_1)_D$ and $w_1x_i \in (w_1, X)_D$ as $(w_1, X)_D \neq \emptyset$, where $i, j \in [2]$.

First assume that $(w_2, X \cup \{w_1\})_D \neq \emptyset$. Set $w_2w_2^+ \in (w_2, X \cup \{w_1\})_D$. Then $(B^+ + y_jx_1x_2 + w_1^-w_1 + w_2^-w_2, B^- + w_1x_i + y_1y_2w_2w_2^+)$ is a good pair of D, where $w_1^- \neq w_2$ and $w_2^- \neq y_2$ as $\lambda(D) \geq 2$, a contradiction.

Next assume that $(w_2, X \cup \{w_1\})_D = \emptyset$. Namely $w_2y_1, w_2y_2 \in A$. Since $\lambda(D) \geq 2$, there exists an arc $e \neq y_jx_1$ which is from $Y \cup \{w_2\}$ to $X \cup \{w_1\}$. Now we find an out-branching of D as $O = B^+ + y_jx_1x_2 + w_1^-w_1 + w_2^-w_2$, where $w_1^-w_1 \neq e$ and $w_2^- \neq y_2$ as $\lambda(D) \geq 2$. Note that $B^- + w_1x_i + y_1y_2w_2y_1 + e$ contains an in-branching I of D. Then (O, I) is a good pair of D, a contradiction.

Since other cases can be derived analogously, we omit them here. ◊

Claim 19.4 *If Y is an independent set and $(w_k, X)_D$ is not empty, then $(Y, w_{3-k})_D = \emptyset$ for any $k \in [2]$.*

Proof. W.l.o.g., assume that $k = 1$. Set $w_1x_i \in A$, where $i \in [2]$. Suppose $(Y, w_2)_D \neq \emptyset$. We distinguish the following two cases.

Case 1: $|(Y, w_2)_D| = 2$.

That is $y_1w_2, y_2w_2 \in A$. Let $y_jy_j^+$ be an out-arc of y_j which is different from y_jw_2, for any $j \in [2]$. Note that $y_j^+ \in X \cup \{w_1\}$ as Y is independent.

Subcase 1.1: $w_2x_1 \in A$.

Let $w_2w_2^+$ be an out-arc of w_2 which is different from w_2x_1.

First assume that $w_2^+ \in X \cup \{w_1\}$. Since $\lambda(D) \geq 2$, w_1 has an in-neighbour $w_1^- \neq w_2$ and there is at least one vertex in Y which is not w_1^-, w.l.o.g., say $y_1 \neq w_1^-$. Then $(B^+ + y_1w_2x_1x_2 + w_1^-w_1, B^- + w_1x_i + y_2w_2w_2^+ + y_1y_1^+)$ is a good pair of D, a contradiction.

Next assume that $w_2^+ \in Y$, w.l.o.g., say $w_2^+ = y_1$, i.e., $w_2y_1 \in A$. Then $(B^+ + y_1w_2x_1x_2 + w_1^-w_1, B^- + w_1x_i + y_2w_2y_1y_1^+)$ is a good pair of D, where $w_1^- \neq y_1$ as $\lambda(D) \geq 2$, a contradiction.

Other cases can be derived analogously, and we just omit them here. ◊

Claim 19.5 *$(W, X)_D = \emptyset$. Moreover, if Y is not an independent set, then $(Y, W)_D = \emptyset$.*

Proof. By the digraph duality, it suffices to prove that $(W, X)_D = \emptyset$. Suppose $(W, X)_D \neq \emptyset$. W.l.o.g., assume that $(w_1, X)_D \neq \emptyset$, i.e., $w_1x_i \in A$, for some $i \in [2]$. Note that $(Y, w_2)_D = \emptyset$ by Claims 19.3 and 19.4.

Case 1: $(w_2, Y)_D \neq \emptyset$.

Set $w_2y_j \in A$, where $j \in [2]$. Since $N^-(X \cup W) = Y$ and $\lambda(D) \geq 2$, any initial strong component of $D[X \cup W]$ has at least two in-arcs from Y. Set $D' = D - w_1x_i - w_2y_j$. Now any initial strong component of $D'[X \cup W]$ has at least two in-arcs from Y, except at most one initial strong component, say X_1', has exactly one in-arc from Y. Note that $x_i \in X_1'$ but $w_1 \notin X_1'$ and any terminal strong component of $D'[Y]$ has at least two out-arcs to $X \cup W$. By Proposition 8, we get $\mathcal{P}_{X\cup W}, T_{X\cup W}$ and $\mathcal{P}_Y, T_Y$ in D. Since $(Y, w_2)_D = \emptyset$, $(B^+ + \mathcal{P}_{X\cup W} + T_{X\cup W}, B^- + \mathcal{P}_Y + T_Y + w_1x_i + w_2y_j)$ is a good pair of D, a contradiction.

Case 2: $(w_2, Y)_D = \emptyset$.

That is $|(w_2, X)_D| \geq 1$. By Claim 19.2, $|(w_2, X)_D| \leq 1$, i.e., $w_2w_1 \in A$ and $|(w_2, X)_D| = 1$. Interchange w_1 and w_2, likewise $(Y, w_1)_D = \emptyset$ by Claims 19.3 and 19.4. Now $(Y, W)_D = \emptyset$. By Cliam 19.1(2), $|(y_j, X)_D| \leq 1$ for any $j \in [2]$, namely $y_1y_2, y_2y_1 \in A$ and $|(y_1, X)_D| = |(y_2, X)_D| = 1$. We also get that $|(Y, x_i)_D| \leq 1$, for any $i \in [2]$. This implies that D has a good pair by Cliam 19.1(3). ◊

Now we are ready to finish the proof of Lemma 19. By Claim 19.1(2), $|(y_j, X)_D| \leq 1$ for any $j \in [2]$, i.e. $|(Y, X)_D| \leq 2$. Since $(W, X)_D = \emptyset$, $D[X] = C_2$ and $y_jx_1, y_{3-j}x_2 \in A$ for some $j \in [2]$. Note that Y is an independent set by Cliam 19.1(3). W.l.o.g., assume that $j = 1$, i.e., $y_1x_1, y_2x_2 \in A$, which implies that $y_2x_1, y_1x_2 \notin A$. Since $(W, X)_D = \emptyset$, $|(w_2, Y)_D| \geq 1$ as $\lambda(D) \geq 2$. W.l.o.g., assume that $w_2y_2 \in A$.

First assume $y_1w_k, w_ky_2 \in A$, for some $k \in [2]$. W.l.o.g., assume $k = 1$. Then $(B^+ + y_1x_1x_2 + w_2^-w_2 + w_1^-w_1, B^- + y_1w_1y_2x_2 + w_2w_2^+)$ is a good pair of D, where $w_2^-, w_2^+ \neq w_1$ and $w_1^- \neq y_1$, a contradiction.

Next assume $y_1w_2 \notin A$. As $\lambda(D) \geq 2$, $y_1w_1 \in A$, likewise $w_1y_2 \notin A$. It follows that $w_1w_2, w_1y_1 \in A$. Then $(B^+ + y_1x_1x_2 + w_2^-w_2 + w_1^-w_1, B^- + y_1w_1w_2y_2x_2)$ is a good pair of D, where $w_1^- \neq y_1$ and $w_2^- \neq w_1$ as $\lambda(D) \geq 2$, a contradiction.

This completes the proof of Lemma 19. □

Lemma 20. *Let $D=(V,A)$ be a 2-arc-strong digraph on n vertices that contains a subdigraph Q on $n-6$ vertices with a good pair. Set $X=N_D^-(Q)=\{x_1,x_2\}$ and $Y=N_D^+(Q)=\{y_1,y_2\}$ with $X\cap Y=\emptyset$, and $W=V-X-Y-V(Q)=\{w_1,w_2\}$. If X,Y are both independent sets, then D has a good pair except for the case below:*

($*$) *$(Y,X)_D=\{y_jx_i,y_{3-j}x_{3-i}\}$ for some $i,j\in[2]$, $D[W]=C_2$ and $N_W^+(y_j)\cap N_W^+(y_{3-j})=N_W^-(x_i)\cap N_W^-(x_{3-i})=\emptyset$ while $N_W^+(y_j)\cap N_W^-(x_i)\neq\emptyset$ and $N_W^+(y_{3-j})\cap N_W^-(x_{3-i})\neq\emptyset$.*

Proof. Suppose that D has no good pair. Follow the definitions of B_Q^+, B_Q^-, B^+ and B^- in the proof of Lemma 19. Observe that Claim 19.1 still holds here. We distinguish four cases below depending on $|(Y,X)_D|$. By Claim 19.1, $|(Y,X)_D|\leq 3$.

Case 1: $|(Y,X)_D|=2$.

Subcase 1.1: $(Y,X)_D=\{y_jx_1,y_jx_2\}$, for some $j\in[2]$.

W.l.o.g., assume that $j=1$. It implies that y_2 dominates both vertices in W. Note that there exists an arc from W to x_i, say e_{x_i}, where $i\in[2]$.

If e_{x_1} is adjacent to e_{x_2}, say w_1 is the common vertex, then $(B^++y_1x_1+y_2w_1x_2+w_2^-w_2, B^-+y_1x_2+w_1x_1+y_2w_2w_2^+)$ is a good pair of D, where $w_2^-, w_2^+\neq y_2$ as $\lambda(D)\geq 2$, a contradiction.

Hence e_{x_1} is non-adjacent to e_{x_2}, w.l.o.g., say $w_1x_2, w_2x_1\in A$. If $D[W]\neq C_2$, w.l.o.g., say $w_1w_2\notin A$, then $(B^++y_1x_1+y_2w_1x_2+w_2^-w_2, B^-+y_1x_2+y_2w_2x_1+w_1w_1^+)$ is a good pair of D, where $w_2^-\neq y_2$ and $w_1^+\neq x_2$ as $\lambda(D)\geq 2$, a contradiction. If $D[W]=C_2$, then $(B^++y_1x_2+y_2w_1w_2x_1, B^-+y_1x_1+y_2w_2w_1x_2)$ is a good pair of D, a contradiction.

By the digraph duality, we also get a contradiction when $(Y,X)_D=\{y_1x_i,y_2x_i\}$, for some $i\in[2]$.

Since other cases can be derived analogously, we just omit them here. □

We use $D\supseteq E_3$ ($D\not\supseteq E_3$) to denote that D contains an arbitrary orientation (no orientation) of E_3 as a subdigraph. (E_3 is a mixed graph and only the two edges are to be oriented.) E_3 is shown in Figure 2.

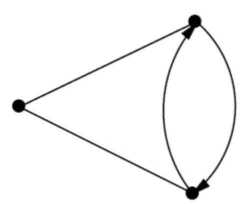

Figure 2: E_3.

Lemma 21. *Let $D=(V,A)$ be a 2-arc-strong digraph on n vertices that contains a subdigraph Q on $n-6$ vertices with a good pair. Set $X=N_D^-(Q)=\{x_1,x_2\}$ and $Y=N_D^+(Q)=\{y_1,y_2\}$ with $X\cap Y=\emptyset$, and $W=V-X-Y-V(Q)=\{w_1,w_2\}$. If $n=8$ or 9 and X,Y are both independent, then D has a good pair.*

Proof. By contradiction, suppose that D has no good pair. Follow the definitions of B_Q^+, B_Q^-, B^+ and B^- in the proof of Lemma 19. Observe that Claim 19.1 still holds here.

From Lemma 20, it suffices to consider the case ($*$). Recall that $(Y,X)_D=\{y_jx_i,y_{3-j}x_{3-i}\}$ for some $i,j\in[2]$, $D[W]=C_2$ and $N_W^+(y_j)\cap N_W^+(y_{3-j})=N_W^-(x_i)\cap N_W^-(x_{3-i})=\emptyset$ while $N_W^+(y_j)\cap N_W^-(x_i)\neq\emptyset$ and $N_W^+(y_{3-j})\cap N_W^-(x_{3-i})\neq\emptyset$. W.l.o.g., assume that $i=j=1$ and $w_k\in N_W^+(y_k)\cap N_W^-(x_k)$ for any $k\in[2]$. Note that $|Q|$ is 2 or 3.

Suppose $|Q|=2$, then $Q=C_2$. Set $V(Q)=\{q_1,q_2\}$ and $x_1q_1,x_2q_2\in A$. If $q_1y_2,q_2y_1\in A$, then D has a good pair $(B_{x_1}^+,B_{x_1}^-)$ as $B_{x_1}^+=x_1q_1q_2y_1w_1w_2x_2+y_2^-y_2$ and $B_{x_1}^-=x_2q_2q_1y_2w_2w_1x_1+y_1x_1$, where $y_2^-\neq q_1$ as $\lambda(D)\geq 2$, a contradiction. If $q_1y_1,q_2y_2\in A$, then D has a good pair $(B_{w_2}^+,B_{x_1}^-)$ as $B_{w_2}^+=w_2w_1x_1q_1q_2y_2x_2+y_1^-y_1$ and $B_{x_1}^-=w_1w_2x_2q_2q_1y_1x_1+y_2w_2$, where $y_1^-\neq q_1$ as $\lambda(D)\geq 2$, a contradiction.

Now $|Q|=3$. By Proposition 12, $|E(Q)|\geq 4$. Set $V(Q)=\{q_1,q_2,q_3\}$. Note that $Q\supseteq E_3$ or Q contains a bidigon as a subdigraph, i.e., $C_2\subset Q$. W.l.o.g., assume $C_2=q_1q_2q_1$. Set $B^+=w_1w_2x_2+w_1x_1$ and

$B^- = y_1w_1 + y_2w_2w_1$. Since $X = N_D^-(Q)$ and $Y = N_D^+(Q)$, there exists an arc from x_i to Q and an arc from Q to y_j, say $x_iq_{x_i}$ and $q_{y_j}y_j$, respectively, where $i, j \in [2]$.

Claim 21.1 If Q has a good pair $(B^+_{q_{x_i}}, B^-_{q_{y_j}})$, then $N^+(x_i) = \{q_{x_i}, y_j\}$ and $N^-(y_j) = \{q_{y_j}, x_i\}$, where $i, j \in [2]$.

Proof. Suppose that x_i has an out-neighbour $x_i^+ \notin \{q_{x_i}, y_j\}$ or y_j has an in-neighbour $y_j^- \notin \{q_{y_j}, x_i\}$, then $(B^+ + x_iq_{x_i} + B^+_{q_{x_i}} + y_j^-y_j + q_{y_{3-j}}y_{3-j}, B^- + q_{y_j}y_j + B^-_{q_{y_j}} + x_ix_i^+ + x_{3-i}q_{x_{3-i}})$ is a good pair of D, a contradiction. $\Diamond$

We distinguish several cases as follows.

Case 1: Q contains a bidigon.

Set $q_2q_3, q_3q_2 \in A$.

Subcase 1.1: $x_iq_1 \in A$, for some $i \in [2]$.

W.l.o.g., assume $i = 1$.

A. $(q_1, Y)_D \neq \emptyset$.

Assume $q_1y_j \in A$, $j \in [2]$. Note that Q has a good pair $B^+_{q_1} = q_1q_2q_3$ and $B^-_{q_1} = q_3q_2q_1$. This implies that $N^+(x_1) = \{q_1, y_j\}$ and $N^-(y_j) = \{q_1, x_1\}$ by Claim 21.1.

We first show $q_3y_{3-j}, x_2q_3 \in A$. If $q_3y_{3-j} \notin A$, i.e., $q_3q_1 \in A$, then Q has a good pair $(B^+_{q_i}, B^-_{q_1})$ for any $i \in [3]$ (see Figure 3). It follows that $(B^+ + x_1y_j + x_2q_{x_2} + B^+_{q_{x_2}} + q_{y_{3-j}}y_{3-j}, B^- + x_1q_1y_j + B^-_{q_1} + x_2x_2^+)$ is a good pair of D, where $x_2^+ \neq q_{x_2}$ as $\lambda(D) \geq 2$, a contradiction. Hence $q_3y_{3-j} \in A$. If $x_2q_3 \notin A$, namely $q_1q_3 \in A$, then $(B^+ + x_1q_1y_j + q_1q_3q_2 + y_{3-j}^-y_{3-j}, B^- + x_1y_j + q_1q_2q_3y_{3-j} + x_2q_{x_2})$ is a good pair of D, where $y_{3-j}^- \neq q_3$ as $\lambda(D) \geq 2$, a contradiction.

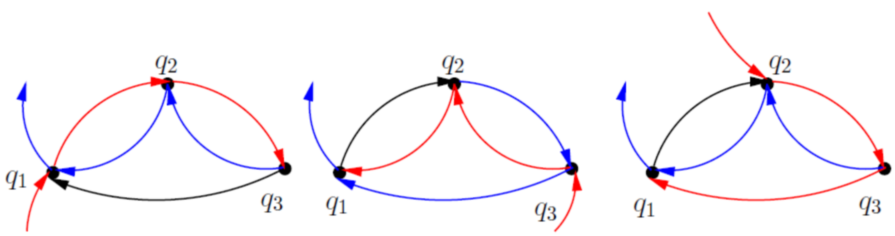

Figure 3: Good pairs $(B^+_{q_i}, B^-_{q_1})$ of Q, for any $i \in [3]$.

Now $q_3y_{3-j}, x_2q_3 \in A$. Note that Q has a good pair $B^+_{q_3} = q_3q_2q_1$ and $B^-_{q_3} = q_1q_2q_3$. This implies that $N^+(x_2) = \{q_3, y_{3-j}\}$ and $N^-(y_{3-j}) = \{q_3, x_2\}$ by Claim 21.1. If $j = 1$, then D has a good pair $(B^+_{x_1}, B^-_{x_1})$ with $B^+_{x_1} = x_1q_1q_2q_3 + x_1y_1w_1w_2x_2y_2$ and $B^-_{x_1} = w_2w_1x_1 + y_2x_2q_3q_2q_1y_1x_1$, a contradiction. If $j = 2$, then D has a good pair $(B^+_{x_1}, B^-_{x_1})$ with $B^+_{x_1} = x_1q_1q_2q_3 + x_1y_2x_2y_1w_1w_2$ and $B^-_{x_1} = x_1q_3y_1x_1 + q_2q_1y_2w_2w_1x_1$, a contradiction.

Therefore, $q_1q_3 \in A$. We omit the proof of other analogous cases.

Case 1 implies that D has a good pair when Q contains a bidigon as a subdigraph.

Case 2: $Q \supseteq E_3$.

First assume $q_1q_3, q_2q_3 \in A$. Since Q has no bidigon as subdigraph and $\lambda(D) \geq 2$, $q_3y_1, q_3y_2 \in A$ and $(X, q_1)_D \neq \emptyset$. Set $x_iq_1 \in A$. Note that Q has a good pair $B^+_{q_1} = q_1q_2q_3$ and $B^-_{q_3} = q_2q_1q_3$. By Claim 21.1, $N^+(x_i) = \{q_1, y_1\} = \{q_1, y_2\}$, a contradiction. The case of $q_3q_1, q_3q_2 \in A$ is analogous. Hence $q_2q_3, q_3q_1 \in A$. We distinguish several subcases as follows.

Subcase 2.1: $x_iq_1 \in A$, for some $i \in [2]$.

W.l.o.g., assume $i = 1$. Since Q has no bidigon, $q_1q_3 \notin A$, which implies that $(q_1, Y)_D \neq \emptyset$. Set $q_1y_j \in A$, $j \in [2]$. Note that Q has a good pair $B^+_{q_1} = q_1q_2q_3$ and $B^-_{q_1} = q_3q_1 + q_2q_1$. This implies that $N^+(x_1) = \{q_1, y_j\}$ and $N^-(y_j) = \{q_1, x_1\}$ by Claim 21.1. By Case 1, $x_2q_3, x_2q_2, q_3y_{3-j} \in A$. Then

$(B^+ + x_1y_j + x_2q_3q_1q_2 + y^-_{3-j}y_{3-j}, B^- + x_1q_1y_j + x_2q_2q_1 + q_3y_{3-j})$ is a good pair of D, where $y^-_{3-j} \neq q_3$ as $\lambda(D) \geq 2$, a contradiction.

By the digraph duality, we also get a contradiction when $q_2y_j \in A$, where $j \in [2]$. This implies that $(X, q_1)_D = (q_2, Y)_D = \emptyset$.

Since the case when "$x_iq_2 \in A$, for some $i \in [2]$" is analogous, we just omit it here. □

Proposition 22 ([3]). *A digraph D has an out-branching (resp. in-branching) if and only if it has precisely one initial (resp. terminal) strong component. In that case every vertex of the initial (resp. terminal) strong component can be the root of an out-branching (resp. in-branching) in D.*

We use T^+_x (resp. T^-_x) to denote an out-tree (resp. in-tree) rooted at x.

Proposition 23. *Let D be an oriented graph on n vertices. Let $P_D = x_1x_2\dots x_n$ be the Hamilton dipath of D and $D' = D - A(P)$. Assume that there are exactly two disjoint strong components I_1 and I_2 in D'. Set $q \in \{2, 3, n-1, n\}$. If for some q, x_{q-1} and x_q are respectively in I_1 and I_2, then D has a good pair.*

Proof. W.l.o.g., assume that $x_{q-1} \in I_1$ and $x_q \in I_2$. Since I_i is strong, $\delta^0(I_i) \geq 1$, for any $i \in [2]$.

First assume $q \in \{n-1, n\}$. Let x be an in-neighbour of x_q in I_2. We get an out-branching of D as $B^+_{x_1} = P_D - x_{q-1}x_q + xx_q$. Then we will show that there is an in-branching B^-_x in $D - A(B^+_{x_1})$. Since I_2 is strong, $I_2 - xx_q$ is connected and has only one terminal srong component which contains x. This implies that there is an in-branching T^-_x in $I_2 - xx_q$. Note that there exists an in-branching $T^-_{x_{q-1}}$ in I_1, as I_1 is strong. Then $B^-_x = T^-_x + x_{q-1}x_q + T^-_{x_{q-1}}$, which implies that $(B^+_{x_1}, B^-_x)$ is a good pair of D.

Now we assume $q \in \{2, 3\}$. Let y be an out-neighbour of x_{q-1} in I_1. We get an in-branching of D as $B^-_{x_n} = P_D - x_{q-1}x_q + x_{q-1}y$. Then we will show that there is an out-branching B^+_y in $D - A(B^-_{x_n})$. Since I_1 is strong, $I_1 - x_{q-1}y$ is connected and has only one initial srong component which contains y. This implies that there is an out-branching T^+_y in $I_1 - x_{q-1}y$. Note that there exists an out-branching $T^+_{x_q}$ in I_2, as I_2 is strong. Then $B^+_y = T^+_y + x_{q-1}x_q + T^+_{x_q}$. So, $(B^+_y, B^-_{x_n})$ is a good pair of D. □

Proposition 24. *Let D be a 2-arc-strong oriented graph on at least seven vertices. Then D has a dipath P_6.*

Proof. Suppose that there is no P_6 in D. Assume that P_t is the longest dipath in D, then $t \geq 4$, as there is no digon in D and $\lambda(D) \geq 2$. Observe that there is no C_t in D, otherwise D has a longer dipath P_{t+1}.

First assume that $t = 4$ and set $P_4 = x_1x_2x_3x_4$. Since $d^+_D(x_4) \geq 2$ and D has no digon, the out-neighbourhood of x_4 either contains x_1 or contains a vertex in $V - V(P_4)$. This implies that there is a P_5 in D, a contradiction.

Henceforth we may assume that $t = 5$ and set $P_5 = x_1x_2x_3x_4x_5$. Since $\lambda(D) \geq 2$, $d^+_D(x_5) \geq 2$ and $d^-_D(x_1) \geq 2$. Then we get $N^+_D(x_5) = \{x_2, x_3\}$ and $N^-_D(x_1) = \{x_3, x_4\}$, as P_5 is the longest dipath in D and D has no digon. Observe that there exsits a different 4-length dipath, $x_4x_5x_3x_1x_2$, in D. Likewise, $N^+_D(x_2) = \{x_3, x_5\}$, which implies that $D[\{x_2, x_5\}]$ is a digon, a contradiction. □

4 Good pairs in digraphs of order 7

Proposition 25. *A 2-arc-strong oriented graph D on n vertices without good pair has a P_7, where $7 \leq n \leq 9$.*

Proof. Since the idea of the proof is similar to Proposition 24, we omit it here. □

Now we are ready to prove Theorem 4. For convenience, we restate it here.

Theorem 4. *Every 2-arc-strong digraph on 7 vertices has a good pair.*

Proof. Suppose that D has no good pair. Let R be a largest clique in D. By Lemma 16 and Proposition 13, $|R| = 3$. Moreover, R is a tournament by Lemma 17 and Proposition 12.

Claim 4.1 *D is an oriented graph.*

Proof. Suppose that there is a digon Q in D with $V(Q) = \{s, t\}$. Observe that Q has a good pair. Since R is a tournament with three vertices, both in- and out-neibourhoods of Q in D have at least two vertices. This implies that D has a good pair by Lemma 18, a contradiction. $\Diamond$

Assume that $P_D = x_1x_2 \dots x_7$ is a Hamilton dipath of D by Proposition 25. Set $D' = D - A(P_D)$. Let I_i and T_j respectively be the initial and terminal strong component in D', where $i \in [a]$ and $j \in [b]$. Note that $a, b \geq 2$ by Proposition 22. Since D is an oriented graph and $\lambda(D) \geq 2$, $|I_i|, |T_j| \geq 3$, for any $i \in [a], j \in [b]$. Thus there are only two disjoint strong components in D', say I_1 and I_2, with $|I_1| = 3$ and $|I_2| = 4$. Note that $|N^-_{D'}(x_1)| \geq 2$ and $|N^+_{D'}(x_7)| \geq 2$ as $\lambda(D) \geq 2$, which implies that $x_1, x_7 \in I_2$. Moreover, $x_2, x_6 \in I_1$ by Claim 4.1. Then D has a good pair by Proposition 23. $\square$

5 Good pairs in digraphs of order 8

The digraph E_3 used in the next proposition is shown in Figure 2.

Proposition 26 ([2]). *Let D be a 2-arc-strong digraph without any subdigraph on order 4 that has a good pair. If D contains an orientation Q of E_3 as a subdigraph, then $N^+_D(Q) \cap N^-_D(Q) = \emptyset$, $|N^+_D(Q)| \geq 2$ and $|N^-_D(Q)| \geq 2$.*

Proposition 27. *Let D be a 2-arc-strong oriented graph on n vertices without good pair, where $8 \leq n \leq 9$. If D does not have K_4 as a subdigraph, but has two disjoint cycles C^1 and C^2 which cover 7 vertices, then D contains a P_8.*

Proof. Suppose that P_7 is the longest dipath of D by Proposition 25. In fact there exist arcs between C^1 and C^2 from both directions, otherwise D has a P_8 as $\lambda(D) \geq 2$. W.l.o.g., assume $|C^1| \geq |C^2|$. Then $|C^1| = 4$ and $|C^2| = 3$. Let $C^1 = x_1x_2x_3x_4x_1$, $C^2 = x_5x_6x_7x_5$, $P_7 = x_1x_2 \dots x_7$ and y_j be the vertex in $V - V(C^1 \cup C^2)$, where $j = 1$ when $n = 8$ and $j \in [2]$ when $n = 9$. From the maximality of P_7 in D, we have the following facts.

Fact 27.1. For any j, at least one of $(C^i, y_j)_D$ and $(y_j, C^{3-i})_D$ is empty for any $i \in [2]$.

Fact 27.2. For any j, at least one of arcs x_iy_j and y_jx_{i+1} is not in A for any $i \in [6]$.

Fact 27.3. For $n = 9$, let $y_jy_{3-j} \in A$. If $x_iy_j \in A$, then $y_{3-j}x_{i+1}, y_{3-j}x_{i+2} \notin A$, where $j \in [2]$ and $i \in [5]$.

Since D is oriented, there are at least three arcs between y_j and C^i, for some i, by Fact 27.1. W.l.o.g., assume $i = 1$. Note that $d^+_{C^1}(y_j) \geq 1$ and $d^-_{C^1}(y_j) \geq 1$. Then $N(y_j) \subset \{y_{3-j}\} \cup C^1$ when $n = 9$ and $N(y_j) \subset C^1$ when $n = 8$.

If y_j is not adjacent to y_{3-j} or $n = 8$, then $N^+(y_j) = \{x_1, x_2\}$ and $N^-(y_{3-j}) = \{x_3, x_4\}$ by Fact 27.2, which implies that D has a P_8 as $y_jx_1 \in A$, a contradiction.

Hence $n = 9$ and y_1 is adjacent to y_2. W.l.o.g., assume that $y_1y_2 \in A$. If $x_1y_1 \in A$, then $N^+(y_2) = \{x_1, x_4\}$ by Fact 27.3 and $\lambda(D) \geq 2$, which implies that D has a Hamilton dipath as $y_2x_1 \in A$, a contradiciton. Hence x_1 is not adjacent to y_1. By Fact 27.2, $N^+(y_1) = \{x_2, x_9\}$ and $N^-(y_1) = \{x_3, x_4\}$. By Fact 27.3 and the longestness of P_7, $N^+(y_2) = \{x_2, x_3\}$. It implies that $D[\{x_2, x_3, y_1, y_2\}]$ is a K_4, a contradiction. $\square$

Proposition 28. *Let $D = (V, A)$ be a 2-arc-strong digraph on n vertices without good pair, where $8 \leq n \leq 9$. If D is an oriented graph without K_4 as a subdigraph, then D has a P_8.*

Proof. Suppose that $P = x_1x_2 \dots x_7$ is the longest dipath in D by Proposition 25. Let $X = V(P)$ and $Y = V - X$. We have the following fact.

Fact 28.1. $N^-(x_1), N^+(x_7) \subset X$ and $x_7x_1 \notin A$.

We will now show the following notes.

Note 28.1 Let B_X^+ be an out-branching of $D[X]$, and let B^- contain two disjoint in-tree $T_{r_1}^-$ and $T_{r_2}^-$ in $D[X] - A(B_X^+)$. Let $H_i = D[T_{r_i}^-]$ for any $i \in [2]$. For some $y \in Y$, if there exists an arc $r_i y$ and an out-arc of y to H_{3-i}, then D has a good pair.

Proof. Let $e_1 = r_1 y_1$ and e_2 be an out-arc of y_1 to H_2. It is trivial when $n = 8$ or $y_2 y_1 \notin A$ as $\lambda(D) \geq 2$. It suffices to consider the case when $n = 9$ and $y_2 y_1 \in A$. Set $B^+ = B_X^+ + y_2 y_1 + y_2^- y_2$ and $B^- = T_{r_1}^- + T_{r_2}^- + e_1 + e_2 + y_2 y_2^+$ where $y_2^-, y_2^+ \neq y_1$ as $\lambda(D) \geq 2$. Therefore, (B^+, B^-) is a good pair of D.

Note 28.2 Let B_X^+ and T_X^- respectively be an out-branching and an in-tree of $D[X]$, with $V(T_X^-) = X - v$, for arbitrary $v \in X$. If v has an out-arc which is not into $B_X^+ \cup T_X^-$, then D has a good pair.

Proof. It can be seen as a special case of Note 28.1, in which let $T_{r_1}^- = v$ and $T_{r_2}^- = T_X^-$.

The discussion of several cases depending on the in-neighbours of x_1 can be found in [9] for interested readers. □

Now we are ready to show Theorem 5. For convenience, we restate it here.

Theorem 5. *Every 2-arc-strong digraph on 8 vertices has a good pair.*

Proof. Suppose that D has no good pair. Let R be a largest clique in D. By Lemma 17 and Proposition 13, $|R| = 3$.

Claim 5.1 *No subdigraph of D of order at least 3 has a good pair.*

Proof. By Lemma 17, it suffices to show that there is no $Q \subset D$ on 3 vertices with a good pair. Suppose that Q has a good pair. If Q is an orientation of E_3, then we use Lemma 18 to find a good pair of D by Proposition 26, a contradiction. Now assume that Q is a bidigon. Set $V(Q) = \{x, y, z\}$ with $Q[\{x, y\}] = C_2$ and $Q[\{y, z\}] = C_2$. If there exists a vertex w in $N_D^+(Q) \cap N_D^-(Q)$, then $D[Q \cup \{w\}]$ has a good pair by Lemma 10. Thus $N_D^+(Q) \cap N_D^-(Q) = \emptyset$. If $N_D^-(Q) = \{w\}$, then $D[Q \cup \{w\}]$ has a good pair as $B_w^+ = wzyx$ and $B_z^- = wxyz$. By symmetry, this implies that $|N_D^+(Q)| \geq 2$ and $|N_D^-(Q)| \geq 2$. Thus by Lemma 18, D has a good pair, a contradiction. ♢

By the claim above, R is a tournament.

Claim 5.2 *D is an oriented graph.*

Proof. Suppose that there is a digon Q in D with $V(Q) = \{s, t\}$. Observe that Q has a good pair. Since R is a tournament with 3 vertices, both in- and out-neibourhoods of Q in D have at least two vertices with $N_D^+(Q) \cap N_D^-(Q) = \emptyset$. This implies that D has a good pair by Lemmas 11, 19, 20 and 21, and Corollary 9, a contradiction. ♢

By Proposition 28, assume that $P_D = x_1 x_2 \ldots x_8$ is a Hamilton dipath of D. Set $D' = D - A(P_D)$. Let I_i and T_j respectively be the initial and terminal strong component in D', where $i \in [a]$ and $j \in [b]$. Note that $a, b \geq 2$ by Proposition 22. Since D is an oriented graph and $\lambda(D) \geq 2$, $|I_i|, |T_j| \geq 3$ for any $i \in [a], j \in [b]$. Thus there are only two disjoint strong components in D', say I_1 and I_2, as $n = 8$. Since $\lambda(D) \geq 2$, x_1 has at least two in-neighbours and one out-neighbour in D', while x_8 has at least two out-neighbours and one in-neighbour in D'. If $|I_1| = 3$ and $|I_2| = 5$, then $x_1, x_8 \in I_2$ and $|A(I_2)| \geq 6$. Note that at least one of x_2 and x_7 is in I_1 as $|R| = 3$. Then we use Proposition 23 to get a good pair of D. Now assume $|I_1| = |I_2| = 4$. If $x_8 \in I_1$ then $x_7 \in I_2$ by Claim 5.2. By Proposition 23, D has a good pair. □

6 Good pairs in digraphs of order 9

We have several generalizations of Proposition 8 here, which are easy to check as they satisfy the conditions in Proposition 8.

Proposition 29. *Let $D = (V, A)$ be a digraph and Q be a subdigraph of D with good pair (O_Q, I_Q). Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$ and $X \cup Y = V - V(Q) - W$, where $W = \{w_1, w_2\}$. Let e_1 be an arc from w_1 to X and e_2 be an arc from Y to w_2. Set $X' = X \cup w_1$, $Y' = Y \cup w_2$ and $D' = (V, A')$ with $A' = A - \{e_1, e_2\}$. Let $\mathcal{X}$ be the set of initial strong components in $D'[X']$ and $\mathcal{Y}$ be the set of terminal strong components in $D'[Y']$. Assume that there exists X_0 and Y_0 in $\mathcal{X}$ and $\mathcal{Y}$ respectively such that $d_Y^-(X_0) = 1$ and $d_X^+(Y_0) = 1$. Let e_x and e_y be arcs from Y to X_0 and from Y_0 to X respectively. If one of the following holds, then D has a good pair.*

1. *$e_x \neq e_y$, but at least one of $\mathcal{X}$ or $\mathcal{Y}$ has only one element.*
2. *e_x (or e_y) is adjacent to some Y_x (or X_y) in $\mathcal{Y}$ (or $\mathcal{X}$), such that $d_X^+(Y_x) \geq 3$ (or $d_Y^-(X_y) \geq 3$).*
3. *e_x (or e_y) is adjacent to $Y' - V(\mathcal{Y})$ (or $X' - V(\mathcal{X})$).*
4. *e_x (or e_y) is adjacent to some $Y_x \neq Y_0$ (or $X_y \neq X_0$) in $\mathcal{Y}$ (or $\mathcal{X}$), such that there exists an arc from Y_x (or $Y' - V(\mathcal{Y})$) to $X' - V(\mathcal{X})$ (or X_y).*

Lemma 30. *Let D be a 2-arc-strong digraph on 9 vertices that contains a digon Q. Assume that D has no subdigraph with a good pair on 3 or 4 vertices. Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$. If $|X| = 3$ and $|Y| = 2$, then D has a good pair.*

Proof. Let $D = (V, A)$ and $W = V - X - Y - V(Q) = \{w_1, w_2\}$. Set $V(Q) = \{q_1, q_2\}$, $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$. By contradiction, suppose that D has no good pair. Since $\lambda(D) \geq 2$, w.l.o.g., assume $q_1y_1, q_2y_2 \in A$. Let $B^+ = q_1q_2y_2 + q_1y_1$ and B^- be an in-tree rooted at q_1 such that $A(B^-) \subseteq \{q_2q_1\} \cup (X, Q)_D$. By Proposition 13, we get the following fact.

Fact 30.1. There is no digon in the induced subdigraph of X or Y.

We will now show some notes below.

Note 30.1. There is no C_3 in the induced subdigraph of X.

Proof. W.l.o.g., assume that x_1q_1, x_2q_1, $x_3q_2 \in A$ and $D[X] = C_3 = x_1x_2x_3x_1$. Then $D[Q \cup X]$ has a good pair as $B_{x_1}^+ = x_1x_2x_3q_2q_1$ and $B_{q_2}^- = x_3x_1q_1 + x_2q_1q_2$. By Lemma 17, D has a good pair, a contradiction. ♢

Note 30.2. None of the following holds:

1. There exists a subset X_1 in X with $|X_1| = 2$, such that $|(Y, X_1)_D| = 4$.
2. Y is not independent and there exists a vertex in X which is dominated by both vertices in Y.
3. X is not independent, w.l.o.g., set that X_1 contains two arbitrarily adjacent vertices in X, and there exists a vertex in Y which dominates both vertices in X_1.
4. Both X and Y are not independent, say $x_{i_1}x_{i_2}, y_jy_{3-j} \in A$, and $y_jx_{i_1}, y_{3-j}x_{i_2} \in A$, where $i_1, i_2 \in [3]$ and $j \in [2]$.
5. Both X and Y are not independent, say $x_{i_1}x_{i_2}, y_jy_{3-j} \in A$, and $y_jx_{i_2}, y_{3-j}x_{i_1} \in A$, where $i_1, i_2 \in [3]$ and $j \in [2]$.

Proof. The proof is similar as that of Claim 19.1. ♢

Note 30.3. There exists respectively an arc from Y to W and an arc from W to X.

Proof. Let X_i be the initial strong components in $D[X]$ and Y_j be the terminal strong components in $D[Y]$, $i \in [a]$ and $j \in [b]$. Note that $1 \leq a \leq 3$ and $1 \leq b \leq 2$.

Case 1: $(W, X)_D = \emptyset$.

This implies that there are at least two arcs from W to Y as $\lambda(D) \geq 2$. For any i, j, $d_{X \cup W}^+(Y_j) \geq 2$ and $d_Y^-(X_i) \geq 2$ since $\lambda(D) \geq 2$ and $(W, X)_D = \emptyset$. By Proposition 8, we get $\mathcal{P}_X$ and T_X.

If $d^+_X(Y_j) \geq 2$ for any j, then we get $\mathcal{P}_Y$ and T_Y with $\mathcal{P}_X \cap \mathcal{P}_Y = \emptyset$ by Proposition 8. It follows that $(B^+ + \mathcal{P}_X + T_X, B^- + \mathcal{P}_Y + T_Y)$ is a good pair of $D - W$, which implies that D has a good pair by Lemma 10, a contradiction. Hence there exists a terminal strong component in $D[Y]$, say Y_1, such that $d^+_X(Y_1) \leq 1$. This implies that $|Y_1| = 1$, w.l.o.g., say $Y_1 = \{y_1\}$. Note that there exists a dipath P^1 from y_1 to y_2 with $P^1 - Y \subseteq W$ and $(y_2, X)_D \neq \emptyset$.

If y_2 is not a terminal strong component in $D[Y]$ or y_2 is a terminal strong component in $D[Y]$ with $d^+_X(y_2) \geq 2$, then we get $\mathcal{P}_Y$ and T_Y with $\mathcal{P}_X \cap \mathcal{P}_Y = \emptyset$ by Proposition 8. It follows that $D - W$ has a good pair $(B^+ + \mathcal{P}_X + T_X, B^- + \mathcal{P}_Y + T_Y)$, a contradiction.

Hence y_2 is a terminal strong component in $D[Y]$ with $d^+_X(y_2) \leq 1$. This implies that $d^+_X(y_1) = d^+_X(y_2) = 1$. Set $(y_i, X)_D = \{e_i\}$ for any $i \in [2]$. Note that $(Y, X)_D = \{e_1, e_2\}$, then there is only one initial strong component in $D[X]$ as $\lambda(D) \geq 2$, say X_1. Now $(Y, X_1)_D = \{e_1, e_2\}$. By Proposition 8, we get $\mathcal{P}_X$ and T_X, such that $\mathcal{P}_X = \{e_1\}$. Let $P_+ = e_1 + T_X$ and $P_- = e_2 + P^1$. If $P^1 = y_1 w_i w_{3-i} y_2$, then $(B^+ + P_+ + w_i^- w_i + w_{3-i}^- w_{3-i}, B^- + P_-)$ is a good pair of D, where $w_i^- \neq y_1$ and $w_{3-i}^- \neq w_i$ as $\lambda(D) \geq 2$, a contradiction. If $P^1 = y_1 w_i y_2$, then $(B^+ + P_+ + w_i^- w_i + w_{3-i}^- w_{3-i}, B^- + P_- + w_{3-i} w_{3-i}^+)$ is a good pair of D, where $w_i^- \neq y_1$ and $w_{3-i}^-, w_{3-i}^+ \neq w_i$ as $\lambda(D) \geq 2$, a contradiction.

Case 2: $(Y, W)_D = \emptyset$.

Note that $|(Y, X)_D| \geq 3$ by Fact 30.1.

First assume that Y is not an independent set, w.l.o.g., say $y_1 y_2 \in A$. Now $d^+_X(y_1) \geq 1$ and $d^+_X(y_2) \geq 2$. By Note 30.2, y_1 and y_2 can not dominate the same vertex in X, which implies that $d^+_X(y_1) = 1$ and $d^+_X(y_2) = 2$. W.l.o.g., assume $y_1 x_1, y_2 x_2, y_2 x_3 \in A$. Then X is independent by Note 30.2. Since $(Y, W)_D = \emptyset$ and $\lambda(D) \geq 2$, there exists a dipath P^1 from some vertex x in $\{x_1, x_2\}$ to x_3 with $V(P^1) - \{x, x_3\} \subseteq W$. Let $P_+ = y_1 x_1 + y_2 x_2 + P^1$ and $P_- = y_1 y_2 x_3$.

Next assume that Y is an independent set, i.e., $d^+_X(y_i) \geq 2$ for any $i \in [2]$. By Note 30.2, at most one vertex in X can be dominated by both vertices in Y, namely $|(Y, X)_D| = 4$. W.l.o.g., say x_1 is dominated by both vertices in Y and $y_1 x_2, y_2 x_3 \in A$. Now x_1 is non-adjacent to x_2 or x_3 by Note 30.2 and there exists at most one arc between x_2 and x_3 by Fact 30.1, w.l.o.g., say $x_2 x_3 \notin A$. Likewise, there exists a dipath P^1 from some vertex x in $\{x_1, x_2\}$ to x_3 with $V(P^1) - \{x, x_3\} \subseteq W$ as $(Y, W)_D = \emptyset$ and $\lambda(D) \geq 2$. Let $P_+ = y_1 x_2 + y_2 x_1 + P^1$ and $P_- = y_1 x_1 + y_2 x_3$.

If $P^1 = x w_1 w_2 x_3$, then $(B^+ + P_+, B^- + P_- + w_1 w_1^+ + w_2 w_2^+)$ is a good pair of D, where $w_1^+ \neq w_2$ and $w_2^+ \neq x_3$, a contradiction. If $P^1 = x w_1 x_3$, then $(B^+ + P_+ + w_2^- w_2, B^- + P_- + w_1 w_1^+ + w_2 w_2^+)$ is a good pair of D, where $w_1^+ \neq x_3$ and $w_2^-, w_2^+ \neq w_1$, a contradiction. ◊

W.l.o.g., assume that $e_1 = w_1 x_1$ is an arc from w_1 to X. First assume that $e_2 = y_1 w_2$ is an arc from Y to w_2. Set $D' = (V, A')$ with $A' = A - \{e_1, e_2\}$ and $X' = X \cup \{w_1\}$, $Y' = Y \cup \{w_2\}$. Let $\mathcal{X}$ be the set of initial strong components in $D'[X']$ and $\mathcal{Y}$ be the set of terminal strong components in $D'[Y']$.

By Proposition 8, if there exists at most one strong component S in $\mathcal{X} \cup \mathcal{Y}$ which satisfies that $S \in \mathcal{X}$ (or $S \in \mathcal{Y}$) with $d^-_Y(S) = 1$ (or $d^+_X(S) = 1$), then we can find a good pair of D, a contradiction. We get the fact below.

Fact 30.2. There exist two strong components $X_0 = x_1$ and $Y_0 = y_1$ respectively in $\mathcal{X}$ and $\mathcal{Y}$ such that $d^-_Y(X_0) = 1$ and $d^+_X(Y_0) = 1$.

That is $y_1 y_2, x_2 x_1, x_3 x_1 \notin A$. Let e_x and e_y be the arc from Y to X_0 and from Y_0 to X, respectively. One can easily check the note below.

Note 30.4 *Assume that $e_x = e_y = y_1 x_1$ and there are at least two arcs from w_2 to X' in D. Let e be an arc from w_2 to X and $D'' = (V, A'')$ with $A'' = A - \{e_x, e\}$. Then there exists at least one arc from Y' to each initial strong component of $D''[X']$ respectively.*

Claim 30.1 *$D[\{w_2, y_2\}]$ is not a digon.*

Proof. Suppose $D[\{w_2, y_2\}] = C_2$. Since $D \not\supseteq E_3$, $A(D[Y']) = \{y_1 w_1, w_2 y_2, y_2 w_2\}$. Set $D'' = (V, A'')$ with $A'' = A - \{y_2 w_2, w_1 x_1\}$. Note that $D''[Y']$ has only one terminal strong component $\{y_2\}$. By Fact 30.2, $d^+_X(y_2) = 1$, say $e_{y_2} = (y_2, X)_D$. Then $e_{y_2} = e_x$ by Proposition 29.1.

Observe that $I_D = B^- + y_1w_2y_1x_1 + w_1w_1^+$ is an in-branching of D, where $w_1^+ \neq x_1$ as $\lambda(D) \geq 2$. If $w_1^+ \in Y'$, then any initial strong component in $D[X']$ has at least one in-arc from Y' which is different from y_1x_1. This implies that D has an out-branching $O_D = B^+ + y_2w_2 + T_{X'} + \mathcal{P}_{X'}$ by Proposition 8. Thus (O_D, I_D) is a good pair in D, a contradiction.

Otherwise $w_1^+ \in \{x_2, x_3\}$. Set $D^* = (V, A^*)$ with $A^* = A - \{y_2w_2, w_1w_1^+\}$. By Fact 30.2, y_2 and w_1^+ is respectively a terminal and an initial strong component in $D^*[Y']$ and $D^*[X']$ with $d_X^+(y_2) = d_Y^-(w_1^+) = 1$. Since $e_{y_2} = y_2x_1 \neq y_2w_1^+$, D has a good pair by Proposition 29.1, a contradiction. ◊

Claim 30.2 $w_2y_2 \notin A$.

Proof. Suppose to the contrary that $w_2y_2 \in A$.

Case 1: y_1 is adjacent to y_2 in D.

That is $y_2y_1 \in A$. Now $D'[Y']$ has only one terminal strong component Y_0, which implies that $e_y = e_x = y_1x_1$. Then $I_D = B^- + w_2y_2y_1x_1 + w_1w_1^+$ is an in-branching of D, where $w_1^+ \neq x_1$ as $\lambda(D) \geq 2$. If $w_1^+ \in Y'$, then any initial strong component in $D[X']$ has at least one in-arc from Y' which is different from y_1x_1. This implies that D has an out-branching $O_D = B^+ + y_1w_2 + T_{X'} + \mathcal{P}_{X'}$ by Proposition 8. Thus (O_D, I_D) is a good pair in D, a contradiction.

Otherwise $w_1^+ \in \{x_2, x_3\}$. Set $D^* = (V, A^*)$ with $A^* = A - \{y_1w_2, w_1w_1^+\}$. By Fact 30.2, y_1 and w_1^+, respectively, is a terminal and an initial strong component in $D^*[Y']$ and $D^*[X']$ with $d_X^+(y_1) = d_Y^-(w_1^+) = 1$. Since $e_y = y_1x_1 \neq y_1w_1^+$, D has a good pair by Proposition 29.1, a contradiction.

Case 2: y_1 is not adjacent to y_2 in D.

That is $|(y_2, X')_D| \geq 2$.

Subcase 2.1: $|\mathcal{X}| = 1$.

Namely $\mathcal{X} = \{X_0\}$. By Proposition 29.1, $e_y = e_x = y_1x_1$. By Proposition 8, we get $T_{X'}$ of $D'[X']$. Then $(B^+ + y_1x_1 + T_{X'} + w_2^-w_2, B^- + y_1w_2y_2y_2^+ + w_1x_1)$ is a good pair of D, where $w_2^- \neq y_1$ and $y_2^+ \neq w_2$, a contradiction.

Other cases can be derived analogously, and we omit them here. ◊

Claim 30.3 $y_2w_2 \notin A$.

Proof. Suppose $y_2w_2 \in A$. Set $D'' = (V, A'')$ with $A'' = A - \{w_1x_1, y_2w_2\}$. Note that y_2 is a terminal strong component in $D''[Y']$ by Fact 30.2. Moreover $y_2y_1 \notin A$, i.e., y_1 is not adjacent to y_2 in D. Let e_{y_2} be the arc from y_2 to X'.

First assume $e_y = e_x = y_1x_1$. Now y_1 is not in some terminal strong component of $D''[Y']$, which implies that D has a good pair by Proposition 29.3, a contradiction.

Hence $e_y \neq e_x$. Analogously $e_{y_2} \neq e_x$. Then e_y (resp. e_{y_2}) is adjacent to some initial strong component in $D'[X']$ (resp. $D''[X']$) other than X_0 (resp. x_1). This implies that $e_x = w_2x_1$ and $|\mathcal{X}| \geq 2$. If $|\mathcal{X}| \geq 3$, then $|(Y', X')_D| \geq 5$, i.e., $d_{X'}^+(w_2) \geq 3$. By Proposition 29.2, D has a good pair, a contradiction. If $|\mathcal{X}| = 2$, then we get $T_{X'}$ of $D'[X']$ by Proposition 8. It follows that $(B^+ + y_1w_2x_1 + e_{y_2} + T_{X'}, B^- + w_1x_1 + e_y + y_2w_2w_2^+)$ is a good pair of D, where $w_2^+ \neq x_1, y_2$ by $\lambda(D) \geq 2$ and Claim 30.2, a contradiction. ◊

Therefore y_2 is not adjacent to w_2 in D.

Claim 30.4 y_1 is not adjacent to y_2 in D.

Proof. Suppose to the contrary that y_1 is adjacent to y_2, i.e., $y_2y_1 \in A$.

Case 1: $w_2y_1 \in A$.

Now $D'[Y']$ has only one terminal strong component Y_0, that is $e_y = e_x = y_1x_1$ by Proposition 29.1. Then $I_D = B^- + w_2y_1 + y_2y_1x_1 + w_1w_1^+$ is an in-branching of D, where $w_1^+ \neq x_1$ as $\lambda(D) \geq 2$.

If $w_1^+ \in Y'$, then any initial strong component in $D[X']$ has at least one in-arc from Y' which is different from y_1x_1. This implies that D has an out-branching $O_D = B^+ + y_1w_2 + T_{X'} + \mathcal{P}_{X'}$ by Proposition 8. Thus (O_D, I_D) is a good pair in D, a contradiction.

Otherwise $w_1^+ \in \{x_2, x_3\}$. Set $D^* = (V, A^*)$ with $A^* = A - \{y_1w_2, w_1w_1^+\}$. By Fact 30.2, y_1 is the only terminal strong component of $D^*[Y']$ and w_1^+ is an initial strong component in $D^*[X']$ with $d_X^+(y_1) = d_Y^-(w_1^+) = 1$. Since $e_y = y_1x_1 \neq y_1w_1^+$, D has a good pair by Proposition 29.1, a contradiction.

Thus $d_{X'}^+(w_2) \geq 2$.

Case 2: $|\mathcal{X}| = 1$.

By Proposition 29.1, $e_y = e_x = y_1x_1$. By Proposition 8, we get $T_{X'}$. It follows that $(B^+ + y_1x_1 + T_{X'} + w_2^-w_2, B^- + y_2y_1w_2w_2^+ + w_1x_1)$ is a good pair of D, where $w_2^- \neq y_1$ and $w_2^+ \in X'$ as $\lambda(D) \geq 2$ and $d_{X'}^+(w_2) \geq 2$, a contradiction.

Case 3: $|\mathcal{X}| = 2$.

Set $\mathcal{X} = \{X_0, X_1\}$. Recall that $X_0 = x_1$. By Proposition 8, we get $T_{X'}$.

Subcase 3.1: $e_x = e_y = y_1x_1$.

If $(y_2, X_1)_D \neq \emptyset$, say $e' \in (y_2, X_1)_D$, then D has a good pair $(B^+ + y_1x_1 + e' + T_{X'} + w_2^-w_2, B^- + y_2y_1w_2w_2^+ + w_1x_1)$, where $w_2^- \neq y_1$ and $w_2^+ \in X'$, a contradiction.

Otherwise $d_{X_1}^+(w_2) \geq 2$, namley $|X_1| \geq 2$. As $(y_2, X' - V(X_0 \cup X_1))_D \neq \emptyset$, $|X_1| = 2$. This implies that $D[\{w_2\} \cup X_1] \supseteq E_3$, a contradiction to that $D \not\supseteq E_3$.

Subcase 3.2: $e_y \neq e_x$.

By Proposition 29.3, $e_x = w_2x_1$ and $(y_1, X_1)_D = \{e_y\}$. By Proposition 29.4, $(y_2, X_1)_D = \emptyset$ since y_2 is not in any terminal strong component in $D'[Y']$. That is $(w_2, X_1)_D \neq \emptyset$ as $\lambda(D) \geq 2$. By Proposition 29.2, $d_{X'}^+(w_2) = 2$. Note that $|X_1| \leq 2$ as $(y_2, X' - V(X_0 \cup X_1))_D \neq \emptyset$. If $|X_1| = 2$, then $X_1 = w_1x_iw_1$ where $i \in \{2, 3\}$ by Fact 30.1. Let $D'' = D' + w_1x_1 - w_1x_i$. Since $D''[X']$ has only one initial strong component, D has a good pair by Case 2, a contradiction.

Hence $|X_1| = 1$. Set $X_1 = \{a\}$. We first show that $(w_1, X - x_1)_D = \emptyset$. Suppose that $(w_1, X - x_1)_D \neq \emptyset$, w.l.o.g., say $w_1x_2 \in A$. Then $a \neq x_2$. Set $D'' = D' + w_1x_1 - w_1x_2$. By Fact 30.2, x_2 is an initial strong component of $D''[X']$ with $d_{Y'}^-(x_2) = 1$. Let e_{x_2} be the arc from Y' to x_2. Note that $e_{x_2} = y_2x_2$ as $a \neq x_2$ and $d_{X'}^+(w_2) = 2$. Then D has a good pair by Proposition 29.3.

Thus $N_{D-x_1}^+(w_1) \subset Y'$. If $a = w_1$, then $w_1y_2 \in A$, as $D \not\supseteq E_3$. Then change Y' from $Y \cup \{w_2\}$ to $Y \cup \{w_1\}$ and change X' from $X \cup \{w_1\}$ to $X \cup \{w_2\}$. By Claim 30.2, D has a good pair, a contradiction. If $a \in \{x_2, x_3\}$, then $D[X']$ only has one initial strong component. We get $T_{X'}$ of $D[X']$ by Proposition 8. It follows that $(B^+ + y_1w_2a + T_{X'}, B^- + y_2y_1a + w_2x_1 + w_1w_1^+)$ is a good pair of D, where $w_1^+ \in Y'$, a contradiction.

Other cases can be derived analogously, thus we omit them. $\Diamond$

Now $|(y_2, X')_D| \geq 2$. Similar to Note 30.4 we get the note below.

Note 30.5 *Assume that $e_x = e_y = y_1x_1$, $(w_1, X)_D = \{w_1x_1\}$ and there are at least two arcs from y_2 to X' in D. Let e be an arc from y_2 to X and $D'' = (V, A'')$ with $A'' = A - \{e_x, e\}$. Then there exists at least one arc from Y' to each initial strong component of $D''[X']$ respectively.*

Then we distinguish several cases as follows.

Case 1 $e_x = e_y$.

That is $e_x = e_y = y_1x_1$.

Subcase 1.1 $w_1x_2 \in A$.

Now change e_1 from w_1x_1 to w_1x_2, then $X_0 = x_2$, or D has a good pair. If x_1 is not in some component in $\mathcal{X}$, then D has a good pair by Proposition 29.3, a contradiction. Thus we assume that $x_1 \in X_1 \in \mathcal{X}$. Note that $w_1 \in X_1$ as $w_1x_1 \in A$. If $X_1 = C_3$, then $V(X_1) = \{x_1, x_3, w_1\}$. As $\lambda(D) \geq 2$, $|(Y', X_1)_D| \geq 3$, which implies that D has a good pair by Proposition 29.2, a contradiction.

Hence $X_1 = C_2 = x_1w_1x_1$. Note that $|(Y', X)_D| \geq 3$ as $|\mathcal{X}| \geq 2$. Since now $X_0 = x_2$ and $y_1x_1 \in A$, $(Y', X_1) = \{y_1x_1, w_1^-w_1\}$ and $(Y', X_0) = \{x_2^-x_2\}$, where $w_1^-, x_2^- \in Y'$. If $w_2y_1 \in A$, then $x_2^- \neq w_2$ by Proposition 29.3. Since D has no subdigraph with a good pair on 4 vertices, $w_2x_3, y_2w_1, y_2x_2 \in A$. Let $P_+ = y_1x_1w_1 + y_2x_2 + w_2^-w_2 + x_3^-x_3$ and $P_- = y_1w_2x_3 + y_2w_1x_2$, where $w_2^- \neq y_1$ and $x_3^- \neq w_2$ as $D \not\supseteq E_3$. Then $(B^+ + P_+, B^- + P_-)$ is a good pair of D, a contradiction. Henceforth $|(w_2, X')_D| \geq 2$. It follows that $w_2x_3, y_2x_3 \in A$ and $x_3 \in X_2 \in \mathcal{X}$ by Proposition 29.4. Let $P_+ = y_1x_1w_1x_2 + y_2x_3 + w_2^-w_2$ and

$P_- = y_1w_2x_3 + w_1x_1 + y_2y_2^+$ where $w_2^- \neq y_1$ and $y_2^+ \neq x_3$ as y_2 is not adjacent to w_2. Then $(B^+ + P_+, B^- + P_-)$ is a good pair of D, a contradiction.

The case of $w_1x_3 \in A$ can be proved analogously.

Subcase 1.2 $(w_1, X)_D = \{w_1x_1\}$.

That is $|(w_1, Y')_D| \geq 1$. Let $D'' = D - y_1w_2$. By Proposition 8, we get arc-disjoint $\mathcal{P}_{X'}$, $\mathcal{P}_{Y'}$ and $T_{X'}$, $T_{Y'}$ of D''. Let $P_+ = \mathcal{P}_{X'}$ and $P_- = \mathcal{P}_{Y'}$. Then $O_D = B^+ + P_+ + T_{X'} + y_1w_2$ is an out-branching of D. Let $I_D = B^- + P_- + T_{Y'} + w_1w_1^+$, then I_D is arc-disjoint with O_D and $V(I_D) = V$. **We will show that I_D is an in-branching of D**, i.e., there is no digon in $P_- + w_1w_1^+$.

If $w_2y_1 \in A$, then let $P_- = w_2y_1x_1 + y_1y_1^+$ where $y_1^+ \neq w_1$, which is possible by Note 30.5. Henceforth assume $N^+(w_2) \cap Y = \emptyset$, i.e., $|(w_2, X')_D| \geq 2$. Now $|(Y', X')_D| \geq 5$.

If $w_1y_1 \in A$, then $w_1w_1^+ = w_1y_1$. Since $\lambda(D) \geq 2$, there are at least two arcs from Y' to any initial strong component in $D''[X']$. Then let $P_- = y_1x_1 + w_2w_2^+ + y_2y_2^+$ where $w_2^+, y_2^+ \in X'$, such that $P_- \cap P_+ = \emptyset$ as $(y_1, X')_D = \{y_1x_1\}$.

Thus assume $w_1^+ \in \{w_2, y_2\}$, w.l.o.g., say $w_1^+ = w_2$. If $D''[X']$ is strong, then let $P_+ = e$, where e is an arbitrary out-arc of y_2 and $P_- = w_2w_2^+ + y_2y_2^+$ where $w_2^+ \in X$ and $y_2y_2^+$ is an out-arc of y_2 which is different from e as $\lambda(D) \geq 2$.

If $D''[\{x_1, w_1, x_2\}]$ is strong, then there are at most two initial strong components in $D''[X']$. Set $V_0 = \{x_1, w_1, x_2\}$. It follows that y_2 and w_2 to any initial strong component has at least one arc respectively by the fact $|(Y', X')_D| \geq 5$ and $\lambda(D) \geq 2$. If $D''[X']$ has only one initial strong component, then assume that $e \neq y_1x_1$ is an in-arc of the initial strong component from y_2. Let $P_+ = e$ and $P_- = w_2w_2^+ + y_2y_2^+$, where $w_2^+ \neq w_1$ and $y_2y_2^+ \neq e$. Otherwise $D''[X']$ has exactly two initial strong components, i.e., V_0 and x_3. Then let $P_+ = y_2x_3 + w_2w_2^+$ and $P_- = w_2x_3 + y_2y_2^+$ where $w_2^+, y_2^+ \in V_0$. The case when $D''[\{x_1, w_1, x_3\}]$ is strong can be proved analogously.

If $D''[\{x_1, w_1\}]$ is strong, i.e., $D''[\{x_1, w_1\}] = x_1w_1x_1$, then $w_2w_1 \notin A$ as D has no subdigraph on 3 vertices with a good pair. This implies that $P_- + w_1w_2$ has no digon, as required.

Henceforth x_1 is a strong component in $D''[X']$. If $w_2w_1 \notin A$, then $P_- + w_1w_2$ has no digon as required. Thus it is suffices to consider the case of $w_2w_1 \in A$. Note that x_1 is not an initial strong component in $D''[X']$ as $w_1x_1 \in A$. Since $|(w_2, X')_D| \geq 2$, there exists at least one out-arc of w_2 to $\{x_2, x_3\}$, w.l.o.g., say w_2x_2. Let $P_- = w_2x_2 + y_2y_2^+$ and $P_+ = x_2^-x_2 + x_3^-x_3 + w_2w_1$ where $x_2^- \neq w_2$, $y_2^+ \neq x_2$ and $x_3^- \neq y_2$ as $\lambda(D) \geq 2$.

The case when "$e_x \neq e_y$" can be derived analogously, and we omit it here.

The discussion above implies that D has a good pair when there exists an arc from Y to w_i and an arc from w_{3-i} to X, where $i \in [2]$.

Henceforth assume that $e_2 = y_1w_1$ and there is no arc from Y to w_2 or from w_2 to X. This implies that there exists an in-arc e^- of w_2 from X and an out-arc e^+ of w_2 to Y. Set $D' = (V', A')$ with $V' = V - w_2$, $A' = A - e_1$ and $X' = X \cup \{w_1\}$. Let $\mathcal{X}$ be the set of initial strong components in $D'[X']$ and $\mathcal{Y}$ be the set of terminal strong components in $D'[Y]$. Observe that there is at most one strong component S_x in $\mathcal{X}$ with $d_Y^-(S_x) = 1$, and meanwhile for an arbitrary strong component S in $\mathcal{X} \cup \mathcal{Y} - S_x$, we have $d_Y^-(S) \geq 2$ when $S \in \mathcal{X}$ and $d_{X'}^+(S) \geq 2$ when $S \in \mathcal{Y}$. Now by Proposition 8, we get two arc sets $\mathcal{P}_{X'}$ and $\mathcal{P}_Y$ with $\mathcal{P}_{X'} \cap \mathcal{P}_Y = \emptyset$, and $T_{X'}, T_Y$. It follows that D has a good pair $(B^+ + \mathcal{P}_{X'} + T_{X'} + e^-, B^- + \mathcal{P}_Y + T_Y + e_1 + e^+)$, a contradiction. This completes the proof. □

Lemma 31. *Let $D = (V, A)$ be a 2-arc-strong digraph on 9 vertices that contains a digon Q. Assume that D has no subdigraph with a good pair on at least 3 vertices. Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$ and $W = V - V(Q) - X - Y$. Assume that $|X| = |Y| = 2$ and there is an arc $e = st \in A$ such that $s \in Y$ and $t \in W$ (resp. $s \in W$ and $t \in X$). If there are at least three arcs in $D[Y \cup \{t\}]$ (resp. $D[X \cup \{s\}]$), then D has a good pair.*

Proof. Suppose D has no good pair. Set $V(Q) = \{q_1, q_2\}$, $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $W = \{w_1, w_2, w_3\}$. By the digraph duality, it suffices to prove the case when $s \in Y$ and $t \in W$. W.l.o.g., let $s = y_1$ and $t = w_1$.

Set $D[V(Q) \cup Y \cup \{w_1\}] = H$. Since both $D[\{q_1, q_2, y_1\}]$ and $D[\{q_1, q_2, y_2\}]$ has at most three arcs by Proposition 12, there are six possible cases of H by symmetry, which are depicted in Figure 4. We partition each H_i into two parts as follows:

H_i	$B^+(H_i)$	$F^-(H_i)$	a_i
H_1	$q_2q_1y_1w_1 + q_2y_2$	$q_1q_2 + y_1y_2 + w_1y_2$	an out-arc of y_2
H_2	$B^+(H_1)$	$q_1q_2 + y_1y_2w_1$	an out-arc of w_1
H_3	$B^+(H_1)$	$q_1q_2 + w_1y_1y_2$	an out-arc of y_2
H_4	$B_1^+(H_4) = B^+(H_1)$	$F_1^-(H_4) = q_1q_2 + w_1y_2y_1$	a_4^1: an out-arc of y_1
	$B_2^+(H_4) = q_2q_1y_1w_1y_2$	$F_2^-(H_4) = q_1q_2y_2y_1 + w_1$	a_4^2: an out-arc of w_1
H_5	$B_2^+(H_4)$	$q_1q_2y_2 + w_1y_1$	an out-arc of y_1
H_6	$B_1^+(H_6) = B^+(H_1)$	$F_1^-(H_6) = q_1q_2 + y_2w_1y_1$	a_6^1: an out-arc of y_1
	$B_2^+(H_6) = q_1q_2y_2w_1y_1$	$F_2^-(H_6) = q_2q_1y_1w_1 + y_2$	a_6^2: an out-arcs of y_2

Notice that, for each case, $B^+(H_i)$ is always an out-branching of H_i, whose arcs are in blue, $F^-(H_i)$ is always an in-forest of H_i, whose arcs are in red (see Figure 4), and a_i is some arc from $\{y_1, y_2, w_1\}$ to $\{x_1, x_2, w_1, w_2\}$, where $1 \le i \le 6$. And for $i = 4$ and 6, $B^+(H_i)$ (resp. $F^-(H_i)$, a_i) denotes $B_1^+(H_i)$ (resp. $F_1^-(H_i)$, a_i^1) or $B_2^+(H_i)$ (resp. $F_2^-(H_i)$, a_i^2).

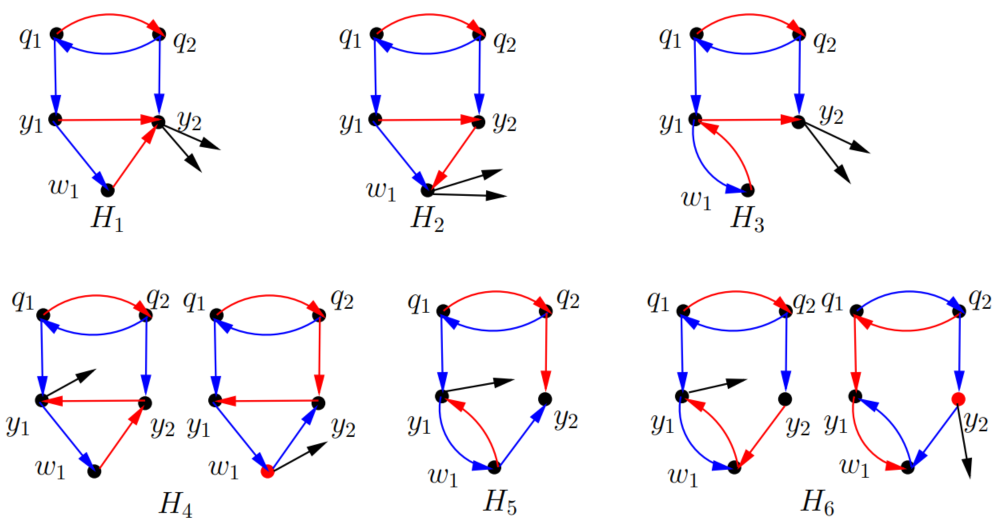

Figure 4: The six possible cases of H. The arcs of $B^+(H_i)$ are in blue and the arcs of $F^-(H_i)$ are in red for $1 \le i \le 6$.

The digraph $H_2 - y_1y_2 + y_2y_1$ is isomorphic to H_2. And, if $H = H_4 - w_1y_2 + w_1y_1$, then for the digon $D[\{y_1, w_1\}] = \hat{Q}$, $N^-(\hat{Q})$ contains the vertices q_1, y_2 and an in-neighbour of w_1 from $\{x_1, x_2, w_2\}$. Thus, $|N^-(\hat{Q})| \ge 3$. Since $|N^+(\hat{Q})| \ge 2$, D has a good pair by Corollary 9, and Lemmas 11 and 30.

Let e_{x_j} be an arc from x_j to Q, for any $j \in [2]$.

Note 31.1 For each $1 \le i \le 6$, if the head of a_i is x_1 or x_2, then $F^-(H_i) + e_{x_1} + e_{x_2} + a_i$ is an in-branching of $D \setminus \{w_2, w_3\}$.

Set $X' = X \cup \{w_2, w_3\}$ and $Y' = Y \cup w_1$.

Note 31.2 Set $D' = (V, A')$ with $A' = A - \{a_i, e_{w_2}, e_{w_3}\}$ and each initial strong component in $D'[X']$ has an in-arc from Y'. If $F^-(H_i) + e_{x_1} + e_{x_2} + a_i + e_{w_2} + e_{w_3}$ is an in-branching of D, then D has a good pair.

Proof. Obviously $B^+(H_i)$ can be extended to an out-branching of D. ◊

Next, we distinguish three cases:

Case 1: $d^+_{Y'}(w_j) \geq 1$ for both $j = 2$ and 3.

Now there exists an out-arc of w_j with head in Y'. Since $\lambda(D) \geq 2$ and $N^+_D(Q) = Y$, each initial strong component in $D[X']$ has at least two in-arcs from Y'. Moreover, since w_j (for both $j = 2$ and 3) has an out-arc with head in Y', denoted by e_{w_j}, for $D' = (V, A')$ with $A' = A - \{a_i, e_{w_2}, e_{w_3}\}$, each initial strong component in $D'[X']$ has at least one in-arc from Y'. Thus, if there exists an arc a_i whose head is x_1 or x_2, then D has a good pair (O_D, I_D) by Notes 31.1 and 31.2, where $I_D = F^-(H_i) + e_{x_1} + e_{x_2} + a_i + e_{w_2} + e_{w_3}$ and O_D is obtained by extending $B^+(H_i)$, for $1 \leq i \leq 6$. Henceforth, we may assume that the head of a_i can only be w_2 or w_3.

Claim 31.1 *If there exists an arc a_i with head w_j and an out-arc of w_j with head x_k, where $1 \leq i \leq 6$, $j \in \{2, 3\}$ and $k \in [2]$, then D has a good pair.*

Proof. W.l.o.g., suppose that there exists an arc a_i with head w_2 and $w_2x_1 \in A$. Let $e_{w_2} = w_2x_1$ and e_{w_3} be an out-arc of w_3 with head in Y'. For $D' = (V, A')$ with $A' = A - e_{w_2}$, if there exists an initial strong component X_0 in $D'[X']$ such that $d^-_{Y'}(X_0) = 1$, then $x_1 \in X_0$ and $w_2 \notin X_0$. Hence, for $D'' = (V, A'')$ with $A'' = A' - \{a_i, e_{w_3}\}$, each initial strong component in $D''[X']$ has at least one in-arc from Y', since the head of a_i is not in X_0 and the head of e_{w_3} is not in X'. Then D has a good pair (O_D, I_D) by Note 31.2, where $I_D = F^-(H_i) + e_{x_1} + e_{x_2} + a_i + w_2x_1 + e_{w_3}$ and O_D is obtained by extending $B^+(H_i)$, for $1 \leq i \leq 6$. The proof is complete. ◊

For $1 \leq i \leq 3$, a_i has two choices e_1 and e_2, both of which are from y_2 (for $i = 1$ and 3) or w_1 (for $i = 2$). Since the head of a_i can only be w_2 or w_3 and there is no multiple arc in D, w.l.o.g., let $e_1 = y_2w_2$ (resp. w_1w_2) and $e_2 = y_2w_3$ (resp. w_1w_3) for $i = 1$ and 3 (resp. $i = 2$).

We first consider H_1 and H_2. Now, $D[Y']$ is a tournament of order 3. If $d^+_{Y'}(w_j) \geq 2$ ($j = 2$ or 3), then $D[Y' \cup \{w_j\}]$ is a tournament of order 4 or contains a subdigraph on 3 vertices with 4 arcs, a contradiction. Hence, $d^+_{Y'}(w_j) = 1$ for both $j = 2$ and 3. Moreover, if $D[\{w_2, w_3\}]$ is a digon, then $D[\{y_2, w_2, w_3\}]$ or $D[\{w_1, w_2, w_3\}]$ is a subdigraph on 3 vertices with 4 arcs, a contradiction. Thus, at least one of the vertices w_2 and w_3, say w_2, has an out-neighbour in X. Let $a_i = e_1$. Now, by Claim 31.1, D has a good pair.

The discussion of other H_i is analogous, where $i \in \{3, 4, 5, 6\}$.

Case 2: $d^+_{Y'}(w_j) = 0$ for both $j = 2$ and 3.

Now each out-arc of w_j has head in X'. Since the proof method of this case is not difficult, we just omit the discussion.

Case 3: $d^+_{Y'}(w_j) = 0$ and $d^+_{Y'}(w_{5-j}) \geq 1$ for $j = 2$ or 3.

W.l.o.g., assume that $d^+_{Y'}(w_2) = 0$ (i.e., $d^+_{X'}(w_2) \geq 2$) and $d^+_{Y'}(w_3) \geq 1$. Let e_{w_3} be an out-arc of w_3 with head in Y'. There are two situations: (a) $w_2x_1, w_2x_2 \in A$ and (b) w.l.o.g., $w_2x_1, w_2w_3 \in A$ and $w_2x_2 \notin A$.

Claim 31.2 *There exists a_i with head in $\{x_1, x_2, w_2\}$, for each $1 \leq i \leq 6$.*

Proof. We omit the proof here. ◊

Next, by Claim 31.2, we distinguish three subcases. Note that, we always let $D' = (V, A')$ with $A' = A - \{e_{w_2}, e_{w_3}\}$ and $D'' = (V, A'')$ with $A'' = A' - \{a_i\}$.

Subcase 3.1: a_i with head x_1.

For w_2 and Situation (a), let $e_{w_2} = w_2x_2$, and for Situation (b), let $e_{w_2} = w_2w_3$. Clearly, for any $1 \leq i \leq 6$, $F^-(H_i) + e_{x_1} + e_{x_2} + a_i + e_{w_2} + e_{w_3} +$ is always an in-branching of D. If there exists an initial strong component X_0 in $D'[X']$ such that $d^-_{Y'}(X_0) = 1$, then for Situation (a) (resp. Situation (b)), $x_2 \in X_0$ (resp. $w_3 \in X_0$) and $w_2 \notin X_0$. Moreover, $x_1 \notin X_0$, since $w_2x_1 \in A(D')$ for both situations. Since the head of a_i is x_1, in D'', each initial strong component of $D''[X']$ has always at least one in-arc from Y'. By Note 31.2, D has a good pair.

We omit the discussion of other two subcases.

The proof is complete. □

Lemma 32. *Let D be a 2-arc-strong digraph on 9 vertices that contains a digon Q. Assume that D has no subdigraph with a good pair on 3 or 4 vertices. Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$ with $X \cap Y = \emptyset$. If $|X| = 2$ and $|Y| = 2$, then D has a good pair.*

Proof. Let $D = (V, A)$, $Q = q_1q_2q_1$ and $W = V - X - Y - V(Q) = \{w_1, w_2, w_3\}$. Set $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. By contradiction, suppose that D has no good pair when $|X| = |Y| = 2$. Let B^+ be an out-tree rooted at q_1 such that $A(B^+) \subseteq \{q_1q_2\} \cup (Q, Y)_D$ and B^- be an in-tree rooted at q_1 such that $A(B^-) \subseteq \{q_2q_1\} \cup (X, Q)_D$.

By Propositions 12 and 13, and Lemma 30, we get the following facts.

Fact 32.1. There is no digon in the induced subdigraph of X or Y.

Fact 32.2. There are exactly two vertices respectively in the disjoint in- and out-neibourhood of any C_2 in D.

Fact 32.3. $|E(W)| \leq 3$.

Fact 32.4. $|(W, D - W)_D| \geq 3$ and $|(D - W, W)_D| \geq 3$. That is at least 2 vertices in W have out-arcs to $X \cup Y$, analogously at least 2 vertices in W have in-arcs from $X \cup Y$.

Let v (resp. u) be the vertex in W which does not have any out-neighbour (resp. in-neighbour) in $X \cup Y$ if it exists, and arbitrarily otherwise. From Claim 19.1 and Note 30.2, we have the fact below by Lemmas 16 and 17.

Fact 32.5 None of the following holds:

1. $|(Y, X)_D| = 4$;
2. X is not independent and there exists a vertex in Y which dominates each vertex in X;
3. Analogously Y is not independent and there exists a vertex in X which is dominated by each vertex in Y;
4. Both X and Y are not independent, say $x_ix_{3-i}, y_jy_{3-j} \in A$, and $y_jx_i, y_{3-j}x_{3-i} \in A$, where $i, j \in [2]$.
5. Both X and Y are not independent, say $x_ix_{3-i}, y_jy_{3-j} \in A$, and $y_jx_{3-i}, y_{3-j}x_i \in A$, where $i, j \in [2]$.

Now assume that there is no good pair in $D - W$, i.e., $|(Y, X)_D| \leq 3$. We will now show the note below.

Note 32.1. There exists an arc from Y to W and an arc from W to X.

Proof. Assume that $(Y, W)_D = \emptyset$, i.e., $|(Y, X)_D| \geq 3$ by Fact 32.1. By Fact 32.5, $|(Y, X)_D| \neq 4$, then $|(Y, X)_D| = 3$, which implies that Y is not an independent set. W.l.o.g., assume $y_1y_2 \in A$. Now $d_X^+(y_1) \geq 1$ and $d_X^+(y_2) \geq 2$, namely y_2 dominates both vertices in X and y_1 dominates at least one vertex in X. This is impossible by Fact 32.5, a contradiction. Thus $(Y, W)_D \neq \emptyset$. Likewise $(W, X)_D \neq \emptyset$. ◊

Claim 32.1 *If $|(Y, X)_D| = 3$ or $|(Y, X)_D| = 2$ with a P_3 in $D[X \cup Y]$, then D has a good pair.*

Proof. Suppose that D has no good pair. If $|(Y, X)_D| = 3$, then w.l.o.g., assume $(Y, X)_D = \{y_1x_1, y_1x_2, y_2x_2\}$. By Fact 32.5, both X and Y are independent sets. Let $P^1 = y_1x_1 + y_2x_2$ and $P^2 = y_1x_2$. Note that $V(B^+ + P^1) = X \cup Y$ and $V(B^- + P^2) = X \cup Y - y_2$.

We do the similar process to the case when $|(Y, X)_D| = 2$ with a P_3 in $D[X \cup Y]$. Since there exists a P_3 in $D[X \cup Y]$ when $|(Y, X)_D| = 2$, exactly one of X and Y is not an independent set by Fact 32.5. W.l.o.g., assume $x_2x_1 \in A$. If $(Y, X)_D = \{y_1x_1, y_2x_2\}$, then let $P^1 = y_2x_2x_1$ and $P^2 = y_1x_1$. If $(Y, X)_D = \{y_1x_2, y_2x_2\}$, then let $P^1 = y_2x_2x_1$ and $P^2 = y_1x_2$. Since $\lambda(D) \geq 2$, y_2 has an out-arc to W, say y_2w_1. We want to add y_2 to B^- by y_2w_1.

If there exists an arc from w_1 to $X \cup Y - y_2$, say e_1, then let $P_- = y_2w_1 + e_1$. If there exists an arc from $X \cup Y - y_2$ to w_1, say e_2, then let $P_+ = e_2$. Note that $(B^+ + P^1 + P_+, B^- + P^2 + P_-)$ is a good pair of $D \setminus \{w_2, w_3\}$, which implies that D has a good pair by Lemma 10 as $\lambda(D) \geq 2$, a contradiction.

Hence assume that at least one of e_1 and e_2 does not exist. We distinguish several cases below.

Case 1: $N^+(w_1) \subseteq W$.

That is $w_1w_2, w_1w_3 \in A$. Since $|E(W)| \leq 3$, there is at least one more arc in $E(W)$ other than these two. This implies that at least one of w_2 and w_3 has both in- and out-neighbours in $X \cup Y$, say w_2. Let e_1 be an arc from $X \cup Y$ to w_2.

First assume that $(w_2, X \cup Y - y_2)_D \neq \emptyset$ and $(D - w_3, w_1)_D \neq \emptyset$, say $e_2 \in (w_2, X \cup Y - y_2)_D$ and $e_3 \in (D - w_3, w_1)_D$. Let $P_+ = e_1 + e_3$ and $P_- = y_2w_1w_2 + e_2$. Note that $(B^+ + P^1 + P_+, B^- + P^2 + P_-)$ is a good pair of $D - w_3$, which implies that D has a good pair as $\lambda(D) \geq 2$, a contradiction.

Next assume that $w_2y_2 \in A$ and $N^+_{D-y_2}(w_2) \subseteq W$. If $w_2w_1 \in A$, then w_3 has at least one in-neighbour and two out-neighbours in $X \cup Y$ as $|E(W)| \leq 3$, which implies that D has a good pair by the discussion above, a contradiction. That is $w_2w_3 \in A$. Now w_3 has at least one out-neighbour $w_3^+ \neq y_2$ in $X \cup Y$ and w_1 has at least one in-neighbour w_1^- in $X \cup Y$. Let $P_+ = w_1^-w_1w_3 + e_1$ and $P_- = y_2w_1w_2w_3$. It follows that $(B^+ + P^1 + P_+, B^- + P^2 + P_-)$ is a good pair of D, a contradiction.

Henceforth, assume $w_3w_1 \in A$. Since $|E(W)| \leq 3$, w_2 has at least one out-neighbour $w_2^+ \neq y_2$ in $X \cup Y$ and w_3 respectively has an in-neighbour w_3^- and an out-neighbour w_3^+ in $X \cup Y$. Let $P_+ = e_1 + w_3^-w_3w_1$ and $P_- = y_2w_1w_2w_2^+ + w_3w_3^+$. It follows that $(B^+ + P^1 + P_+, B^- + P^2 + P_-)$ is a good pair of D, a contradiction.

We omit the proof of other two analogous cases here. $\Diamond$

Claim 32.2 *At least one of X and Y is an independent set.*

Proof. W.l.o.g., assume $q_1y_1, q_2y_2 \in A$. Suppose that both X and Y are not independent sets, i.e., $|E(X)| = |E(Y)| = 1$ by Fact 32.1. W.l.o.g., assume $y_1y_2, x_1x_2 \in A$. By Lemma 31, $|[Y, w_i]_D| \leq 1$ and $|[X, w_i]_D| \leq 1$, for arbitrary $w_i \in W$.

Case 1: $|(Y, X)_D| = 0$.

Since $\lambda(D) \geq 2$, $|(Y, W)_D| = 3$ with $|N^+_W(y_1)| = 1$ and $|N^+_W(y_2)| = 2$ by the fact that $|[Y, w_i]_D| \leq 1$. W.l.o.g., assume $y_1w_1, y_2w_2, y_2w_3 \in A$. Note that $(W, Y)_D = \emptyset$. Likewise, $|(W, X)_D| = 3$ and $(X, W)_D = \emptyset$ with $|N^-_W(x_1)| = 2$ and $|N^-_W(x_2)| = 1$. This implies that $D[W] = C_3$ by Fact 32.3. Set $C_3 = w_1w_{j+1}w_{4-j}w_1$, where $j \in \{1, 2\}$.

If $w_1x_1 \notin A$, then $w_2x_1, w_3x_1, w_1x_2 \in A$. It follows that $(B^+ + y_1w_1w_{j+1} + y_2w_{4-j}x_1x_2, B^- + y_1y_2w_{j+1}w_{4-j}w_1x_2)$ is a good pair of D, a contradiction. If $w_1x_1 \in A$, then set $w_ix_1 \in A$, where $i \in \{2, 3\}$. It follows that $(B^+ + y_1w_1w_{j+1} + y_2w_{4-j} + w_ix_1x_2, B^- + y_1y_2w_{j+1}w_{4-j}w_1x_1)$ is a good pair of D, a contradiction.

We omit the discussion of the case when "$|(Y, X)_D| = 1$". $\Diamond$

Claim 32.3 *If $|E(X)| + |E(Y)| = 1$, then $|(Y, X)_D| \leq 1$.*

Proof. Suppose to the contrary that $|(Y, X)_D| = 2$ by Claim 32.1. W.l.o.g., assume $E(X) = \{x_2x_1\}$. By Claim 32.1, assume $(Y, X)_D = \{y_1x_1, y_2x_1\}$. Let $P^1 = x_2x_1$ and $P^2 = y_1x_1 + y_2x_1$. Then $B^- + P^2$ is an in-tree containing Y but $B^+ + P^1$ is not an out-tree. Since $|[W, X \cup Y]_D| \geq 4$ and $|W| = 3$, at least one vertex in W is adjacent to two vertices in $X \cup Y$, say w_1.

Case 1: $y_1w_1, y_2w_1 \in A$.

Subcase 1.1: $w_1x_2 \in A$.

If w_1 has an out-neighbour w_1^+ in $X \cup Y$ such that $w_1^+ \neq x_2$, then $(B^+ + P^1 + y_2w_1x_2, B^- + P^2 + w_1w_1^+)$ is a good pair of $D - \{w_2, w_3\}$. This implies that D has a good pair by Lemma 10, a contradiction. Otherwise, $N^+_{D-x_2}(w_1) \subset W$. W.l.o.g., assume $w_1w_2 \in A$. If $(w_2, X \cup Y)_D \neq \emptyset$ and $(X \cup Y, w_2)_D \neq \emptyset$, then set $e_1 \in (w_2, X \cup Y)_D$ and $e_2 \in (X \cup Y, w_2)_D$. Let $P_+ = y_2w_1x_2 + e_2$ and $P_- = w_1w_2 + e_1$. Then $(B^+ + P^1 + P_+, B^- + P^2 + P_-)$ is a good pair of $D - w_3$, a contradiction. Now we discuss situations when e_1 or e_2 does not exist.

First assume that e_1 does not exist, namely $N^+(w_2) \subset W$. Then $E(W) = \{w_1w_2, w_2w_3, w_2w_1\}$. Let $P_+ = y_2w_1x_2 + w_2^-w_2 + w_3^-w_3$ and $P_- = w_1w_2w_3w_3^+$, where $w_2^-, w_3^-, w_3^+ \in X \cup Y$ by Fact 32.3. It follows that $(B^+ + P^1 + P_+, B^- + P^2 + P_-)$ is a good pair of D, a contradiction.

Next assume that e_1 exists, but e_2 does not, that is $w_3w_2 \in A$. Let $P_+ = y_2w_1x_2 + w_3^-w_3w_2$ and $P_- = w_1w_2 + e_1 + w_3w_3^+$, where $w_3^-, w_3^+ \neq w_2$ as $\lambda(D) \geq 2$. It follows that $(B^+ + P^1 + P_+, B^- + P^2 + P_-)$ is a good pair of D, a contradiction.

Since other cases can be derived analogously, we just omit them. ◊

Claim 32.4 *Both X and Y are independent sets.*

Proof. W.l.o.g., assume $q_1y_1, q_2y_2 \in A$. Suppose that one of X and Y is not an independent set by Claim 32.2. W.l.o.g., assume that $|E(Y)| = 1$ by Fact 32.1, say $y_1y_2 \in A$. Now $|(Y,X)_D| \leq 1$ by Claim 32.3.

Case 1: $|(Y,X)_D| = 1$.

Subcase 1.1: $(Y,X)_D = \{y_1x_i\},\ i \in \{1,2\}$.

W.l.o.g., assume $i = 2$. Then $N^-(x_1) \subseteq W$, $d_W^-(x_2) \geq 1$ and $d_W^+(y_2) \geq 2$. Assume $w_1x_1, w_2x_1 \in A$, then $|(y_2, \{w_1,w_2\})_D| \geq 1$, w.l.o.g., say $y_2w_1 \in A$.

First assume $w_1x_2 \in A$. Let $B_{q_2}^+ = q_2q_1y_1y_2w_1x_2$ and $B_{y_2}^-$ be an in-tree rooted at y_2 such that $A(B_{y_2}^-) \subseteq \{q_1q_2y_2, y_1x_2, w_1x_1\} \cup (X,Q)_D$. If $N_{D-\{x_1,w_2\}}^-(w_3) \neq \emptyset$, say $w_3^- \in N_{D-\{x_1,w_2\}}^-(w_3)$, then D has a good pair $(B_{q_2}^+ + w_3^-w_3 + w_2^-w_2x_1, B_{y_2}^- + w_3w_3^+ + w_2w_2^+)$, where $w_2^- \neq w_3$, $w_3^+ \neq w_2$ and $w_2^+ \neq x_1$ as $\lambda(D) \geq 2$, a contradiction. Hence $x_1w_3, w_2w_3 \in A$. It follows that $(B_{q_2}^+ + w_2^-w_2x_1w_3, B_{y_2}^- + w_2w_3w_3^+)$ is a good pair of D, where $w_2^- \neq x_1, w_3$ and $w_3^+ \neq x_1, w_2$ as $D \not\supseteq E_3$, a contradiction.

Next assume that $w_1x_2 \notin A$ but $w_2x_2 \in A$. Now there exists at least one arc from y_2 to $\{w_2,w_3\}$. By the discussion above, $y_2w_3 \in A$. Let $B_{q_2}^+ = q_2q_1y_1y_2w_1x_1 + y_2w_3$ and $B_{y_2}^-$ be an in-tree rooted at y_2 such that $A(B_{y_2}^-) \subseteq \{q_1q_2y_2, y_1x_2, w_2x_1\} \cup (X,Q)_D$. It follows that $(B_{q_2}^+ + w_2^-w_2x_2, B_{y_2}^- + w_1w_1^+ + w_3w_3^+)$ is a good pair of D, where $w_2^- \neq w_1, x_2$, $w_1^+ \neq x_1$ and $w_3^+ \neq w_2$ by $\lambda(D) \geq 2$ and Lemma 31, a contradiction.

Henceforth assume that $w_1x_2, w_2x_2 \notin A$ but $w_3x_2 \in A$.

A. $N^-(w_3) \cap \{y_1,y_2,x_2\} \neq \emptyset$.

Set $w_3^- \in \{y_1,y_2,x_2\}$. If $w_2w_1 \in A$, then D has a good pair $(B^+ + y_1x_2 + w_3^-w_3 + w_2^-w_2w_1x_1, B^- + w_3x_2 + w_2x_1 + y_1y_2w_1w_1^+)$, where $w_2^- \neq w_1, x_1$ and $w_1^+ \notin Y$ as $D \not\supseteq E_3$ and Lemma 31, a contradiction. Thus $w_2w_1 \notin A$. Let w_1^- and w_2^- respectively be an in-neighbour of w_1 and an in-neighbour of w_2 such that $w_1^- \neq y_2$ and $w_2^- \neq x_1$ as $\lambda(D) \geq 2$. Since $D \not\supseteq E_3$, at least one of w_1^- and w_2^- is not in $\{x_1, w_1, w_2\}$. It follows that $(B^+ + y_1x_2 + w_3^-w_3 + w_1^-w_1 + w_2^-w_2x_1, B^- + w_3x_2 + y_1y_2w_1x_1 + w_2w_2^+)$ is a good pair of D, where $w_2^+ \neq x_1$, a contradiction.

B. $y_2w_2 \in A$ and $d_{\{x_1,w_1,w_2\}}^-(w_3) \geq 2$.

Since $d_{\{x_1,w_1,w_2\}}^-(w_3) \geq 2$, assume that $w_iw_3 \in A$, where $i \in [2]$. We first show that $w_3w_i \notin A$. If $x_1w_3 \in A$, then $w_3w_i \notin A$ as $D \not\supseteq E_3$. If $w_1w_3, w_2w_3 \in A$, then $w_3w_i \notin A$ by Fact 32.3. Let $B_{q_2}^+ = q_2q_1y_1y_2w_ix_1 + y_2w_{3-i}$ and $B_{y_2}^-$ be an in-tree rooted at y_2 such that $A(B_{y_2}^-) \subseteq \{q_1q_2y_2, y_1x_2, w_{3-i}x_1\} \cup (X,Q)_D$. It follows that $(B_{q_2}^+ + w_3^-w_3x_2, B_{y_2}^- + w_iw_3w_3^+)$ is a good pair of D, where $w_3^- \neq w_i$ and $w_3^+ \neq x_2$ as $\lambda(D) \geq 2$, a contradiction.

We omit the proof of other analogous cases. ◊

Now both X and Y are independent sets.

Claim 32.5 *If both X and Y are independent sets and $(Y,X)_D = \{y_1x_1, y_2x_2\}$, then D has a good pair.*

Proof. Suppose that D has no good pair. Now $|[X \cup Y, W]_D| \geq 4$. Since $|W| = 3$, at least one vertex in W is adjacent to at least two vertices in $X \cup Y$, w.l.o.g., say w_1.

Case 1: $y_1w_1, y_2w_1 \in A$ ($w_1x_1, w_1x_2 \in A$).

By the digraph duality, it suffices to prove the case of $y_1w_1, y_2w_1 \in A$. By Lemma 31, there is no arc from w_1 to Y.

Subcase 1.1: $w_1x_1, w_1x_2 \in A$.

Let $P_+ = y_1x_1 + y_2w_1x_2$ and $P_- = y_1w_1x_1 + y_2x_2$. Then $(B^+ + P_+, B^- + P_-)$ is a good pair of $D - \{w_2, w_3\}$, which implies that D has a good pair by Lemma 10, a contradiction.

Subcase 1.2: $w_1x_2 \in A$ but $w_1x_1 \notin A$ ($w_1x_1 \in A$ but $w_1x_2 \notin A$).

By the digraph duality, it suffices to prove the case when $w_1x_2 \in A$ but $w_1x_1 \notin A$. Then $N^+_{D-x_2}(w_1) \subset W$. Since $(W, x_1)_D \neq \emptyset$, w.l.o.g., assume $w_2x_1 \in A$. Let $P^1 = y_1x_1 + y_2w_1x_2$ and $P^2 = y_2x_2 + y_1w_1 + w_2x_1$.

First assume $w_1w_2 \in A$. If $(X\cup Y, w_2)_D \neq \emptyset$, say $w_2^-w_2 \in (X\cup Y, w_2)_D$, then $(B^+ + P^1 + w_2^-w_2, B^- + P^2 + w_1w_2)$ is a good pair of $D-w_3$, a contradiction. Hence $w_3w_2 \in A$. Let $P_+ = w_3^-w_3w_2$ and $P_- = w_1w_2 + w_3w_3^+$, where $w_3^- \in X \cup Y$ and $w_3^+ \neq w_2$. It follows that $(B^+ + P_+ + P^1, B^- + P_- + P^2)$ is a good pair of D, a contradiction. Thus $w_1w_3 \in A$ but $w_1w_2 \notin A$. Then $(B^+ + P^2 + w_2^-w_2 + w_1w_3, B^- + P^1 + w_2w_2^+ + w_3w_3^+)$ is a good pair of D, where $w_2^-, w_2^+ \neq x_1$ and $w_3^+ \neq w_2$ as $\lambda(D) \geq 2$, a contradiction.

We just omit the discussion of other cases. $\Diamond$

Claim 32.6 *If both X and Y are independent sets and $(Y, X)_D = \{y_1x_1, y_2x_1\}$, then D has a good pair.*

Proof. Suppose that D has no good pair. Now $d^-_W(x_2) \geq 2$ and $d^+_W(y_j) \geq 1$ for any $j \in [2]$. W.l.o.g., assume $w_1x_2, w_2x_2 \in A$.

Case 1: $d^-_Y(w_i) \geq 1$ for any $i \in [2]$.

W.l.o.g., assume $y_1w_1, y_2w_2 \in A$. Since $D \not\supseteq E_3$, $D[\{w_1, w_2\}] \neq C_2$, w.l.o.g., say $w_1w_2 \notin A$. If $w_1w_3, w_3w_2 \in A$, then $(B^+ + y_1w_1w_3w_2x_2 + y_2x_1, B^- + y_1x_1 + y_2w_2w_2^+ + w_1x_1 + w_3w_3^+)$ is a good pair of D, where $w_2^+ \neq x_2$ and $w_3^+ \neq w_2$ as $\lambda(D) \geq 2$, a contradiction. If $w_1w_3, w_3w_2 \notin A$, then $(B^+ + y_1w_1x_2 + y_2x_1 + w_2^-w_2, B^- + y_1x_1 + y_2w_2x_2 + w_1w_1^+)$ is a good pair of D, where $w_2^- \neq y_2$ and $w_1^+ \neq x_2$, a contradiction. Otherwise D has a good pair $(B^+ + y_1w_1x_2 + y_2x_1 + w_3^-w_3 + w_2^-w_2, B^- + y_1x_1 + y_2w_2x_2 + w_1w_1^+ + w_3w_3^+)$, where $w_2^- \neq y_2, w_1^+ \neq x_2, w_3^- \neq w_2$ and $w_3^+ \neq w_1$ as $\lambda(D) \geq 2$, a contradiction.

Then at least one of w_1 and w_2 has no in-neighbours in Y, w.l.o.g., say w_1, that is $N^-(w_1) \cap W \neq \emptyset$ by Lemma 31.

The discussion of other three analogous cases are omitted here. $\Diamond$

By the digraph duality, we also prove the case of $(Y, X)_D = \{y_1x_1, y_1x_2\}$. Now $|(Y, X)_D| \leq 1$.

Claim 32.7 *If both X and Y are independent sets, then $(Y, X)_D = \emptyset$.*

Proof. Suppose $|(Y, X)_D| = 1$. W.l.o.g., assume $y_1x_1 \in (Y, X)_D$. Now $|N^+_W(Y)| \geq 2$ and $|N^-_W(X)| \geq 2$ as $\lambda(D) \geq 2$.

Case 1: $|N^+_W(Y)| = 2$ ($|N^-_W(X)| = 2$).

By the digraph duality, it suffices to prove the case of $|N^+_W(Y)| = 2$. W.l.o.g., assume $y_1w_1, y_2w_1, y_2w_2 \in A$.

Subcase 1.1: $w_1x_2 \in A$.

A. $w_2x_2, w_3x_1 \in A$.

We find a good pair $(B^+ + P_+, B^- + P_-)$ of D as follows, a contradiction.

Case	P_+, P_-	Notation
$w_2w_3 \notin A$	$y_1w_1y_2w_2x_2 + w_3^-w_3x_1, y_1x_1 + y_2w_1x_2 + w_2w_2^+ + w_3w_3^+$	$w_3^-, w_3^+ \neq x_1; w_2^+ \neq x_2$
$w_2w_3 \in A, w_1y_2 \notin A$	$y_1w_1x_2 + y_2w_2w_3x_1, y_1x_1 + y_2w_1w_1^+ + w_2x_2 + w_3w_3^+$	$w_1^+ \neq x_2, w_3^+ \neq x_1$
$w_2w_3, w_1y_2 \in A$	$y_1x_1 + y_2w_1x_2 + w_2^-w_2 + w_3^-w_3, y_1w_1y_2w_2w_3x_1$	$w_2^- \neq y_2, w_3^- \neq w_2$

Since other cases can be derived analogously, we just omit them. $\Diamond$

Now we are ready to finish the proof of Lemma 32. Since $|(Y, X)_D| = 0$ and both X and Y are independent sets, $|(Y, W)_D| \geq 4$ and $|(W, X)_D| \geq 4$. By Lemma 31, for any $w \in W$, $d^-_Y(w) \leq 2$ and $d^+_X(w) \leq 2$. This implies that there exist vertices w_i and w_j in W such that $y_1w_i, y_2w_i, w_jx_1, w_jx_2 \in A$. Note that it is possible that $w_i = w_j$.

Case 1: At least two vertices in W have two in-neighbours in Y.

W.l.o.g., assume $y_1w_1, y_2w_1, y_1w_2, y_2w_2 \in A$. Note that $|(Y, W)_D| \geq 5$.

Subcase 1.1: $j \in \{1, 2\}$.

W.l.o.g., assume $j = 2$.

First assume $d^+_X(w_1) \geq 1$, w.l.o.g., say $w_1x_1 \in A$. Then w_1 has an out-neighbour $w_1^+ \neq x_1, y_2$ as $\lambda(D) \geq 2$ and Lemma 31. Set $P_+ = y_1w_1x_1 + y_2w_2x_2$ and $P_- = y_1w_2x_1 + y_2w_1w_1^+$. If $w_1^+ \neq w_3$, then $(B^+ + P_+, B^- + P_-)$ is a good pair of $D - w_3$, a contradiction. Hence $w_1^+ = w_3$. This implies that $d^+_X(w_3) \geq 1$. It follows that $(B^+ + P_+ + w_3^- w_3, B^- + P_- + w_3w_3^+)$ is a good pair of D, where $w_3^- \neq w_1$ as $\lambda(D) \geq 2$ and $w_3^+ \in X$, a contradiction.

Next assume $d^+_X(w_1) = 0$. This implies that $N^+(w_1) \subset W$ and $d^+_X(w_3) = 2$ by Lemma 31, i.e., $w_1w_2, w_1w_3, w_3x_1, w_3x_2 \in A$. It follows that $(B^+ + y_1w_1w_3x_1 + y_2w_2x_2, B^- + y_1w_2x_1 + y_2w_1w_2 + w_3x_2)$ is a good pair of D, a contradiction.

Subcase 1.2: $d^+_X(w_1) = d^+_X(w_2) = 1$.

By Lemma 31, $d^+_X(w_3) = 2$, i.e., $w_3x_1, w_3x_2 \in A$. W.l.o.g., assume $w_1x_1, w_2x_2 \in A$. It follows that $(B^+ + y_1w_1 + y_2w_2x_2 + w_3^- w_3x_1, B^- + w_3x_2 + y_2w_1x_1 + y_1w_2w_2^+)$ is a good pair of D, where $w_3^- \notin \{w_2\} \cup X$ and $w_2^+ \neq x_2, y_1$ as $\lambda(D) \geq 2$ and Lemma 31, a contradiction.

Case 2: Only w_i in W has $d^-_Y(w_i) = 2$.

W.l.o.g., assume $w_i = w_1$, i.e., $y_1w_1, y_2w_1 \in A$. Note that $|(Y, W)_D| = 4$ and $d^-_Y(w_2) = d^-_Y(w_3) = 1$, w.l.o.g., say $y_1w_2, y_2w_3 \in A$.

Subcase 2.1: $d^+_X(w_2) = d^+_X(w_3) = 2$.

That is $w_2x_1, w_2x_2, w_3x_1, w_3x_2 \in A$. It follows that $(B^+ + y_2w_3x_1 + y_1w_1 + w_2^- w_2x_2, B^- + y_1w_2x_1 + w_3x_2 + y_2w_1w_1^+)$ is a good pair of D, where $w_2^- \neq y_1, x_2$ and $w_1^+ \neq w_2, y_2$ as $\lambda(D) \geq 2$ and Lemma 31, a contradiction.

We omit the discussion of other two subcases here.

This compeltes the proof of Lemma 32. □

Since the idea of the proof of Proposition 33 and Lemma 34 are respectively similar to Proposition 27 and 28, we only give their contents and omit proofs here.

Proposition 33. *Let $D = (V, A)$ be a 2-arc-strong oriented graph on 9 vertices without good pair. Assume that D does not have K_4 as a subdigraph. If D have two cycles C^1 and C^2 with $C^1 \cap C^2 = \emptyset$ which cover 8 vertices, then D contains a Hamilton dipath.*

Lemma 34. *Let $D = (V, A)$ be a 2-arc-strong digraph on 9 vertices without good pair. If D is an oriented graph without K_4 as a subdigraph, then D has a Hamilton dipath.*

Now we are ready to show Theorem 6. For convenience, we restate it here.

Theorem 6. *Every 2-arc-strong digraph on 9 vertices has a good pair.*

Proof. By contradiction, suppose that D has no good pair.

Claim 6.1 *No subdigraph of D of order at least 4 has a good pair.*

Proof. Suppose that a subdigraph Q of D of order at least 4 has a good pair. If $|Q| \geq 5 = n - 4$, then D has a good pair by Lemmas 10, 16 and 17. Thus, assume $|Q| = 4 = n - 5$. Set $X = N^-_D(Q)$ and $Y = N^+_D(Q)$.

If $X \cap Y \neq \emptyset$, then there is a vertex v in $D - Q$ which has both an in-neighbour and an out-neighbour in Q. By Lemma 10, $D[V(Q) \cup \{v\}]$ has a good pair. Since $|Q \cup \{v\}| = 5 = n - 4$, D has a good pair by Lemma 17. Henceforth, assume $X \cap Y = \emptyset$.

If $|X| \geq 2$ or $|Y| \geq 2$, then D has a good pair by Lemma 18. Therefore, $|X| = |Y| = 1$. Set $Q = \{q_1, q_2, q_3, q_4\}$, $X = \{x\}$, $Y = \{y\}$ and $V - V(Q) - X - Y = W = \{w_1, w_2, w_3\}$. Moreover, let (B^+_s, B^-_t) be a good pair of Q, where $s, t \in V(Q)$. Since $\lambda(D) \geq 2$, $N^-_D(Q) = \{x\}$ and $N^+_D(Q) = \{y\}$, there are at least two out-neighbours of x and at least two in-neighbours of y in Q.

Note 6.1 $xs, ty \notin A$.

Proof. By the digraph duality, it suffices to prove the case of $xs \notin A$. Suppose $xs \in A$. Let $e_x(\neq xs)$ be another arc from x to Q. Then $(xs + B_s^+, e_x + B_t^-)$ is a good pair of $D[V(Q) \cup \{x\}]$, which implies that D has a good pair since $|Q \cup \{x\}| = 5$, a contradiciton. ♢

Next, we distinguish two cases:

Case 1: $s = t$.

W.l.o.g., suppose $s = t = q_1$. By Note 6.1, neither x nor y is adjacent to q_1. Thus, $N_D(q_1) \subseteq \{q_2, q_3, q_4\}$. Since $\lambda(D) \geq 2$, $d^+(q_1) \geq 2$ and $d^-(q_1) \geq 2$, which implies that there is a vertex $q_i \in \{q_2, q_3, q_4\}$ such that $D[\{q_1, q_i\}]$ is a digon. W.l.o.g., let $i = 2$. By Proposition 7, Q has a good pair $(B_{q_2}^+, B_{q_2}^-)$. Again by Note 6.1, neither x nor y is adjacent to q_2. Thus, $N_D(\{q_1, q_2\}) = \{q_3, q_4\}$ and $N^+(x) \cap Q = N^-(y) \cap Q = \{q_3, q_4\}$. Note that, it is impossible that $D[\{q_i, q_j\}]$ is a digon, where $i = 1$ or 2 and $j = 3$ or 4. Otherwise, by Proposition 7, Q has a good pair $(B_{q_j}^+, B_{q_j}^-)$. This implies that $xq_j \in A$ and $q_j y \in A$ ($j = 3$ and 4), contradicting Note 6.1. Hence, one of the vertices q_3 and q_4 is an out-neighbour of q_1 (resp. q_2) and the other an in-neighbour of q_1 (resp. q_2). By symmetry, assume $q_1q_3, q_4q_1 \in A$.

Now, if q_2q_3, $q_4q_2 \in A$, then $D[V(Q) \cup \{x\}]$ has a good pair $(B_x^+, B_{q_3}^-)$ with $B_x^+ = xq_4q_2q_1q_3$ and $B_{q_3}^- = q_4q_1q_2q_3 + xq_3$. Since $|Q \cup \{x\}| = 5 = n - 4$, by Lemma 17, D has a good pair, a contradiction. Hence, q_3q_2, $q_2q_4 \in A$. Let $H = Q \cup X \cup Y$. Figure 5 shows an out-branching $\hat{B}_x^+ = xq_4q_1q_2 + xq_3y$ of H and an in-branching $\hat{B}_y^- = q_1q_3q_2q_4y$ of $H \setminus \{x\}$. If $yx \in A$, then $(\hat{B}_x^+, \hat{B}_y^- + yx)$ is a good pair of H, which implies that D has a good pair by Lemma 16, a contradiction. Thus, assume $yx \notin A$.

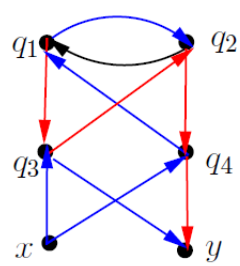

Figure 5: The arcs of $\hat{B}_x^+$ are in blue and the arcs of $\hat{B}_y^-$ are in red.

Since $N_D^-(Q) = \{x\}$ and $N_D^+(Q) = \{y\}$, Q contains no out-neighbours of y or in-neighbours of x. Moreover, since $yx \notin A$, $N_D^+(y) \subseteq W = \{w_1, w_2, w_3\}$ and $N_D^-(x) \subseteq W$. Because $\lambda(D) \geq 2$, $d^+(y) \geq 2$ and $d^-(x) \geq 2$. Thus, there is a vertex $w_i \in W$ such that $yw_i, w_ix \in A$. W.l.o.g., let $i = 1$, i.e., $yw_1, w_1x \in A$.

If $xw_1 \in A$, then $(\hat{B}_x^+ + xw_1, \hat{B}_y^- + yw_1x)$ is a good pair of $D[V(H) \cup \{w_1\}]$, which implies that D has a good pair by Lemma 10, a contradiction. By the digraph duality, we also get a contradiction when $w_1y \in A$. Therefore, assume $xw_1, w_1y \notin A$.

Now, suppose that there is another vertex $w_i(\neq w_1) \in W$ such that $yw_i, w_ix \in A$. W.l.o.g., let $i = 2$. If $w_jw_{3-j} \in A$ ($j = 1$ or 2), then $(w_jx + \hat{B}_x^+ + yw_{3-j}, \hat{B}_y^- + yw_jw_{3-j}x)$ is a good pair of $D[V(H) \cup \{w_1, w_2\}]$, which implies that D has a good pair by Lemma 10, a contradiction. Hence, w_1 and w_2 are not adjacent. Note that, $xw_1, w_1y \notin A$ and for a similar reason, $xw_2, w_2y \notin A$. For both $i = 1$ and 2, since $d^+(w_i) \geq 2$ and $d^-(w_i) \geq 2$, $D[\{w_i, w_3\}]$ is a digon. Then, $(w_1w_3w_2x + \hat{B}_x^+, \hat{B}_y^- + yw_1x + w_2w_3w_1)$ is a good pair of D, a contradiction. Thus, w_1 is the unique vertex which is both an out-neighbour of y and an in-neighbour of x. Now, w.l.o.g., suppose that $yw_2 \in A$ and $w_3x \in A$, and moreover, $w_2x \notin A, yw_3 \notin A$.

Next, consider the following two situations.

(A): $w_2y \in A$.

Since $d^+(w_2) \geq 2$ and $w_2x \notin A$, at least one of the vertices w_1 and w_3 is an out-neighbour of w_2. If $w_2w_1 \in A$, then $(\hat{B}_x^+ + yw_2w_1, \hat{B}_y^- + yw_1x + w_2y)$ is a good pair of $D - \{w_3\}$, which implies that D has a good pair by Lemma 10, a contradiction. If $w_2w_1 \notin A$ but $w_2w_3 \in A$, $w_3w_1 \in A$ since $d^-(w_1) \geq 2$ and $xw_1 \notin A$. Then, $(\hat{B}_x^+ + yw_2w_3w_1, \hat{B}_y^- + yw_1x + w_2y + w_3x)$ is a good pair of D, a contradiction.

(B): $w_2y \notin A$.

Since $d^+(w_2) \geq 2$ and $w_2y, w_2x \notin A$, $w_2w_1, w_2w_3 \in A$. Moreover, since $d^+(w_1) \geq 2$ and $w_1y \notin A$, at least one of the vertices w_2 and w_3 is an out-neighbour of w_1. If $w_1w_2 \in A$, then $(\hat{B}_x^+ + yw_1w_2w_3, \hat{B}_y^- + yw_2w_1x + w_3x)$ is a good pair of D, a contradiction. If $w_1w_3 \in A$, then $(\hat{B}_x^+ + yw_2w_1w_3, \hat{B}_y^- + yw_1x + w_2w_3x)$ is a good pair of D, a contradiction.

Therefore, in this case, we can always obtain a contradiction.

Case 2: $s \neq t$.

W.l.o.g., suppose that $s = q_1$ and $t = q_2$, i.e., Q has a good pair $(B_{q_1}^+, B_{q_2}^-)$. By Note 6.1, $xq_1 \notin A$. But $N^-(Q) = \{x\}$ and so $N_D^-(q_1) \subseteq \{q_2, q_3, q_4\}$, namely $d_Q^-(q_1) \geq 2$. Moreover, both $B_{q_1}^+$ and $B_{q_2}^-$ contain an out-arc of q_1. So, $d_Q^+(q_1) \geq 2$. Similarly, $d_Q^+(q_2) \geq 2$ and $d_Q^-(q_2) \geq 2$.

Since the root of any out-branching has in-degree zero and the root of any in-branching has out-degree zero, if $q_2q_1 \in A$, then $q_2q_1 \notin B_{q_1}^+ \cup B_{q_2}^-$. Set $\tilde{B}_{q_1}^- = B_{q_2}^- + q_2q_1 - e$, where e is the unique out-arc of q_1 in $B_{q_2}^-$. Obviously, $\tilde{B}_{q_1}^-$ is an in-branching of Q with root q_1. So, Q has a good pair $(B_{q_1}^+, \tilde{B}_{q_1}^-)$, whose out-branching and in-branching have the same root q_1, which is Case 1. Thus, assume $q_2q_1 \notin A$.

Since $d_Q^-(q_1) \geq 2$ and $q_2q_1 \notin A$, $N_D^-(q_1) = \{q_3, q_4\}$. Moreover, since $d_Q^+(q_1) \geq 2$, D contains at least one of the arcs q_1q_3 and q_1q_4. W.l.o.g., let $q_1q_3 \in A$, that is $D[\{q_1, q_3\}]$ is a digon. Similarly, since $d_Q^+(q_2) \geq 2$ and $q_2q_1 \notin A$, $N_D^+(q_2) = \{q_3, q_4\}$. Moreover, since $d_Q^-(q_2) \geq 2$, there is an arc e from $\{q_1, q_3\}$ to q_2. Set $\tilde{B}_{q_1}^+ = q_1q_3 + e + q_2q_4$ and $\tilde{B}_{q_1}^- = q_2q_3q_1 + q_4q_1$. Then, $(\tilde{B}_{q_1}^+, \tilde{B}_{q_1}^-)$ is a good pair of Q, whose out-branching and in-branching have the same root q_1, which is Case 1. The proof is complete. $\lozenge$

Let R be a largest clique in D. Then R has three vertices by Claim 6.1 and Proposition 13.

Claim 6.2 *No subdigraph of D of order at least 3 has a good pair.*

Proof. By Lemma 18, it suffices to show that there is no $Q \subset D$ on 3 vertices with good pair. Suppose to the contrary that Q has a good pair. Analogous to Claim 5.1, $|N_D^+(Q)| \geq 2$ and $|N_D^-(Q)| \geq 2$ with $N_D^+(Q) \cap N_D^-(Q) = \emptyset$. Thus by Lemma 19, D has a good pair, a contradiction. $\lozenge$

By the claim above, R is a tournament.

Claim 6.3 *D is an oriented graph.*

Proof. Suppose that D has a digon Q. Set $X = N_D^-(Q)$ and $Y = N_D^+(Q)$. By Claim 6.2, $X \cap Y = \emptyset$. Since $\lambda(D) \geq 2$, both X and Y have at least two vertices. If $|X| + |Y| = 4$, then D has a good pair by Lemma 32, a contradiction. If $|X| + |Y| = 5$, then D has a good pair by Lemma 30 and the digraph duality, a contradiction. If $|X| + |Y| = 6$, then D has a good pair by Lemma 11, a contradiction. If $|X| + |Y| = 7$, then D has a good pair by Corollary 9, a contradiction. $\lozenge$

Now we are ready to finish the proof of Theorem 5. By Lemma 34, assume that $P_D = x_1x_2 \ldots x_9$ is a Hamilton dipath of D. Set $D' = D - A(P_D)$. Let $I_i,\ i \in [a]$, be the initial strong components in D' and let $T_j,\ j \in [b]$, be the terminal strong components in D'. Note that $a, b \geq 2$ by Proposition 22. Since D is an oriented graph and $\lambda(D) \geq 2$, $|I_i|, |T_j| \geq 3$, for any $i \in [a], j \in [b]$. Since $\lambda(D) \geq 2$, x_1 has at least two in-neighbours and one out-neighbour in D' and x_9 has at least two out-neighbours and one in-neighbour in D'. Thus there are only two disjoint strong components in D', say I_1 and I_2, as $n = 9$ and D is an oriented graph. We distinguish two cases below.

Case 1: $|I_1| = 4$ and $|I_2| = 5$.

If $x_9 \in I_1$, then $x_8 \in I_2$ as $|R| = 3$. Analogously, if $x_1 \in I_1$, then $x_2 \in I_2$. By Proposition 23, D has a good pair for each cases. Henceforth, both x_1 and x_9 are in I_2. Note that at least one of x_2 and x_8 is in I_1 as $|R| = 3$. By Proposition 23, D has a good pair, a contradiction.

Case 2: $|I_1| = 3$ and $|I_2| = 6$.

In this case, $x_1, x_9 \in I_2$ and $|A(I_2)| \geq 7$. If one of x_2 and x_8 is in I_1, then D has a good pair by Proposition 23. Thus both x_2 and x_8 are in I_2. Then $V(I_1) = \{x_3, x_5, x_7\}$, which implies that D has a good pair by Proposition 23, a contradiction.

This completes the proof of Theorem 6. $\square$

Acknowledgments. Gu was supported by National Natural Science Foundation of China (No. 11701143). Li was supported by the Natural Science Foundation of Ningbo, China (No. 202003N4148). Shi and Taoqiu are supported by the National Natural Science Foundation of China (Nos. 11922112 and 12161141006), the Natural Science Foundation of Tianjin (Nos. 20JCJQJC00090 and 20JCZDJC00840) and the Fundamental Research Funds for the Central Universities, Nankai University.

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