Monochromatic-degree conditions for properly colored cycles in edge-colored complete graphs[1]

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Abstract

Let G be an edge-colored graph and v be a vertex of G. Define the monochromatic-degree $d^{mon}(v)$ of v to be the maximum number of edges with the same color incident with v in G, and the maximum monochromatic-degree $\Delta^{mon}(G)$ of G to be the maximum value of $d^{mon}(v)$ over all vertices v of G. A cycle (path) in G is called *properly colored* if any two adjacent edges of the cycle (path) have distinct colors. Wang et al. in 2014 showed that an edge-colored complete graph K_n^c with $\Delta^{mon}(K_n^c) < \lfloor \frac{n}{2} \rfloor$ contains a properly colored cycle of length at least $\lceil \frac{n}{2} \rceil + 2$. In this paper, we obtain a generalization of their result that an edge-colored complete graph K_n^c of order n with $\Delta^{mon}(K_n^c) = d \leq n - 2$ contains a properly colored cycle of length at least $n - d + 1$.

Keywords: edge-colored (complete) graph; (minimum) color-degree; (maximum) monochromatic-degree; properly colored cycle (path).

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1 Introduction

An *edge-coloring* of a graph is an assignment of colors to the edges of the graph. An *edge-colored graph* is a graph with an edge-coloring. Let K_n^c denote an edge-colored

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complete graph with an edge-coloring c. A cycle (path) in an edge-colored graph G is *properly colored*, or *PC* for short, if any two adjacent edges of the cycle (path) have distinct colors. For other notation and terminology not defined here, we refer to [4].

In an edge-colored graph G, the *color-degree* of a vertex v of G is the number of colors on the edges incident with v in G, denoted by $d^c(v)$. Let $\delta^c(G)$ denote the minimum value of $d^c(v)$ over all vertices $v \in V(G)$, called the *minimum color-degree* of G. Actually, there are many results on the color-degree conditions for the existence of PC cycles, for which we refer the reader to [9, 10].

In this paper, we consider the monochromatic-degree conditions for the existence of PC cycles. The *monochromatic-degree* of a vertex v of G is the maximum number of edges with the same color incident with v in G, denoted by $d^{mon}(v)$. Let $\Delta^{mon}(G)$ denote the maximum value of $d^{mon}(v)$ over all vertices $v \in V(G)$, called the *maximum monochromatic-degree* of G. In recent years, many people have worked on the conditions for the existence of a PC Hamilton cycle in an edge-colored graph. In 1976, Bollobás and Erdős in [3] posed the following famous conjecture.

Conjecture 1 ([3]). *If $\Delta^{mon}(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then K_n^c contains a PC Hamiltonian cycle.*

Li et al. in [9] studied long PC cycles in K_n^c and proved that if $\Delta^{mon}(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then K_n^c contains a PC cycle of length at least $\lceil \frac{n+2}{3} \rceil + 1$. Later on, Wang et al. in [15] improved the bound on the lengths of PC cycles.

Theorem 2 ([15]). *If $\Delta^{mon}(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then K_n^c contains a PC cycle of length at least $\lceil \frac{n}{2} \rceil + 2$.*

In this paper, we obtain a bound on the lengths of PC cycles under monochromatic-degree conditions.

Theorem 3. *If $\Delta^{mon}(K_n^c) = d \le n-2$, then K_n^c contains a PC cycle of length at least $n - d + 1$.*

Remark. Theorem 3 can be seen as a generalization of Theorem 2, since from $\Delta^{mon}(K_n^c) = d < \lfloor \frac{n}{2} \rfloor$, we have

$$d \le \begin{cases} \frac{n-3}{2} & n \text{ is odd;} \\ \frac{n-2}{2} & n \text{ is even,} \end{cases}$$

and then $n - d + 1 \ge \lceil \frac{n}{2} \rceil + 2$.

The main idea is the rotation-extension technique of Pósa [12], which was used on edge-colored graphs in [10, 15].

Since $\Delta^{mon}(K_n^c) + \delta^c(K_n^c) \leq n$, we can get the following corollary.

Corollary 4. *If $\delta^c(K_n^c) \geq 2$, then K_n^c contains a PC cycle of length at least $\delta^c(K_n^c) + 1$.*

Thus we completely solve the problem "Does every edge-colored complete graph K_n^c with $\delta^c(K_n^c) \geq 2$ contain a PC cycle of length at least $\delta^c(K_n^c)$?", which was posed by Li et al. in [7].

The paper is organized as follows. In Section 2, we give some notation and tools. In Section 3 we prove our main result Theorem 3. In Section 4, we give a remark concerning the lengths of PC cycles in Theorem 3 and pose two conjectures.

2 Preliminaries

Grossman and Häggkvist in [6] gave a condition for the exitance of a PC cycle in an edge-colored graph with two colors, and later on, Yeo in [16] extended the result to an edge-colored graph with any number of colors.

Theorem 5 ([6, 16])**.** *Let G be an edge-colored graph containing no PC cycles. Then G contains a vertex v such that no component of $G - v$ is joined to v with edges of more than one color.*

Li et al. [8] observed that in an edge-colored complete graph G, for any PC cycle C, each vertex $v \in V(C)$ is contained in a PC cycle C' of length at most 4 such that $V(C') \subseteq V(C)$. Combining this observation and Theorem 5, they got the following result.

Theorem 6 ([8])**.** *If $\Delta^{mon}(K_n^c) \leq n - 2$, then K_n^c contains a PC cycle of length at most 4.*

For convenience, let the vertices of K_n^c be labeled from 1 to n. A path of length $\ell - 1$ is considered to be an ℓ-tuple, $(i_1, i_2, \cdots, i_\ell)$, where $i_1, i_2, \cdots, i_\ell$ are distinct. Let $[a, b]$ and $[b]$ denote the sets $\{i \in \mathbb{N} : a \leq i \leq b\}$ and $\{i \in \mathbb{N} : 1 \leq i \leq b\}$, respectively.

Given a longest PC path $P = (i_1, i_2, \cdots, i_\ell)$, we define two sets

$$X(P) = \{j \in [\ell] : c(i_1, i_j) \neq c(i_1, i_2)\},$$

$$Y(P) = \{j \in [\ell] : c(i_\ell, i_j) \neq c(i_\ell, i_{\ell-1})\},$$

of indices and two subsets

$$N^c(i_1; P) = \{i_x : x \in X(P)\},$$

$$N^c(i_\ell; P) = \{i_y : y \in Y(P)\}$$

of vertices. Clearly, $\min\{|X(P)|, |Y(P)|\} \geq n - \Delta^{mon}(G) - 1$. Apparently, as P is a longest PC path, $N^c(i_1; P), N^c(i_\ell; P) \subseteq V(P)$. We say that P has a *crossing* if there exist x and y with $1 \leq y < x \leq \ell$ such that $y \in Y(P)$ and $x \in X(P)$. If $i_j \in N^c(i_\ell; P)$ and $c(i_\ell, i_j) \neq c(i_j, i_{j-1})$, then $(i_1, i_2, \cdots, i_j, i_\ell, i_{\ell-1}, \cdots, i_{j+1})$ is also a PC path, which is called *a rotation of P with endpoint i_1 and pivot point i_j*. A *reflection* of P is simply the PC path $(i_\ell, i_{\ell-1}, \cdots, i_1)$. The set of PC paths that can be obtained by a sequence of rotations and reflections of P is denoted by $\mathcal{R}(P)$. Note that if P is a longest PC path, then all paths in $\mathcal{R}(P)$ are longest PC paths. Let $q(P) = \max\{j : j \in X(P)\}$ and $r(P) = \min\{j : j \in Y(P)\}$. Then the next lemmas follow easily.

Lemma 1. *Let $\Delta^{mon}(K_n^c) = d \leq n - 2$. Suppose $P = (i_1, i_2, \cdots, i_\ell)$ is a longest PC path in K_n^c. If there does not exist a PC cycle of length at least $n - d + 1$, then $c(i_1, i_{q(P)}) = c(i_{q(P)}, i_{q(P)-1})$ and $c(i_l, i_{r(P)}) = c(i_{r(P)}, i_{r(P)+1})$.*

Proof. Suppose not, then $(i_1, i_2, \cdots, i_{q(P)}, i_1)$ and $(i_{r(P)}, i_{r(P)+1}, \cdots, i_\ell, i_{r(P)})$ are PC cycles containing $N^c(i_1; P) \cup \{i_1, i_2\}$ and $N^c(i_\ell; P) \cup \{i_\ell, i_{\ell-1}\}$, respectively, a contradiction. □

Lemma 2. *Let $\Delta^{mon}(K_n^c) = d \leq n - 2$. Let P be a longest PC path in K_n^c. If there does not exist a PC cycle of length at least $n - d + 1$, then each path in $\mathcal{R}(P)$ has a crossing.*

Proof. Suppose, to the contrary, that there is a path $Q = (i_1, i_2, \cdots, i_\ell)$ in $\mathcal{R}(P)$ such that Q does not have a crossing. Then we have $q(Q) \leq r(Q)$. Since $d \leq n-2$, we have $r(Q) \leq \ell - 2$. Hence, $q(Q) \leq \ell - 2$. Therefore, $c(i_1, i_{\ell-1}) = c(i_1, i_\ell) = c(i_1, i_2) \neq c(i_1, i_{q(Q)})$. From Lemma 1, $c(i_1, i_{q(Q)}) = c(i_{q(Q)}, i_{q(Q)-1}) \neq c(i_{q(Q)}, i_{q(Q)+1})$. Then $(i_1, i_{q(Q)}, i_{q(Q)+1}, \cdots, i_\ell, i_1)$ or $(i_1, i_{q(Q)}, i_{q(Q)+1}, \cdots, i_{\ell-1}, i_1)$ is a PC cycle containing $N^c(i_\ell; Q) \cup \{i_1, i_{\ell-1}\}$, a contradiction. □

Given a longest PC path $P = (i_1, i_2, \cdots, i_\ell)$, $X(P)$ and $Y(P)$, we define some indices on P, which can be regarded as functions of P.

$r(P) = \min\{y : y \in Y(P)\}$;

$s(P) = \max\{s' : s' \in Y(P)$ such that $c(i_\ell, i_y) = c(i_y, i_{y+1})$ for every $y \in Y(P) \cap [s']\}$;

$u(P) = \max\{u' : u' \in X(P) \setminus \{\ell\}$ such that $c(i_1, i_x) = c(i_x, i_{x+1})$ for every $x \in X(P) \cap [s(P) + 1, u']\}$;

$w(P) = \min\{x : x \in X(P) \cap [u(P) + 1, \ell]\}$.

Note that $s(P), u(P), w(P)$ exist not for an arbitrary P. If $s(P)$ exists, then we further define the set $S(P)$ to be $\{i_y : y \in Y(P) \cap [s(P)]\}$ and $t(P) = u(P) - |S(P)| + 1$. In the following lemma, we show that $r(P), s(P), u(P), w(P), t(P)$ exist for all longest PC paths. For simplicity, we use r, s, u, w, t to denote them.

Lemma 3. *Let $\Delta^{mon}(K_n^c) = d \leq n-2$ and let $P = (i_1, i_2, \cdots, i_\ell)$ be a longest PC path in K_n^c. If there does not exist a PC cycle of length at least $n-d+1$, then r, s, u, w exist.*

Moreover, the following statements hold:

(a) $1 \leq r \leq s < u < w \leq \ell$ and $u \leq n-d$;

(b) $c(i_1, i_y) = c(i_y, i_{y+1})$, for all $i_y \in S(P)$;

(c) $c(i_1, i_x) = c(i_x, i_{x+1})$, for all $x \in [r+1, u] \cap X(P)$;

(d) $c(i_1, i_w) \neq c(i_w, i_{w+1})$ if $w < \ell$;

$c(i_1, i_w) = c(i_w, i_{w-1})$ if $w = \ell$.

Proof. From Lemma 1, $c(i_\ell, i_r) = c(i_r, i_{r+1})$. Hence s exists with $r \leq s \leq \ell - 2$. Next we prove a claim to show that u exists.

Claim 1. $s < q$.

We may assume $p \leq \ell - 2$. Let $y \in Y(P)$ be the maximum such that $y < q$. Since P has a crossing by Lemma 2, y exists. If $c(i_\ell, i_y) = c(i_y, i_{y+1}) \neq c(i_y, i_{y-1})$, then $(i_1, i_2, \cdots, i_y, i_\ell, i_{\ell-1}, \cdots, i_q, i_1)$ is a PC cycle containing $N^c(i_\ell; P) \cup \{i_\ell, i_{\ell-1}\}$, a contradiction. Hence, $c(i_\ell, i_y) \neq c(i_y, i_{y+1})$. Thus, according to the definition of s, $s < y$. Hence, $s < q$.

Let $x \in X(P)$ be the minimum such that $s < x$. By Claim 1, x exists. If $x = \ell$, then $c(i_1, i_2) \neq c(i_\ell, i_1) = c(i_\ell, i_{\ell-1}) \neq c(i_\ell, i_s)$. Since $c(i_\ell, i_s) = c(i_s, i_{s+1}) \neq c(i_s, i_{s-1})$, $(i_1, i_2, \cdots, i_s, i_\ell, i_1)$ is a PC cycle containing $N^c(i_1; P) \cup \{i_1, i_2\}$, a contradiction. Then, $x \leq \ell - 1$. Suppose, to the contrary, that u does not exist. Then, $c(i_1, i_x) \neq c(i_x, i_{x+1})$. If $s \neq 1$, then $(i_1, i_2, \cdots, i_s, i_\ell, i_{\ell-1}, \cdots, i_x, i_1)$ is a PC cycle containing $N^c(i_1; P) \cup \{i_1, i_2\}$, a contradiction. If $s = 1$, then from Lemma 1, $c(i_\ell, i_{\ell-1}) \neq c(i_1, i_\ell) = c(i_1, i_2) \neq c(i_1, i_x)$. Thus, $(i_1, i_x, i_{x+1}, \cdots, i_\ell, i_1)$ is a PC cycle containing $N^c(i_1; P) \cup \{i_1, i_\ell\}$, a contradiction. So, u exists. According to Lemma 1, w exists. Since $c(i_1, i_u) = c(i_u, i_{u+1}) \neq c(i_u, i_{u-1})$, $(i_1, i_2, \cdots, i_u, i_1)$ is a PC cycle of length at least u. Hence, $u \leq n - d$. Therefore, from the definitions of r, s, u, w, (a), (b) and (d) hold.

Next we show that $c(i_1, i_j) = c(i_j, i_{j+1})$ for $j \in [r+1, s+1] \cap X(P)$. Otherwise, if there exists an $x \in [r+1, s+1] \cap X(P)$ such that $c(i_1, i_x) \neq c(i_x, i_{x+1})$, letting y be the maximum such that $y \in [1, s] \cap Y(P)$ and $y < x$, then $(i_1, i_2, \cdots, i_y, i_\ell, i_{\ell-1}, \cdots, i_x, i_1)$ is a PC cycle containing $N^c(i_\ell; P) \cup \{\ell - 1, \ell\}$, a contradiction. Then, let u be the maximum such that $c(i_1, i_j) = c(i_j, i_{j+1})$ for all $j \in [s+1, u] \cap X(P)$ and $s < u < \ell$. Thus (c) holds. $\square$

According to Lemma 3, for any longest PC path Q, we have $S(Q) \neq \emptyset$. Now given a PC path P and the set $\mathcal{R}(P)$, without loss of generality, assume that $|S(P)|$ is maximum over all the longest PC paths. In the next lemma, we find a longest PC cycle C_0 in an edge-colored complete graph which does not have PC cycles of length at least $n - \Delta^{mon}(G) + 1$, and get some useful properties.

Lemma 4. *Let G be an edge-colored complete graph K_n such that $\Delta^{mon}(G) = d \leq n-2$, and let $P = (i_1, i_2, \cdots, i_\ell)$. If there does not exist a PC cycle of length at least $n-d+1$, then the following statements are true (for simplicity, we use r, s, u, w, t instead of $r(P), s(P), u(P), w(P), t(P)$):*

(a) $C_0 = (i_1, i_2, \cdots, i_s, i_\ell, i_{\ell-1}, \cdots, i_w, i_1)$ is a PC cycle (see Fig.1).

(b) $|C_0| = n - d$, $|X(P)| = n - d + 1$ and $S(P) = \{i_y : y \in [r, s]\}$.

(c) $t \geq max\{3, r+1\}$ and $X(P) = \begin{cases} [3, r] \cup [t, u] \cup [w, \ell], & if\ r \geq 3, \\ [t, u] \cup [w, \ell], & if\ r = 2, \\ [t, u] \cup [w, \ell - 1], & if\ r = 1. \end{cases}$ *where all the intervals are non-empty and pairwise disjoint.*

(d) $c(i_1, i_x) = c(i_x, i_{x+1})$ for all $t \leq x \leq u$.

(e) Given an integer a with $r \leq a \leq s$, the path $P^* = (i_{a+1}, i_{a+2}, \cdots, i_\ell, i_a, i_{a-1}, \cdots, i_1) \in \mathcal{R}(P)$; moreover, if $a < t$, then $N^c(i_1; P^*) = N^c(i_1; P)$ and $S(P^*) = \{i_y : y \in [t, u]\}$.

(f) If $P^* \in \mathcal{R}(P)$ with $|S(P^*)| = |S(P)|$, then the corresponding statements of (a)-(e) hold.

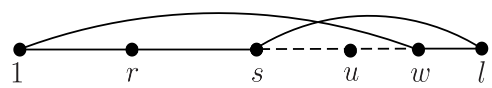

Figure 1: $C_0 = (i_1, i_2, \cdots, i_s, i_\ell, i_{\ell-1}, \cdots, i_w, i_1)$

Proof. From Lemma 3, (a) holds.

Since $c(i_\ell, i_r) = c(i_r, i_{r+1}) \neq c(i_r, i_{r-1})$, $P_1 = (i_{r+1}, i_{r+2}, \cdots, i_\ell, i_r, i_{r-1}, \cdots, i_1) \in \mathcal{R}(P)$. Clearly, $N^c(i_1; P_1) = N^c(i_1; P)$. By Lemma 3 (c), $c(i_1, i_x) = c(i_x, i_{x+1})$ for all $x \in [r+1, u] \cap X(P)$. Then, $\{i_y : y \in [r+1, u] \cap X(P)\} \subseteq S(P_1)$. By the maximality of $S(P)$, we have $|[r,s]\cap Y(P)| = |S(P)| \geq |S(P_1)| \geq |[r+1,u]\cap X(P)|$. Then

$$\begin{aligned}
|C_0| &= |[1,s]| + |[w,\ell]| \\
&= |[1,s]\cap X(P)| + |[1,s]\setminus X(P)| + |[w,\ell]\cap X(P)| + |[w,\ell]\setminus X(P)| \\
&= |X(P)| - |[s+1,u]\cap X(P)| + |[1,s]\setminus X(P)| + |[w,\ell]\setminus X(P)| \\
&= |X(P)| - |[r+1,u]\cap X(P)| + |[r+1,s]| + |[1,r]\setminus X(P)| + |[w,\ell]\setminus X(P)| \\
&\geq |X(P)| - |[r,s]\cap Y(P)| + |[r,s]| + |[2,r]\setminus X(P)| + |[w,\ell]\setminus X(P)| \\
&\geq |X(P)| + |[2,r]\setminus X(P)| + |[w,\ell]\setminus X(P)| \\
&\geq |X(P)| + 1 \\
&\geq n-d.
\end{aligned}$$

Since $|C_0| \leq n-d$, we have $|C_0| = n-d$. Therefore, all the inequalities become equalities. Then $|X(P)| = n-d-1$, and

$$|[2,r]\setminus X(P)| + |[w,\ell]\setminus X(P)| = 1, \tag{1}$$

$$|[r,s]| = |[r,s]\cap Y(P)| = |[r+1,u]\cap X(P)|. \tag{2}$$

Moreover, as $2 \notin X(P)$, (1) implies that

$$\begin{cases} [3,r]\cup[w,\ell] \subseteq X(P), & if\ r \geq 3, \\ [w,\ell] \subseteq X(P), & if\ r = 2, \\ [w,\ell-1] \subseteq X(P), & if\ r = 1, \end{cases}$$

and (2) implies that $S(P) = \{i_y : y \in [r,s]\cap Y(P)\} = \{i_y : y \in [r,s]\}$ and $S(P_1) = \{i_y : y \in [r+1,u]\cap X(P)\}$. By the definition of u, we have $c(i_1,i_u) = c(i_u,i_{u+1})$ and $c(i_1,i_u) \neq c(i_1,i_2)$. Thus, $i_u \in S(P_1)$. Since $|S(P_1)| = |S(P)|$, we deduce that $S(P_1)$ is also an interval by taking $P = P_1$. Then, $[r+1,u]\cap X(P) = [t,u]$. Therefore,

$$X(P) = \begin{cases} [3,r]\cup[t,u]\cup[w,\ell], & if\ r \geq 3, \\ [t,u]\cup[w,\ell], & if\ r = 2, \\ [t,u]\cup[w,\ell-1], & if\ r = 1. \end{cases} \tag{3}$$

So far, (b)-(d) hold.

Next, we are going to prove (e). If $a = r$, then there is nothing to prove. Hence, suppose $r < a \leq s$. Since $a \in S(P)$, $c(i_l,i_a) = c(i_a,i_{a+1})$. Then P^* is a PC path. Note that P^* is obtained from P by a rotation with endpoint i_1 and pivot point i_a followed by a reflection. Therefore, $P^* \in \mathcal{R}(P)$. Further, if $a < t$, clearly $N^c(i_1;P^*) = N^c(i_1;P)$. We can get $\{i_y : y \in [t,u]\} \subseteq S(P^*)$. By the maximality of $|S(P)|$, $S(P^*) = \{i_y : y \in [t,u]\}$ and so (e) holds. Apparently, (f) follows from (a)-(e). □

Now we are ready to give the proof of Theorem 3.

3 Proof of Theorem 3

If $d = n-2$, then the result follows from Theorem 6. Then, we may assume $d \leq n-3$. Suppose, to the contrary, that each PC cycle in K_n^c is of length at most $n-d$. Let P be a longest PC path in K_n^c, and for simplicity, we label the the vertices of P by $(1, 2, \cdots, \ell)$ and $P' = (\ell, \ell-1, \cdots, 1)$. According to Lemma 3, we know that $r(P), s(P), t(P), u(P)$ and $w(P)$ do exist. For convenience, we use r, s, t, u, w instead. Without loss of generality, assume that P is a longest PC path satisfying that $|S(P)|$ is maximum over all the longest PC paths. Since P is a longest PC path, $N^c(1;P) \cup N^c(\ell;P) \subseteq V(P)$. Thus, $\ell \geq n-d+1$. Moreover, if $\ell \in N^c(1;P)$ and $1 \in N^c(\ell;P)$, then $(1, 2, \cdots, \ell, 1)$ is a PC cycle of length $\ell \geq n-d+1$. Hence, $\ell \notin N^c(1;P)$ or $1 \notin N^c(\ell;P)$. So, $\ell \geq n-d+2$. Note that if $\ell - 1 \in X(P)$, then $\ell - 1 \in S(P)$; otherwise, $(1, 2, \cdots, \ell-1, 1)$ is a PC cycle of length $n-d+1$, a contradiction. In the following, we show some claims which will be used in our proof.

Claim 1. If $|S(P')| = |S(P)|$, then $r \in \{1, 2\}$ and $X(P) = \begin{cases} [t,u] \cup [w,\ell], & r = 2, \\ [t,u] \cup [w, \ell-1], & r = 1. \end{cases}$ Moreover, if $r = 1$ then $r(P') = 2$, and if $r = 2$ then $r(P') = 1$.

Proof. Let $P' = (v_1, v_2, \cdots, v_\ell)$. Since $|S(P')| = |S(P)|$, by Lemma 4 (f) and (c), we have

$$X(P') = \begin{cases} [3, r(P')] \cup [t(P'), u(P')] \cup [w(P'), \ell], & if\ r(P') \geq 2, \\ [t(P'), u(P')] \cup [w(P'), \ell-1], & if\ r(P') = 1. \end{cases} \tag{4}$$

Suppose, to the contrary, that $r \geq 3$. Then, $\ell \in X(P)$. Therefore, $c(1,\ell) = c(\ell, \ell-1)$, which implies that $1 \notin N^c(\ell;P)$. Noticing that $\ell = v_1$, we have $r(P') = 1$. Hence, by (4), $v_{\ell-1} = 2 \in N^c(\ell;P') = N^c(\ell;P)$, which implies that $r = 2$, a contradiction. Hence, $r \in \{1, 2\}$. Moreover, if $r = 1$ then $r(P') = 2$, and if $r = 2$ then $r(P') = 1$. □

Claim 2. For each $y \in N^c(\ell;P) \cap [s+1, w-1]$, we have that $c(\ell, y) = c(y, y-1)$ and $|N^c(\ell;P) \cap [s+1, w-1]| \leq |S(P)|$.

Proof. Since $c(1,w) \neq c(w, w+1)$, we have that $Q = (w-1, w-2, \cdots, s+1, s, \cdots, 1, w, w+1, \cdots, \ell)$ is a longest PC path. Clearly, $N^c(\ell;P) = N^c(\ell;Q)$. Since $|C_0| = n-d$, for any $y \in N^c(\ell;P) \cap [s+1, w-1]$ we have $c(\ell, y) = c(y, y-1)$; otherwise, $(1, 2, \cdots, s, s+1, \cdots, y, \ell, \ell-1, \cdots, w, 1)$ is a PC cycle of length at least $n-d+1$, a contradiction. Then, $N^c(\ell;P) \cap [s+1, w-1] \subseteq S(Q)$. Therefore, $|N^c(\ell;P) \cap [s+1, w-1]| \leq |S(Q)|$. By the maximality of $|S(P)|$, we have $|N^c(\ell;P) \cap [s+1, w-1]| \leq |S(P)|$. □

Claim 3. $|S(P)| \geq 3$.

Proof. Suppose, to the contrary, that $|S(P)| \leq 2$. Assume $r \neq 1$. Since if $r = 1$, by Lemma 4 (e) we take $P = (2, 3, \cdots, \ell, 1)$. Then we have $N^c(1; P) = [3, r] \cup [t, u] \cup [w, \ell]$. We divide the proof into cases, depending on the value of w.

Case 1. $w \leq \ell - 1$.

Now we consider $P' = (\ell, \ell-1, \cdots, 1)$. Note that $N^c(1; P) = N^c(1; P')$ and $N^c(\ell; P) = N^c(\ell; P')$. Since $w \leq \ell - 1$, we have $\ell, \ell-1 \in S(P')$. By the maximality of $|S(P)|$, we have $|S(P)| = |S(P')| = 2$. According to Claim 1, we have $r = 2$ and $s = 3$. Then, $N^c(1; P) = [t, u] \cup [w, \ell]$, and

$$c(\ell, \ell-1) \neq c(\ell, 3) = c(3, 4) \neq c(3, 2). \quad (5)$$

Let $P_1 = (3, 4, \cdots, \ell, 2, 1) = (v_1, v_2, \cdots, v_\ell) \in \mathcal{R}(P)$.

Subcase 1.1. $w = \ell - 1$.

In this subcase, it follows that $N^c(1; P) = \{t, t+1, \ell-1, \ell\}$, $n - d = 5$ and $t + 1 \leq 5$. Since $t \geq 3$, we have $t = 3$ or 4.

If $t = 3$, then by Lemma 4 (e), $S(P_1) = \{3, 4\}$. Thus, $r(P_1) = 1$. Then, applying Lemma 4 (f) and (c) with $P^* = P_1$, we have $X(P_1) = [t(P_1), t(P_1) + 1] \cup [\ell - 2, \ell - 1]$. Therefore, $\ell, 2 \in N^c(3; P_1)$. Hence, $c(3, \ell) \neq c(3, 4)$, a contradiction to (5).

If $t = 4$, then $S(P_1) = [4, 5]$ and $r(P_1) = 2$. Applying Lemma 4 (f) and (c) with $P^* = P_1$, we have $X(P_1) = [t(P_1), t(P_1) + 1] \cup [\ell - 1, \ell]$. By Lemma 4 (d),

$$c(3, 4) \neq c(3, v_{t(P_1)}) = c(v_{t(P_1)}, v_{t(P_1)+1}) \neq c(v_{t(P_1)}, v_{t(P_1)-1}). \quad (6)$$

Noticing that $\ell = v_{\ell-2}$, and $\ell \notin N^c(3; P_1)$ by (5), we have $t(P_1) \in [3, \ell - 4]$ and $v_{t(P_1)} \in [5, \ell - 2]$. According to Lemma 4 (e), $P_2 = (4, 5, \cdots, \ell, 3, 2, 1) \in \mathcal{R}(P)$, $N^c(1; P_2) = \{4, 5, \ell-1, \ell\}$ and $S(P_2) = \{4, 5\}$. Thus, $r(P_2) = 1$. Applying Lemma 4 (f) and (e) with $P^* = P_2$, we have $\ell - 1 \in X(P_2)$, that is, $2 \in N^c(4; P_2)$. Then, we have

$$c(4, 5) \neq c(4, 2) = c(2, 3) \neq c(1, 2). \quad (7)$$

Recalling that $\ell - 1 \in S(P')$ and $3 \in S(P)$, we have

$$c(1, 2) \neq c(1, \ell-1) = c(\ell-1, \ell-2) \neq c(\ell-1, \ell) \neq c(3, \ell) = c(3, 4). \quad (8)$$

Since $4 = t < u < w = \ell - 1$, we have $\ell \geq 7$. Therefore, combining (5), (6), (7) and (8), we can get that $(1, 2, 4, 5, \cdots, v_{t(P_1)-1}, v_{t(P_1)}, 3, \ell, \ell-1, 1)$ is a PC cycle of length at least 6 (see Figure 2), a contradiction.

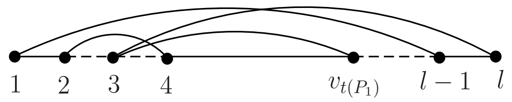

Figure 2: A PC cycle of length at least 6: $(1, 2, 4, 5, \cdots, v_{t_1-1}, v_{t_1}, 3, \ell, \ell-1, 1)$

Subcase 1.2. $w \leq \ell - 2$.

In this subcase, it follows that $\ell - 2 \notin S(P')$, and

$$c(1, 2) \neq c(1, \ell-2) \neq c(\ell-2, \ell-3). \tag{9}$$

Hence, $C_1 = (1, 2, \cdots, \ell-2, 1)$ is a PC cycle. Clearly, $|C_1| = \ell - 2 \leq n - d$. Then, $\ell = n - d + 2$. Applying Lemma 4 (f) and (b), $|N^c(\ell; P')| = n - d - 1 = \ell - 3$. Since $1, \ell - 1 \notin N^c(\ell; P')$, we have $N^c(\ell; P') = [2, \ell-2]$. According to Lemma 4 (a), $|C_0| = n - d = \ell - 2$ and $s = 3$, we have that $w = 6$ and $\ell \geq 8$.

If $\ell = n$, then $|N^c(\ell; P')| = \ell - 3 = n - 3 = n - d - 1$. Thus, $d = 2$. By Lemma 3, we have $3 = s < u < w = 6$. Then, $u = 4$ or 5. Since $d = 2$, we have $s \notin [t, u]$; otherwise, $c(1, s) = c(s, s+1) = c(\ell, s)$ which implies that $d^{mon}(s) \geq 3$, a contradiction. Hence, $t = 4$ and $u = 5$. So, $N^c(1; P) = \{4, 5, \ell-1, \ell\}$. Then, $|X(P)| = n - 3 = 4$, which implies that $n = \ell = 7$, a contradiction.

If $\ell < n$, then there exists a vertex $z \in V(G) \setminus V(P)$. Since $s = 3$, by Lemma 4 (e) $(1, 2, 3, \ell, \ell-1, \cdots, 5, 4) \in \mathcal{R}(P)$. Then

$$c(4, z) = c(4, 5) \neq c(3, 4). \tag{10}$$

Since $\ell \geq 8$, we have $5 \in N^c(\ell; P)$. Since $s = 3$ and $w = 6$, from Claim 2 we have that

$$c(5, \ell) = c(4, 5) \neq c(5, 6). \tag{11}$$

Combining (9), (10) and (11), $(z, 4, 3, 2, 1, \ell-2, \ell-3, \cdots, 5, \ell, \ell-1)$ is a PC path longer than P (see Figure 3), a contradiction.

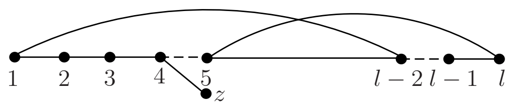

Figure 3: A PC path of length $\ell + 1$: $(z, 4, 3, 2, 1, \ell-2, \ell-3, \cdots, 5, \ell, \ell-1)$

Case 2. $w = \ell$.

We divide this case into subcases, depending on the value of $|S(P)|$.

Subcase 2.1. $|S(P)| = 2$.

In this subcase, it follows that $N^c(1;P) = [3,r] \cup [t,t+1] \cup \{\ell\}$. Since $|N^c(1;P)| = n-d-1$ and $t+1 \leq n-d$, we have $t = n-d-1$ and $r = n-d-2$. From Claim 2, $n-d-1 \leq |N^c(\ell;P)| = |[r,s]| + |N^c(\ell;P) \cap [s+1,\ell-2]| \leq 2|S(P)| = 4$. Then, $n-d-1 \leq 4$. Since $t \geq 3$, we have $n-d = 4$ or 5.

Subcase 2.1.1. $n-d=4$.

In this subcase, it follows that $r=2$, $s=t=3$, $N^c(1;P) = \{3,4,\ell\}$ and

$$c(1,3) = c(3,4). \tag{12}$$

Given a path $Q = (v_1, v_2, \cdots, v_\ell)$, we define the path $\phi(Q) = (v_3, v_4, \cdots, v_\ell, v_2, v_1)$. Set $P_0 = P = (1,2,\cdots,\ell)$. Define P_i to be $\phi(P_{i-1})$, $i \geq 1$. We write p_j^i to be the j^{th} vertex of P_i. We are going to prove following statements for $i \geq 1$.

(i) $P_i \in \mathcal{R}(P_0)$.

(ii) $S(P_i) = \{p_j^i : j \in \{1,2\}\}$.

(iii) $N^c(p_1^i; P_i) = \{p_j^i : j \in \{3,4,\ell-1\}\}$ and $c(p_1^i, p_j^i) = c(p_j^i, p_{j+1}^i)$, $j = 3,4$.

(iv) $N^c(p_2^i; P_i) = \{p_j^i : j \in \{1,4,5\}\}$; moreover, $c(p_2^i, p_j^i) = c(p_j^i, p_{j+1}^i)$, $j = 4,5$.

Firstly, we are going to show (i)-(iii) by induction on i. Note that $N^c(1;P) = \{3,4,\ell\}$ and $s=3$. Then by Lemma 4, $P_1 \in \mathcal{R}(P_0)$, $r(P_1) = 1$ and $S(P_1) = \{3,4\} = \{p_1^1, p_2^1\}$. Since $t(P_1)+1 \leq n-d = 4$ and $t(P_1) \geq 3$, we have $t(P_1) = 3$. Therefore, $N^c(p_1^1; P_1) = \{p_j^1 : j \in \{3,4,\ell-1\}\}$. Thus, the statements hold for $i=1$. Assume that they are true for $i-1$, where $i \geq 2$. For the sake of simplicity, we use r_i, s_i, t_i, u_i, w_i instead of $r(P_i), s(P_i), t(P_i), u(P_i), w(P_i)$.

(i) According to the induction hypothesis, we have $p_2^{i-1} \in S(P_{i-1})$. Then by Lemma 4 (e), $P_i = (p_3^{i-1}, p_4^{i-1}, \cdots, p_\ell^{i-1}, p_2^{i-1}, p_1^{i-1}) \in \mathcal{R}(P_0)$.

(ii) According to the induction hypothesis, we have $S(P_{i-1}) = \{p_j^{i-1} : j \in \{1,2\}\}$ and $N^c(p_1^{i-1}; P_{i-1}) = \{p_j^{i-1} : j \in \{3,4,\ell-1\}\}$. Then, $r_{i-1} = 1$ and $t_{i-1} = 3$. Since $r_{i-1} \leq 2 \leq s_{i-1}$ and $2 < t_{i-1}$, according to Lemma 4 (e), we have $S(P_i) = \{p_j^{i-1} : j \in \{3,4\}\} = \{p_j^i : j \in \{1,2\}\}$.

(iii) Since $r_i = 1$ and $|S(P_i)| = 2$, we have $N^c(p_1^i; P_i) = \{p_j^i : j \in \{t_i, t_i+1, \ell-1\}\}$ ($w_i = \ell-1$ as $|N^c(p_1^i; P_i)| = 4$). Since $t_i + 1 \leq n-d = 4$ and $t_i \geq 3$, we have $t_i = 3$. Hence, $N^c(p_1^i; P_i) = \{p_j^i : j \in \{3,4,\ell-1\}\}$.

(iv) Since $p_1^i \in S(P_i)$, by Lemma 4 (e), $P_i^2 = (p_2^i, p_3^i, \cdots, p_\ell^i, p_1^i) = (v_1, v_2, \cdots, v_\ell) \in \mathcal{R}(P)$, $N^c(p_1^i; P_i) = N^c(p_1^i; P_i^2)$ and $S(P_i^2) = \{p_j^i : j \in \{3,4\}\}$. Then, $r(P_i^2) = 2$. Applying Lemma 4 (f) and (c) with $P^* = P_i^2$, we have $N^c(p_2^i; P_i^2) = \{v_j : j \in \{t(P_i^2), t(P_i^2)+1, \ell\}\}$.

Since $t(P_i^2)+1 \le n-d=4$ and $t(P_i^2)\ge 3$, we have $t(P_i^2)=3$. Therefore, $N^c(p_2^i;P_i^2)=\{v_3,v_4,v_\ell\}=\{p_j^i : j\in\{4,5,1\}\}$. Moreover, by Lemma 4 (d), $c(p_2^i,p_j^i)=c(p_j^i,p_{j+1}^i)$, $j=4,5$.

Since $3=s<u<w=\ell$, $\ell\ge 5$. If ℓ is odd, taking $i=\frac{\ell+1}{2}$, then $P_{\frac{\ell+1}{2}}=(1,4,3,\cdots,\ell-1,\ell-2,2,\ell)$. If ℓ is even, taking $i=\frac{\ell}{2}$, then $P_{\frac{\ell}{2}}=(2,1,4,3,\cdots,\ell,\ell-1)$. By (iii) and (iv), $c(1,3)\ne c(3,4)$, a contradiction to (12).

Subcase 2.1.2. $n-d=5$.

In this subcase, it follows that $t=s=4$, $r=3$. According to Lemma 4 (e), $P_1=(4,5,6,\cdots,\ell,3,2,1)=(v_1,v_2,\cdots,v_\ell)\in\mathcal{R}(P)$ and $S(P_1)=\{4,5\}$. Then, $r(P_1)=1$ and $s(P_1)=2$. Applying Lemma 4 (f) and (c), we have $X(P_1)=\{t(P_1),t(P_1)+1,\ell-1,\ell-2\}$. Since $t(P_1)+1\le n-d=5$ and $t(P_1)\ge 3$, we have $t(P_1)=3$ or 4. Since $r(P_1)\le t(P_1)-2\le s(P_1)$, we have $P_2=(v_{t(P_1)-1},v_{t(P_1)},\cdots,v_\ell,v_{t(P_1)-2},\cdots,v_1)\in\mathcal{R}(P)$ and $S(P_2)=\{v_j : j\in\{t(P_1),t(P_1)+1\}\}$. Then, $r(P_2)=2$ and $s(P_2)=3$. Applying Lemma 4 (f) and (c), we have $X(P_2)=\{t(P_2),t(P_2)+1,\ell-1,\ell\}$. Since $t(P_2)+1\le n-d=5$ and $t(P_2)\ge 3$, we have $t(P_2)=3$ or 4. Hence, we can apply Subcase 1.1 with $P=P_2$. If $t(P_2)=3$, then $c(v_1,v_{t(P_1)+1})\ne c(v_{t(P_1)+1},v_{t(P_1)+2})$, a contradiction. If $t(P_2)=4$, then there is a PC cycle of length at least 6, a contradiction.

Subcase 2.2. $|S(P)|=1$.

According to Lemma 3 and the maximality of $|S(P)|$, $s(P')$ exists and $|S(P')|=1$. Moreover by Claim 1, $r=2$ and $r(P')=1$. Then according to Lemma 4 (f) and (c), $N^c(1;P)=\{t,\ell\}$. Then, $|N^c(1;P)|=2$, which implies that $d^c(1)\le 3$ and $n-d=3$. Hence, $t=t(P')=3$. Then, $N^c(1;P)=\{3,\ell\}$ and $N^c(\ell;P)=\{2,\ell-2\}$. Thus,

$$c(1,2)\ne c(1,3)=c(3,4)\ne c(2,3), \tag{13}$$

$$c(1,2)\ne c(1,\ell)=c(\ell,\ell-1)\ne c(\ell-1,\ell-2), \tag{14}$$

and

$$c(\ell,\ell-1)\ne c(\ell,\ell-2)=c(\ell-2,\ell-3)\ne c(\ell-2,\ell-1). \tag{15}$$

According to Lemma 4 (e), (f) and (c), $P_1=(3,4,\cdots,\ell,2,1)\in\mathcal{R}(P)$ and $N^c(3;P_1)=\{2,5\}$. Then

$$c(3,4)\ne c(3,5)\ne c(5,4). \tag{16}$$

Subcase 2.2.1. $d^c(1)=2$.

In this subcase, it follows that

$$c(3,4)=c(1,3)=c(1,\ell)=c(\ell,\ell-1). \tag{17}$$

If $\ell = 5$, then $c(3,4) = c(4,5)$ by (13), (14) and (17), a contradiction. If $\ell = 6$, then $c(6,4) = c(3,4)$ by (15). Then, $c(6,4) = c(5,6)$ by (13), (14) and (17), a contradiction. Thus, $\ell \geq 7$, and then $c(3,\ell-1) = c(3,4) \neq c(2,3)$. By (17), $c(3,\ell-1) \neq c(\ell-1,\ell-2)$. Combining these with (13), (14), (17), $(1,2,3,\ell-1,\ell-2,\ell,1)$ is a PC cycle of length 6 (see Figure 4), a contradiction.

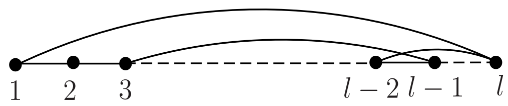

Figure 4: A PC cycle of length 6: $(1,2,3,\ell-1,\ell-2,\ell,1)$

Subcase 2.2.2. $d^c(1) = 3$.

In this subcase, it follows that $c(1,3) \neq c(1,\ell)$. If $\ell = 5$, then by (13), (14) and (15), $(1,3,5,4,1)$ is a PC cycle of length 4, a contradiction. If $\ell = 6$, then by (13), (14), (15) and (16), $(1,3,5,4,6,1)$ is a PC cycle of length 5, a contradiction. Thus, $\ell \geq 7$, and then

$$c(3,\ell-1) = c(3,4). \tag{18}$$

We may assume that

$$c(3,\ell-1) = c(\ell-1,\ell-2); \tag{19}$$

or else, $(1,2,3,\ell-1,\ell-2,\ell,1)$ is a PC cycle of length 6, (see Figure 4), a contradiction. If $\ell = 7$, then $c(3,4) \neq c(3,5) = c(5,6) = c(3,6)$. Since $6 \notin N^c(3,P_1)$, we have $c(3,6) = c(3,4)$, a contradiction. Hence, $\ell \geq 8$. Then, $c(3,4) = c(3,\ell-2)$. Combining (18) and (19), we have $c(3,\ell-2) = c(\ell-1,\ell-2)$. Hence together with (13), (14) and (15), $(1,2,3,\ell-2,\ell,1)$ is a PC cycle of length 5, a contradiction. The proof of Claim 3 is thus complete. $\square$

Claim 4. There exists a path $Q \in \mathcal{R}(P)$ with $|S(Q)| = |S(P)|$ such that $t(Q) \geq r(Q)+3$.

Proof. By contradiction, suppose $t \leq r+2$. Since $|S(P)| \geq 3$, we have $t-1 \in S(P)$. Without loss of generality, we assume $r = 1$; otherwise, consider $(t, t+1, \cdots, \ell, t-1, \cdots, 1)$ instead. Since $max\{3, r+1\} \leq t \leq r+2$, we have $t = 3$. Since $|[t,u]| = |[r,s]|$, we have $u = s+2 \geq 5$. Then, $N^c(1;P) = [3, s+2] \cup [w, \ell-1]$. By Lemma 4, we have

$$c(1,3) = c(3,4). \tag{20}$$

Given a path $Q = (v_1, v_2, \cdots, v_\ell)$, we define the path $\phi(Q) = (v_3, v_4, \cdots, v_\ell)$. Set $P_0 = P = (1,2,\cdots,\ell)$. Define P_i to be $\phi(P_{i-1})$, $i \geq 1$. We write p^i_j to be the j^{th} vertex of P_i. We are going to prove the following statements for $i \geq 0$.

(i) $P_i \in \mathcal{R}(P_0)$.

(ii) $S(P_i) = \{p^i_j : i \in [1, s]\}$.

(iii) $N^c(p^i_1; P_i) = \{p^i_j : j \in [3, s+2] \cup [w, \ell-1]\}$, and $c(p^i_1, p^i_j) = c(p^i_j, p^i_{j+1})$, $j \in [3, s+2]$.

(iv) $N^c(p^i_2; P_i) = \{p^i_j : j \in [4, n-d+1] \cup \{1\}\}$; moreover, $c(p^i_2, p^i_j) = c(p^i_j, p^i_{j+1})$, $j \in [4, n-d+1]$.

Firstly, we are going to show (i)-(iii) by induction on i. The statements are true for $i = 0$. Assume that the statements are true for $i - 1$, where $i > 1$. For the sake of simplicity, we use r_i, s_i, t_i, u_i, w_i instead of $r(P_i), s(P_i), t(P_i), u(P_i), w(P_i)$.

(i) According to the induction hypothesis, we have $p^{i-1}_2 \in S(P_{i-1})$. Then by Lemma 4 (e), we have $P_i = (p^{i-1}_3, p^{i-1}_4, \cdots, p^{i-1}_\ell, p^{i-1}_2, p^{i-1}_1) \in \mathcal{R}(P_0)$.

(ii) According to the induction hypothesis, we have $S(P_{i-1}) = \{p^{i-1}_j : j \in [1, s]\}$ and $N^c(p^{i-1}_1; P_{i-1}) = \{p^{i-1}_j : j \in [3, s+2] \cup [w, \ell-1]\}$. Then, $r_{i-1} = 1$ and $t_{i-1} = 3$. Since $r_{i-1} \le 2 \le s_{i-1}$ and $2 < t_{i-1}$, according to Lemma 4 (e), we have $S(P_i) = \{p^{i-1}_j : j \in [3, s+2]\} = \{p^i_j : j \in [1, s]\}$.

(iii) Since $r_i = 1$ and $|S(P_i)| = |S(P_0)|$, we have $N^c(p^i_1; P_i) = \{p^i_j : j \in [t_i, t_i + |S(P_0)| - 1] \cup [w_0, \ell-1]\}$ ($w_i = w_0$ as $|N^c(p^i_1; P_i)| = |N^c(p^0_1; P_0)|$ by Lemma 4 (b)). If $t_i > 3$, then Claim 4 holds by taking $Q = P_i$. Thus, $t_i = 3$. Then, $N^c(p^i_1; P_i) = \{p^i_j : j \in [3, s+2] \cup [w, \ell-1]\}$. By Lemma 4 (d), $c(p^i_1, p^i_j) = c(p^i_j, p^i_{j+1})$, $j \in [3, s+2]$.

(iv) Since $p^i_1 \in S(P_i)$, by Lemma 4 (e), $P^2_i = (p^i_2, p^i_3, \cdots, p^i_\ell, p^i_1) = (v_1, v_2, \cdots, v_\ell) \in \mathcal{R}(P)$, $N^c(p^i_1; P_i) = N^c(p^i_1; P^2_i)$ and $S(P^2_i) = \{p^i_j : j \in [3, s+2]\}$. Then, $r(P^2_i) = 2$. Applying Lemma 4 (f) and (c) with $P^* = P^2_i$, we have that $N^c(p^i_2; P^2_i) = \{v_j : j \in [t(P^2_i), u(P^2_i)] \cup [w(P^2_i), \ell]\}$ and $|N^c(p^i_2; P^2_i)| = n - d - 1$. Since $p^i_2 \in S(P_i)$, we have $c(p^i_\ell, p^i_2) = c(p^i_2, p^i_3)$. Thus, $p^i_\ell \notin N^c(p^i_2; P^2_i)$. Noticing that $p^i_\ell = v_{\ell-1}$, we have $N^c(p^i_2; P^2_i) = \{p^j_i : j \in [t(P^2_i), u(P^2_i)] \cup \{\ell\}\}$. By Lemmas 3 and 4, we have that $u(P^2_i) \le n - d$ and $t(P^2_i) \ge 3$. Hence, $u(P^2_i) = n - d$ and $t(P^2_i) = 3$. Therefore, $N^c(p^i_2; P^2_i) = \{v_j : j \in [3, n-d] \cup \{\ell\}\} = \{p^j_i : j \in [4, n-d+1] \cup \{1\}\}$. By Lemma 4 (d), we have $c(p^i_2, p^i_j) = c(p^i_j, p^i_{j+1})$, $j \in [4, n-d+1]$.

Since $3 \le s < u < w = \ell$, $\ell \ge 5$. If ℓ is odd, taking $i = \frac{\ell+1}{2}$, then $P_{\frac{\ell+1}{2}} = (1, 4, 3, \cdots, \ell-1, \ell-2, 2, 5)$. If ℓ is even, taking $i = \frac{\ell}{2}$, then $P_{\frac{\ell}{2}} = (2, 1, 4, 3, \cdots, \ell, \ell-1)$. By (iii) and (iv),$c(1, 3) \ne c(3, 4)$, a contradiction to (20). □

According to Claim 4, we assume $t \ge r + 3$.

Claim 5. $c(r+1, r+3) \notin \{c(r+1, r+2), c(r+3, r+4)\}$.

Proof. By Lemma 4 (e), $P_1 = (r+1, r+2, \cdots, \ell, r \cdots, 1) = (v_1^1, v_2^1, \cdots, v_\ell^1) \in \mathcal{R}(P)$ and $S(P_1) = [t, u]$. Since $t \geq r+3$, $r+1 \notin N^c(1; P) = N^c(1; P_1)$. Then, $r(P_1) \geq 3$. Applying Lemma 4 (f) and (c) with $P^* = P_1$, we have $N^c(r+1; P_1) = \{v_j^1 : j \in [3, r(P_1)] \cup [t(P_1), u(P_1)] \cup [w(P_1), \ell]\}$. Noticing $r+3 \in \{v_j^1 : j \in [3, r(P_1)]\}$, we have $r+3 \in N^c(r+1; P_1)$. Hence, $c(r+1, r+3) \neq c(r+1, r+2)$.

Since $|S(P)| \geq 3$, we have $r+2 \in S(P)$. By Lemma 4 (e), $P_2 = (r+3, r+4, \cdots, \ell, r+2, r+1, \cdots, 1) = (v_1^2, v_2^2, \cdots, v_\ell^2) \in \mathcal{R}(P)$ with $S(P_2) = [t, u]$ and $N^c(r+3; P_2) = \{v_j^2 : j \in X(P_2)\}$, where

$$X(P_2) = \begin{cases} [3, r(P_2)] \cup [t(P_2), u(P_2)] \cup [w(P_2), \ell], & t \neq r+3, \\ [t(P_2), u(P_2)] \cup [w(P_2), \ell-1], & t = r+3. \end{cases}$$

Then by Lemma 4 (d), $c(r+3, v_j^2) = c(v_j^2, v_{j+1}^2), t(P_2) \leq j \leq u(P_2)$. Since $r+2 \in S(P)$, we have

$$c(\ell, \ell-1) \neq c(\ell, r+2) = c(r+2, r+3) \neq c(r+3, r+4). \tag{21}$$

Then, $r+2 \in N^c(r+3; P_2)$ and $r+2 \in \{v_j^2 : j \in [w(P_2), \ell-1]\}$. Noticing that $v_{\ell-1}^2 = 2$, we have $[2, r+2] \subseteq N^c(r+3; P_2)$. In particular, $r+1 \in N^c(r+3; P_2)$. Thus, $c(r+1, r+3) \neq c(r+3, r+4)$. This claim is thus complete. □

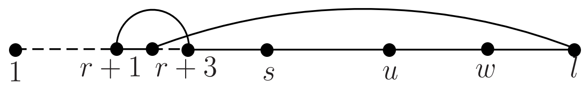

Figure 5: $C = (r+1, r+3, r+4, \cdots, \ell, r+2, r+1)$

According to Claim 5 and (21), $C = (r+1, r+3, r+4, \cdots, \ell, r+2, r+1)$ is a PC cycle containing $N^c(\ell; P) \cup \{\ell, \ell-1\} \setminus \{r\}$ (see Figure 5). Hence, $|C| = n - d$.

If $\ell = n$, then $N^c(\ell; P) = [d, \ell-2]$, which implies $r = d$. Since $1 \notin N^c(\ell; P)$, we have $c(1, \ell) = c(\ell, \ell-1)$, and then $c(\ell, r+2) \neq c(\ell, 1)$. Noticing that $V(P) \setminus V(C_0) = [s+1, w-1]$, we have $w = s+d+1$. Since $|[r, s]| = |[t, u]|$ and $t \geq r+3$, we have $u \geq s+3$. Hence, $d \geq 3$. Note that $c(\ell-1, j) \in \{c(\ell-1, \ell-2), c(j, j+1)\}$ for $j \in [1, r-1]$; or else, $(j, j+1, \cdots, \ell-1, j)$ is a PC cycle of length at least $n-d+1$, a contradiction. If there exists a vertex $j_0 \in [2, r-1]$ such that $c(\ell-1, j_0) \neq c(\ell-1, \ell-2)$, then $c(\ell-1, j_0) = c(j_0, j_0+1) \neq c(j_0, j_0-1)$. Then combining these with Claim 5, $(r+1, r+3, r+4, \cdots, \ell-1, j_0, j_0-1, \cdots, 1, \ell, r+2, r+1)$ is a PC cycle of length at least $n-d+1$ (see Figure 6), a contradiction. Therefore, $c(\ell-1, j) = c(\ell-1, l-2)$ for $j \in [2, r-1]$. If $c(1, \ell-1) \neq c(\ell-1, \ell-2)$, then $c(1, \ell-1) = c(1, 2)$. Hence by Lemma 4 (c),

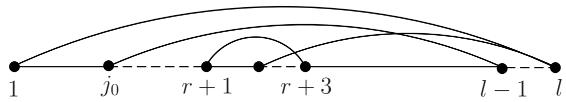

Figure 6: A PC cycle of length at least $n-d+1$: $(r+1, r+3, r+4, \cdots, \ell-1, j_0, j_0-1, \cdots, 1, \ell, r+2, r+1)$

$w=\ell$. Then, $c(1,\ell) \neq c(1,\ell-1)$. Therefore, $(r+1, r+3, r+4, \cdots, \ell-1, 1, l, r+2, r+1)$ is a PC cycle of length $n-d+1$, a contradiction. Since $d^{mon}(\ell-1) \leq d$, we have

$$c(\ell-1, r) \neq c(\ell-1, \ell-2). \tag{22}$$

Then, $c(\ell-1, r) = c(r, r-1)$, or else $(r+1, r+3, r+4, \cdots, \ell-1, r, r-1, \cdots, 1, \ell, r+2, r+1)$ is a PC cycle of length at least $n-d+1$, a contradiction. Then

$$c(\ell, r) = c(r, r+1) \neq c(r, r-1) = c(\ell-1, r). \tag{23}$$

Since $|S(P)| \geq 3$, we have $r+1 \in S(P)$. By Lemma 4 (e), $P_1 = (r+2, r+3, \cdots, \ell, r+1, r, \cdots, 1) = (v_1, v_2, \cdots, v_\ell) \in \mathcal{R}(P)$ with $S(P_1) = [t, u]$ and $N^c(r+2; P_1) = \{v_j : j \in [3, r(P_1)] \cup [t(P_1), u(P_1)] \cup [w(P_1), \ell]\}$. Then by Lemma 4 (d), $c(r+2, v_j) = c(v_j, v_{j+1}), t(P_1) \leq j \leq u(P_1)$. Since $r+1 \in S(P)$, we have $c(\ell, \ell-1) \neq c(\ell, r+1) = c(r+1, r+2) \neq c(r+2, r+3)$. Then, $r+1 \in N^c(r+2; P_1)$ and $r+1 \in \{v_j : j \in [w(P_1), \ell-1]\}$. Noticing that $v^2_{\ell-1} = 2$, we have $[2, r+1] \subseteq N^c(r+2; P_1)$. In particular, $r \in N^c(r+2; P_1)$. Thus, $c(r+2, r) \neq c(r+2, r+3)$. Then, $c(r+2, r) = c(r+1, r)$, or else $(r+2, r+3, \cdots, \ell, r+1, r, r+2)$ is a PC cycle containing $N^c(l; P) \cup \{\ell, \ell-1\}$, a contradiction. Therefore, $c(r+2, r+3) \neq c(r, r+1)$. Since $r, r+2 \in S(P)$, we have

$$c(l, r) \neq c(l, r+2). \tag{24}$$

Hence combining Claim 5 and (22), (23), (24), $(r+1, r+3, r+4, \cdots, \ell-1, r, \ell, r+2, r+1)$ is a PC cycle of length at least $n-d+1$ (see Figure 7), a contradiction.

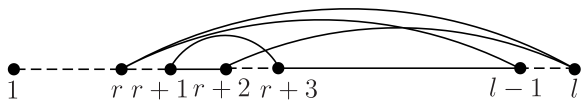

Figure 7: $C = (r+1, r+3, r+4, \cdots, \ell-1, r, \ell, r+2, r+1)$

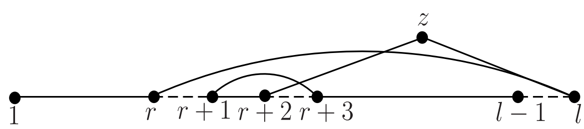

Figure 8: A PC path of length $\ell+1$: $(1, 2, \cdots, r, \ell, z, r+2, r+1, r+3, r+4, \cdots, \ell-1)$

Then we may assume $\ell < n$. Hence, there exists a vertex $z \in V(G) \setminus V(P)$. Note that $c(\ell-1, \ell) = c(\ell, z)$. Since $r+2 \in S(P)$, $(1, 2, \cdots, r+1, \ell, \ell-1, \cdots, r+2)$ is also a longest PC path. Thus, $c(r+2, r+3) = c(r+2, z)$ and $c(r+2, r+3) = c(r+2, \ell) \neq c(\ell, \ell-1) = c(\ell, z)$. Then, $c(r+2, z) \neq c(\ell, z)$. Therefore, $(1, 2, \cdots, r, \ell, z, r+2, r+1, r+3, r+4, \cdots, \ell-1)$ is a PC path longer than P (see Figure 8), a contradiction.

Theorem 3 is thus complete.

4 Concluding remarks

There have been many researchers working on Conjecture 1, which implies that the bound on the length of a PC cycle in Theorem 3 is not sharp. The author in [13] showed that $\Delta^{mon}(K_n^c) \leq \frac{n}{7}$ is sufficient for the existence of a PC Hamiltonian cycle. Up to 2016, Lo [11] showed that for any $\varepsilon > 0$, there exists an integer n_0 such that every edge-colored complete graph K_n^c with $\Delta^{mon}(K_n^c) < (\frac{1}{2} - \varepsilon)n$ and $n \geq n_0$ contains a PC Hamiltonian cycle, which implies a result obtained by Alon and Gutin [1] that for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$, any edge-colored complete graph K_n^c with $\Delta^{mon}(K_n^c) < (1 - \frac{1}{\sqrt{2}} - \varepsilon)n$ and $n \geq n_0$ contains a PC Hamiltonian cycle. Hence, the conjecture of Bollobás and Erdős is true asymptotically.

While the authors in [5] constructed an edge-colored complete graph of order $2m$ with $\delta^c(G) = m$ and $\Delta^{mon}(G) = m$ that does not contain a PC Hamiltonian cycle, which implies that the condition $\Delta^{mon}(K_n^c) < \frac{n}{2}$ in Conjecture 1 is sharp.

As for the bound $\Delta^{mon}(K_n^c) \geq \frac{n}{2}$, we believe that there is also a potential sharp bound in Theorem 3. So, we pose the following conjecture.

Conjecture 7. *Let K_n^c be an edge-colored complete graph such that $\frac{n}{2} \leq \Delta^{mon}(K_n^c) = d \leq n-2$. Then K_n^c contains a PC cycle of length at least $2(n-d-1)$.*

Next we give an example of edge-coloring of a complete graph, supporting the conjecture.

Example 8. *Consider a complete graph of order n with $\Delta^{mon}(K_n^c) = d \geq \frac{n}{2}$. Let x be the vertex with the maximum monochromatic-degree and $N_i(x)$ be the set of vertices which are adjacent to x by color $i = 1, 2$. Then color $G[N_i(x)]$ with i, $i = 1, 2$, respectively, and color the edges in $E[N_1(x), N_2(x)]$ with color 3.*

In particular, Proposition of [11] (in the Arxiv version) provides with constructions to support Conjecture 7. Consider the edge-colored complete graph K_n^c in our Example 8. Clearly, when $n-d-1$ is odd, the longest PC cycle in K_n^c has a length $2(n-d)-1$;

while when $n-d-1$ is even, the longest PC cycle in K_n^c has a length $2(n-d-1)$. Since $\delta^c(K_n^c)+\Delta^{mon}(K_n^c)\leq n$, we have the following conjecture.

Conjecture 9. *Let K_n^c be an edge-colored complete graph such that $2\leq\delta^c(K_n^c)\leq\frac{n}{2}$. Then K_n^c contains a PC cycle of length at least $2\delta^c(K_n^c)-2$.*

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