On Edge-path Eigenvalues of Graphs

Saieed Akbari [a] Seyran Azizi [b] Modjtaba Ghorbani [b] Xueliang Li [c]

[a] Department of Mathematical Sciences, Sharif University of Technology, Tehran, I. R. Iran

[b] Department of Mathematics, Faculty of Science, Shahid Rajaee Teacher Training University, Tehran, 16785 - 136, I. R. Iran

[c] Center for Combinatorics and LPMC-TJKLC, Nankai University, Tianjin 300071, China

Abstract

Let G be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and $EP(G)$ be an $n \times n$ matrix whose (i, j)-entry is the maximum number of internally edge-disjoint paths between v_i and v_j, if $i \neq j$, and zero otherwise. Also, define $\overline{EP}(G) = EP(G) + D$, where D is a diagonal matrix whose i-th diagonal element is the number of edge-disjoint cycles containing v_i. In this paper, we investigate all graphs G, whose $EP(G)$ is a multiple of $J - I$. Among other results, we determine the spectrum and the energy of $\overline{EP}(G)$ for an arbitrary bicyclic graph G.

Keywords: eigenvalue, path energy, edge-connectivity.

1 Introduction

All graphs considered in this paper are simple and connected. Let A be an $n \times n$ matrix. Then $\chi_A(\lambda) = det(\lambda I - A)$ is called the characteristic polynomial of A and the roots of $\chi_A(\lambda)$ are called the eigenvalues of A. In particular, if A is the adjacency matrix of graph G, the eigenvalues of A are the eigenvalues of G. The algebraic multiplicity m of an eigenvalue λ is denoted by $[\lambda]^m$. Let $\lambda_1, \ldots, \lambda_n$ be all eigenvalues of G. The *energy* of the graph G was first defined by Gutman as $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$, for instance see [5, 6, 7]. Let G be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$. In [12] the so-called *path matrix* $P(G)$ of graph G is defined as an $n \times n$ matrix whose (i, j)-entry is the maximum number of internally vertex-disjoint paths between the vertices v_i and v_j, for $i \neq j$ and is zero otherwise. The eigenvalues of $P(G)$ are called the path eigenvalues of G, forming its path spectrum $Spec_P(G)$. The *path energy*, $PE(G)$ is defined as the sum of the absolute values of the eigenvalues of $P(G)$. The basic properties of the path matrix and its eigenvalues were established in references [2, 11, 12]. A *unicyclic graph* and a *bicyclic graph* is a connected graph of order n that contains n and $n+1$ edges, respectively. Notice that there are three types of bicyclic graphs without pendent vertices as depicted in Figure 2, where c is the number of vertices between the cycles C_a and C_b not containing $V(C_a) \cup V(C_b)$. The spectral properties of unicyclic graphs are given in

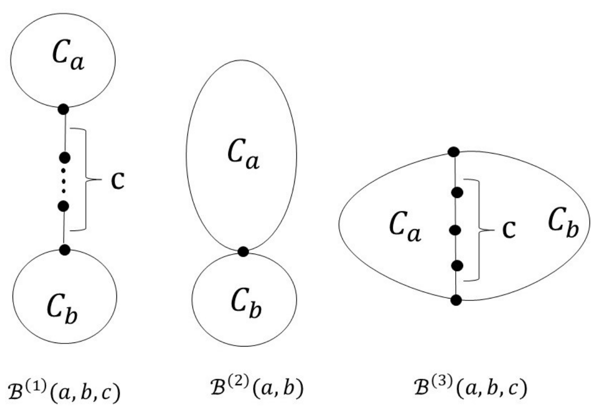

Figure 1: Bicyclic graphs without pendant vertex.

[12]. Also, the path energy of bicyclic graphs are investigated in [1, 2].

In this paper, for a given graph G, we define an edge version of path matrix of G to be a square matrix $EP(G) = (p_{ij})_{n\times n}$, where p_{ij} is the maximum number of edge disjoint-paths between the vertices v_i and v_j for $i \neq j$ and zero if $i = j$. This matrix is called the *edge-path matrix* of G. We call the eigenvalues of $EP(G)$ as the *edge-path eigenvalues* of G, forming its *edge-path spectrum* $Spec_{EP}(G)$. Let $\mu_1, \ldots, \mu_n$ be all eigenvalues of edge-path matrix $EP(G)$. Then the *edge-path energy* of G is defined as $\mathcal{E}_{EP}(G) = \sum_{i=1}^{n} |\mu_i|$.

The paper is organized as follows. In the rest of this section, further definitions are given and known results needed are stated. In Section 2, we provide some preparatory results. In the subsequent section, some properties of the edge-path matrix with respect to the edge-connectivity are established. In Section 4, the generalized edge-path matrix of a graph is defined and the generalized edge-path energy of bicyclic graphs are investigated.

The cycle and the complete graph of order n are denoted by C_n and K_n, respectively. In this paper $J_{r\times s}$ is an $r \times s$ matrix whose all entries are 1. An *ear* of a graph G is a maximal path whose internal vertices have degree two in G. An *ear decomposition* of a graph G is a decomposition of the edges of G into a sequence of ears (paths and cycles) $P_0, P_1, \ldots, P_i, \ldots, P_k$ such that P_0 is a cycle and $P_i (i > 0)$ is an ear of $P_0 \cup P_1 \cup \cdots \cup P_i$. A *block* of G is a maximal connected subgraph of G with no cut-vertex. It is a well-known fact that blocks of a non-trivial tree are the copies of K_2 and, in general, the blocks of a connected graph construct a treelike graph. A block B of graph G is a *leaf block*, if it contains exactly one cut-vertex. An *edge-cut(disconnecting set of edges)* of G is a subset of $E(G)$ of the form $[S, \bar{S}]$, where S is a nonempty proper subset of $V(G)$ and $\bar{S} = V \setminus S$. A *vertex-cut* of G is a subset V' of V such that $G \setminus V'$ is disconnected. By $G \setminus e$, we mean a graph obtained from G by removing the edge e.

An *automorphism* of graph G of order n is a permutation $\alpha \in S_n$, in which $uv \in E(G)$ if and only if $\alpha(u)\alpha(v) \in E(G)$, where the image of α at vertex u is denoted by $\alpha(u)$. A graph is *vertex-transitive* if its automorphism group acts transitively on its vertex set, namely for two distinct vertices $u, v \in V(G)$, there is an automorphism $\alpha \in Aut(G)$, where $\alpha(u) = v$.

Consider a symmetric matrix A with rows and columns indexed by a set V. Assume V is partitioned into m classes $V_1, \ldots, V_m$. Thus, with a suitable ordering of V we may write

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & \cdots & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{pmatrix},$$

where each diagonal block $A_{i,j}$ is symmetric. Such a matrix partition is called *equitable*, whenever each block $A_{i,j}$ has constant row and column sums. Let $b_{i,j}$ denotes the row sum of $A_{i,j}$. Then the $m \times m$ matrix $B = (b_{ij})$ is called the *quotient matrix* of A with respect to the given partition. It is well-known that the spectrum of B is a subset of the spectrum of A, see for example [8]. Let $B = (b_{ij})_{m\times n}$ and $C = (c_{ij})_{m\times n}$ be two matrices in $M_n(\mathbb{R})$. We use the notation $B \leq C$, if for each i, j, $b_{ij} \leq c_{ij}$.

2 Edge-path Spectrum of some Families of Graphs

There are many classes of graphs such as trees and unicycle graphs whose path matrix and edge-path matrix are the same, but in general, they are different. For example, consider the graph G as depicted in Figure 1. The path and the edge-path matrices of G are

$$P(G) = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 1 \\ 2 & 0 & 2 & 1 & 1 & 1 \\ 2 & 2 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

and

$$EP(G) = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 1 \\ 2 & 0 & 2 & 2 & 2 & 1 \\ 2 & 2 & 0 & 2 & 2 & 1 \\ 2 & 2 & 2 & 0 & 2 & 1 \\ 2 & 2 & 2 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

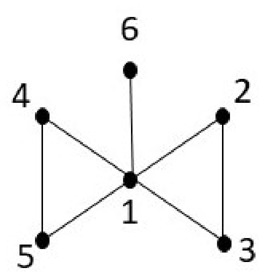

Figure 2: A graph with different path and edge-path matrices.

Theorem 1 [13, p.167](Menger-1927) If x, y are vertices of a graph G and $xy \notin E(G)$, then the minimum size of an x, y-cut equals the maximum number of pairwise internally vertex-disjoint x, y-paths.

Theorem 2 [13, p.168](edge version of Menger's Theorem) If x and y are distinct vertices of a graph G, then the minimum size of an x, y-disconnecting set of edges equals to the maximum number of pairwise edge-disjoint x, y-paths.

Example 3 The cocktail party graph of order $2n$ is a graph formed from the complete graph K_{2n} by removing a perfect matching. Consider two vertices u and v in G.

Two cases can be considered. If u and v are non-adjacent, then the degree of both u and v is $2n-2$ and thus by Theorem 2, the maximum number of edge-disjoint paths between them is $2n-2$. On the other hand, for each vertex $x \in V(G)$, $(x \neq u, v)$, the (u, x, v) is a path of length two between u and v. Hence, there are exactly $2n-2$ distinct paths between u and v of length 2. Therefore $EP(G) = (2n-2)(J-I)$. This yields that

$$Spec_{EP}(G) = \{[(2n-2)(2n-1)]^1, [-(2n-2)]^{2n-1}\},$$

and, $\mathcal{E}_{EP}(G) = 4(n-1)(2n-1)$.

Now, we would like to determine the edge-path spectrum of graph $K_n \setminus e$, where e is an arbitrary edge. We state the following result which was proved in [1].

Lemma 4 *Consider the matrix*

$$A = \begin{pmatrix} p_{11}(J-I) & p_{12}J & \cdots & p_{1k}J \\ p_{21}J & p_{22}(J-I) & \cdots & p_{2k}J \\ \vdots & \vdots & \ddots & \vdots \\ p_{k1}J & p_{k2}J & \cdots & p_{kk}(J-I) \end{pmatrix},$$

where the (i,j) block of A is an $n_i \times n_j$ matrix. Then

$$det(xI - A) = (x+p_{11})^{n_1-1} \cdots (x+p_{kk})^{n_k-1} det(xI - B),$$

where

$$B = \begin{pmatrix} p_{11}(n_1-1) & p_{12}n_2 & \cdots & p_{1k}n_k \\ p_{21}n_1 & p_{22}(n_2-1) & \cdots & p_{2k}n_k \\ \vdots & \vdots & \ddots & \vdots \\ p_{k1}n_1 & p_{k2}n_2 & \cdots & p_{kk}(n_k-1) \end{pmatrix}.$$

Theorem 5 $Spec_{EP}(K_n \setminus e) = \{[2-n]^1, [1-n]^{n-3}, [\frac{1}{2}(n^2-3n+1 \pm \sqrt{\alpha})]^1\}$, *where* $\alpha = n^4 - 2n^3 - 13n^2 + 46n - 39$.

Proof. Let $e = v_1v_2$. Then the number of edge-disjoint paths between the vertex $v_1(v_2)$ and the other vertices is $n-2$ while the number of edge-disjoint paths between any two other vertices v_r and v_s ($\{r,s\} \cap \{1,2\} = \varnothing$) is $n-1$. This yields that

$$EP(K_n \setminus e) = \begin{pmatrix} (n-2)(J-I)_{2\times 2} & (n-2)J_{2\times n-2} \\ (n-2)J_{n-2\times 2} & (n-1)(J-I)_{n-2\times n-2} \end{pmatrix}.$$

By Lemma 4, we obtain the following:

$$det(xI - EP(K_n \setminus e)) = (x+n-2)(x+n-1)^{n-3} det(xI - B),$$

where

$$B = \begin{pmatrix} n-2 & (n-2)\times(n-2) \\ (n-2)\times 2 & (n-1)\times(n-3) \end{pmatrix}.$$

Hence

$$Spec_{EP}(K_n \setminus e) = \{[2-n]^1, [1-n]^{n-3}, [\frac{1}{2}(n^2-3n+1 \pm \sqrt{\alpha})]^1\},$$

where $\alpha = n^4 - 2n^3 - 13n^2 + 46n - 39$.

3 Main Results

Connectivity plays a major role in the existence of paths and cycles in graphs. By studying the path matrices, we observe that for many classes of graphs such as trees, cycles and complete graphs, the path matrix $P(G)$ is equal to a multiple of $J - I$. Therefore characterizing such matrices would be important. In continuing of this paper, we characterize these kinds of matrices for $k = 1$ or $k = 2$ in terms of connectivity. We proceed as follows. The *edge-connectivity* $\kappa'(G)$ of a graph G is the smallest number of edges that by removing them the resulted graph is disconnected. A k-edge connected graph G is *minimally k-edge connected* if for every $e \in E(G)$, the graph $G \setminus e$ is not k-edge connected. We make use the following theorem appeared in [13].

Theorem 6 [13, p.162] A graph is 2-connected if and only if it has an ear decomposition, and every cycle in a 2-connected graph is the initial cycle in some such decomposition.

Theorem 7 *Let G be a graph. Then $P(G) = 2(J - I)$ if and only if G is a cycle.*

Proof. Let $P(G) = 2(J-I)$. Thus by Theorem 1, G is 2-connected and thus it has an ear-decomposition. Hence G has an ear decomposition with cycle P_0 and ears $P_1, \ldots, P_i$. Without loss of generality, let $u, v \in V(P_0) \cap V(P_1)$. Then there are at least three edge-disjoint paths between u and v, a contradiction. This means that G is a cycle P_0, as desired. Now, if G is a cycle, then it is clear that $P(G) = 2(J - I)$. □

Here, we determine the edge-path matrix of trees, unicyclic and bicyclic graphs. It is clear that for a tree T, $P(T) = EP(T)$. If G is a unicyclic graph, then $P(G) = EP(G)$.

Remark 8 Let G be a bicyclic graph of Type $B^{(1)}(a,b,c)$ or $B^{(3)}(a,b,c)$, then $P(G) = EP(G)$, but if G is a bicyclic graph of Type $B^{(2)}(a,b)$, (see Figure 2), then $EP(G) \neq P(G)$.

Here, we characterize all graphs with $EP(G) = k(J - I)$, where $k = 1$ or 2. Clearly, a graph G is a tree if and only if $EP(G) = J - I$.

Theorem 9 *Let G be a graph of order n. Then $EP(G) = 2(J_n - I_n)$ if and only if each block of G is a cycle.*

Proof. Let $C_1, \ldots, C_k$ be k blocks of G which all of them are cycles. Note that each pair C_i and $C_j (i \neq j)$ share at most one vertex in common. By induction on n, we show that $EP(G) = 2(J_n - I_n)$. It is clear that if G is a cycle, $EP(G) = 2(J_n - I_n)$. Now, let $C_1, \ldots, C_k$ be k blocks of G. Without loss of generality, suppose that C_k is a leaf block of G. If we remove $E(C_k)$ from $E(G)$ then, by induction hypothesis, $EP(G \setminus E(C_k)) = 2(J_{n-t+1} - I_{n-t+1})$, where $t = |V(C_k)|$. Now, let $H = G \setminus E(C_k)$ and $v \in V(H) \cap V(C_k)$. For each vertex $v \neq x \in V(C_k)$, there are exactly two edge-disjoint paths between x and v, so by induction hypothesis, $EP(G) = 2(J_n - I_n)$. Conversely, if $EP(G) = 2(J_n - I_n)$, then there are exactly two edge-disjoint paths between each pair of vertices of G. Thus G is 2-edge connected. By Theorem 6, G has an ear-decomposition. It means that G has a cycle P_0 with some ears. If G is a cycle, then we are done. Let P_i be an ear in this decomposition with endpoints u and v, where $u \neq v$. Obviously, there are at least three edge-disjoint paths between u and v, a contradiction. This implies that every ear in this decomposition is a cycle, as desired. $\square$

For a k-edge connected k-regular graph G, $EP(G) = k(J - I)$. Brouwer and Haemers [8] showed that a distance-regular graph of degree k is k-edge connected and strongly regular graphs are distance-regular. Hence, Theorem 2 yields that for such a graph, we obtain $EP(G) = k(J - I)$.

Theorem 10 [9] Let G be a connected vertex-transitive k-regular graph of order n. Then G is k-edge connected

Theorem 10 implies that each vertex-transitive graph satisfies in the equation $EP(G) = k(J - I)$. For example, since all Cayley graphs are vertex-transitive, if G is a Cayley graph, then $EP(G)$ is a multiple of $J - I$. If $EP(G) = k(J - I)$, then determining the structure of G is not an easy task. Here, we give some properties of these kind of graphs.

Remark 11 If $P(G) = k(J - I)$, then G is minimally k-edge connected.

Conjecture 1. *Let $G = (V, E)$ be a graph of order n and $EP(G) \leq k(J-I)$. Then $|E(G)| \leq (k+1)(n-1)/2$.*

Theorem 12 *If $k = 2$, then the Conjecture holds.*

Proof. If $EP(G) = 2(J-I)$, then by Theorem 9 each block of G is a cycle. If $G = C_n$, then clearly the assertion holds. Now, suppose that G is not a cycle and let C_k be a leaf block of G. By induction on n, we show that $E(G) \leq 3(n-1)/2$. Let $H = G \backslash (V(C_k) - v)$, where v is the unique cut-vertex of G contained in $V(C_k)$. Since $EP(H) = 2(J-I)$, by induction hypothesis, $|E(H)| \leq 3(n-(k-1)-1)/2 = 3/2(n-k)$. Therefore, we obtain $|E(G)| \leq 3/2(n-k)+k = (3n-k)/2 \leq 3/2(n-1)$, as desired. Now, if $EP(G) < 2(J-I)$, then by Theorem 2, G is not 2-edge connected and thus G has a cut edge. By induction on n, we show that $E(G) \leq 3/2(n-1)$. Now, let $e = uv$ be a cut edge of G. Suppose that G_1 and G_2 are two components of $G \setminus e$. Let $|V(G_i)| = n_i, i = 1, 2$. By induction hypothesis, $|E(G_i)| \leq 3/2(n_i - 1)$. Therefore $|E(G)| \leq 3/2(n_1 + n_2 - 2) + 1 \leq 3/2(n-1)$. The proof is complete. □

4 The Generalized Edge-Path Matrix

Here, we define the *generalized edge-path matrix* $\overline{EP}(G)$ of graph G as follows. The ij-th entry of this matrix is defined as the maximum number of edge-disjoint paths between two vertices v_i and v_j. Notice that the diagonal entries of this matrix are not zero.

Example 13 The generalized edge-path matrix of C_n is

$$\overline{EP}(C_n) = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & \cdots & 2 \\ 2 & 1 & 2 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 1 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & 1 & 2 & \cdots & 2 \\ 2 & 2 & 2 & 2 & 1 & \cdots & 2 \\ \vdots & 2 & 2 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & 2 & 2 & \cdots & 1 \end{pmatrix} = 2J - I.$$

Hence, its spectrum is

$$Spec_{\overline{EP}(G)}(G) = \{[2n-1]^1, [-1]^{n-1}\}.$$

It is clear that $EP(G) = \overline{EP}(G)$ if and only if G is a tree. Also, the *generalized edge-path energy*, $\mathcal{E}_{\overline{EP}}(G)$ is defined as the sum of absolute value of eigenvalues of $\overline{EP}(G)$. The following theorem will be useful for the characterization of graphs for which $\overline{EP}(G)$ is a multiple of J.

Theorem 14 [10] Let G be an edge-minimally k-edge connected graph. Then G has a vertex of degree k.

Theorem 15 *There is no graph G whose generalized edge-path matrix is a multiple of J.*

Proof. Let $\overline{EP}(G) = kJ$. By Theorem 2, G is k-edge connected. If G is not minimally k-edge connected, then there exists $e = uv \in E(G)$ such that $G \setminus e$ is still k-edge connected. Hence there are at least k edge-disjoint paths between u and v in $G \setminus e$. So, there are $k+1$ edge-disjoint paths between u and v in G, a contradiction. Therefore G is minimally edge connected. Now, by Theorem 14, G has a vertex of degree k, say v_i which yields that $(\overline{EP}(G))_{ii} \neq k$, a contradiction. □

Here, we investigate the generalized edge-path energy of bicyclic graphs.

Theorem 16 *Let $a, b \geq 3$ and G be a bicyclic graph of Type $B^{(1)}(a, b, c)$ of order $n = a + b + c$.*

(i) If $c = 0$, then $\mathcal{E}_{\overline{EP}}(G) = 3n - 4$.

(ii) If $c > 0$, then $\mathcal{E}_{\overline{EP}}(G) = a + b - 3 + |\alpha - 1| + \beta - 1| + |\gamma - 1|$, where α, β, γ are roots of $\lambda^3 - (2a + 2b + c)\lambda^2 + (3ab + ac + bc)\lambda - abc$.

Proof. (i) Let $v_1, \ldots, v_a$ be the vertices of C_a and $v_{a+1}, \ldots, v_{a+b}$ be the vertices of C_b. Then the generalized edge-path matrix of G is

$$\overline{EP}(G) = \begin{pmatrix} 2J_a - I & J \\ J & 2J_b - I \end{pmatrix}.$$

Let $C = \overline{EP}(G) + I$. Then the rank and nullity of C are 2 and $a+b-2$, respectively. It means that matrix $\overline{EP}(G)$ has the eigenvalue -1 with multiplicity $a+b-2$. The quotient matrix B of G is as follows:

$$B = \begin{pmatrix} 2a - 1 & b \\ a & 2b - 1 \end{pmatrix}.$$

The eigenvalues of matrix B are $a + b - 1 \pm \sqrt{a^2 + b^2 - ab}$. This yields that

$$Spec_{\overline{EP}(G)} = \{[-1]^{a+b-2}, [a + b - 1 + \sqrt{a^2 + b^2 - ab}]^1, [a + b - 1 - \sqrt{a^2 + b^2 - ab}]^1\},$$

and thus

$$\mathcal{E}_{\overline{EP}}(G) = a + b - 2 + 2(a + b - 1) = 3a + 3b - 4 = 3n - 4.$$

(ii) Let $v_1, \ldots, v_a$ be the vertices of C_a and $v_{a+1}, \ldots, v_{a+b}$ be the vertices of C_b, and $v_{a+b+1}, \ldots, v_{a+b+c}$ be the other vertices of G. The generalized edge-path matrix of G is

$$\overline{EP}(G) = \begin{pmatrix} 2J_a - I_a & J & J \\ J & J_c - I_c & J \\ J & J & 2J_b - I_b \end{pmatrix}.$$

Let $C = \overline{EP}(G) + I$. Then the rank and nullity of C are 3 and $a + b + c - 3$, respectively. It means that matrix $\overline{EP}(G)$ has the eigenvalue -1 with multiplicity $a + b - 3$. Also, the quotient matrix B of G is as follows:

$$B = \begin{pmatrix} 2a - 1 & c & b \\ a & c - 1 & b \\ a & c & 2b - 1 \end{pmatrix}.$$

On the other hand, the characteristic polynomial of matrix

$$B^{'} = \begin{pmatrix} 2a & c & b \\ a & c & b \\ a & c & 2b \end{pmatrix}$$

is $\chi_{B'}(\lambda) = \lambda^3 - (2a + 2b + c)\lambda^2 + (3ab + ac + bc)\lambda - abc$. Suppose α, β and γ are the roots of $\chi_{B'}(\lambda)$. Then $\alpha - 1, \beta - 1$ and $\gamma - 1$ are the other eigenvalues of B. Therefore $\mathcal{E}_{\overline{EP}}(G) = a + b - 3 + |\alpha - 1| + \beta - 1| + |\gamma - 1|$. □

Theorem 17 *Let $a, b \geq 3$ and G be a bicyclic graph of Type $B^{(2)}(a, b)$ of order $n = a + b - 1$. Then $\mathcal{E}_{\overline{EP}}(G) = n - 2 + \sqrt{4n^2 - 4n + 9}$.*

Proof. Let $v_1, \ldots, v_a$ be the vertices of C_a and $v_{a+1}, \ldots, v_{a+b-1}$ be the vertices of C_b except the common vertex of C_a and C_b. The generalized edge-path matrix of G is as follows:

$$\overline{EP}(G) = \begin{pmatrix} 2 & 2J_{1 \times n-1} \\ 2J_{n-1 \times 1} & 2J_{n-1 \times n-1} - I_{n-1} \end{pmatrix}.$$

Let $C = \overline{EP}(G) + I$. It is not hard to see that, the rank and nullity of the matrix $C = \overline{EP}(G) + I$ are 2 and $a + b - 3$, respectively. It means that $\overline{EP}(G)$ has -1 as an eigenvalue with multiplicity $a+b-3$. Now, the quotient matrix B of G is as follows:

$$B = \begin{pmatrix} 2 & 2n-2 \\ 2 & 2n-3 \end{pmatrix}.$$

The eigenvalues of matrix B are $-\frac{1}{2}+n+\frac{1}{2}\sqrt{4n^2-4n+9}, -\frac{1}{2}+n-\frac{1}{2}\sqrt{4n^2-4n+9}$. This yields that

$$Spec_{\overline{EP}(G)} = \{[-1]^{a+b-3}, [-\frac{1}{2}+n+\frac{1}{2}\sqrt{4n^2-4n+9}]^1, [-\frac{1}{2}+n-\frac{1}{2}\sqrt{4n^2-4n+9}]^1\},$$

and thus

$$\mathcal{E}_{\overline{EP}}(G) = a+b-3+\sqrt{4n^2-4n+9} = n-2+\sqrt{4n^2-4n+9}.$$

□

Theorem 18 *Let $a,b \geq 3$ and G be a bicyclic graph of Type $B^{(3)}(a,b,c)$ of order $n = a+b-c-2$. Then $\mathcal{E}_{\overline{EP}}(G) = n-1+\sqrt{4n^2-4n+17}$.*

Proof. Let $v_1, \ldots, v_a$ and $v_{a+1}, \ldots, v_{a+b-c-2}$ are the vertices of C_a and C_b, respectively. Then the generalized edge-path matrix of $\overline{EP}(G)$ has the following form:

$$\overline{EP}(G) = \begin{pmatrix} 3J_{2\times 2} - 2I_{2\times 2} & 2J_{2\times n-2} \\ 2J_{n-2\times 2} & 2J_{n-2\times n-2} - I_{n-2\times n-2} \end{pmatrix}.$$

A similar argument shows that the rank and nullity of $C = \overline{EP}(G) + I$ are 3 and $a+b-c-5$, respectively. This means that -1 is an eigenvalue of $\overline{EP}(G)$ with multiplicity $a+b-c-5$. The eigenvalues of quotient matrix is

$$B = \begin{pmatrix} 4 & 2n-4 \\ 4 & 2n-5 \end{pmatrix},$$

are $-\frac{1}{2}+n+\frac{1}{2}\sqrt{4n^2-4n+17}, -\frac{1}{2}+n-\frac{1}{2}\sqrt{4n^2-4n+17}$.

Since, $tr(\overline{EP}(G)) = n = a+b-c-2$. Hence, $\lambda_1 = -2$. This yields that,

$$Spec_{\overline{EP}(G)} = \{[-1]^{a+b-c-5}, -2, [-\frac{1}{2}+n+\frac{1}{2}\sqrt{4n^2-4n+17}]^1, [-\frac{1}{2}+n-\frac{1}{2}\sqrt{4n^2-4n+17}]^1\},$$

and thus

$$\mathcal{E}_{\overline{EP}}(G) = a+b-c-5+2+\sqrt{4n^2-4n+17} = n-1+\sqrt{4n^2-4n+17}.$$

□

We close the paper with the following conjecture.

Conjecture 2. *A graph is Eulerian if and only if every entry of $EP(G)$ is even.*

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