Vertex-disjoint rainbow cycles in edge-colored graphs[1]

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Abstract

Let G be an edge-colored graph of order n. The *color-neighborhood* of a vertex u of G is the set of colors of the edges incident with u in G, denoted by $CN_G(u)$, or $CN(u)$ for short. A subgraph F of G is called *rainbow* if any two edges of F have distinct colors. In this paper, we first give a sufficient condition for the existence of rainbow cycles by using color-neighborhood unions of pairs of vertices in G. In 2019, Fujita et al. showed that G contains k vertex-disjoint rainbow cycles if $|CN(x) \cup CN(y)| \geq n/2 + 64k + 1$ for any two vertices x, y of G. We obtain a result that G contains k vertex-disjoint rainbow cycles if $|CN(x) \cup CN(y)| \geq n/2 + 18k + 1$ for any two vertices x, y of G. Furthermore, we give better bounds for $k = 2, 3$. Finally, we show that G contains two vertex-disjoint rainbow cycles of different lengths if $|CN(x) \cup CN(y)| \geq 2n/3 + 6$ for every pair of vertices x, y of G.

Keywords: edge-colored graph; color-neighborhood; vertex-disjoint; rainbow cycle.
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1 Introduction

We only consider finite simple graphs in this paper. For terminology and notations not defined here, we refer the reader to [3, 5]. By an *edge-colored graph* we mean a triple $G = (V(G), E(G), c)$, where $V(G)$ and $E(G)$ are the vertex-set and edge-set of G, respectively, and c is a mapping from $E(G)$ to the natural number set $\mathbb{N}$, called an edge-coloring of G. In an edge-colored graph G, we use $c(e)$ to denote the color of an edge e and $c(G)$ to denote the set of colors of all the edges of G. The set of edges with a same color i is called a color-class, or the color i-class. A subgraph of an edge-colored graph G is called *properly colored* if any two adjacent edges of the subgraph receive distinct

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colors. Similarly, a subgraph of an edge-colored graph G is called *rainbow* if any two edges of the subgraph receive distinct colors. Let $CN_G(u)$ denote the set of colors on the edges incident with a vertex u in G, and $d^c_G(u) = |CN_G(u)|$. We use $CN_G(u)$ and $d^c_G(u)$ to denote the *color-neighborhood* and *color-degree* of a vertex u in G, respectively. When there is no confusion, we write $CN(u)$ and $d^c(u)$ instead of $CN_G(u)$ and $d^c_G(u)$, respectively. Let $\delta^c(G)$ denote the minimum value of $d^c(u)$ over all vertices u in G, called the *minimum color-degree* of an edge-colored graph G. We use $d^{mon}(u)$ to denote the maximum number of edges with a same color incident with a vertex u in G, called the *monochromatic-degree* of u, and let $\Delta^{mon}(G) = \max\{d^{mon}(u) : u \in V(G)\}$, called the *maximum monochromatic-degree* of G.

As usual, we use C_k to denote a cycle of length k. For a subset S of $V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and $G - S$ to denote the subgraph of $G[V(G) \setminus S]$. For any two distinct vertex subsets S and T in G, we use $E(S,T)$ to denote the edge subset of G such that one end of each edge of $E(S,T)$ is in S and the other end is in T. Set $c(S,T) = \{c(e) : e \in E(S,T)\}$. If $S = \{v\}$, then we simply write $E(v,T)$ and $c(v,T)$ for $E(\{v\},T)$ and $c(\{v\},T)$, respectively.

For a digraph D, we use $V(D)$ to denote the vertex-set of D, and $A(D)$ to denote the arc-set of D, respectively. We say that a vertex y is an *out-neighbor* (*in-neighbor*) of a vertex x in D if (x,y) (resp., (y,x)) is an arc of D. $N^+_D(x)$ denotes the set of out-neighbors of x, and $N^-_D(x)$ denotes the set of in-neighbors of x in D. The cardinality of $N^+_D(x)$ is called the *out-degree* $d^+_D(x)$ of x, and the cardinality of $N^-_D(x)$ is called the *in-degree* $d^-_D(x)$ of x in D. Let $\delta^+(D)$ ($\delta^-(D)$) denote the minimum value of $d^+(u)$ ($d^-(u)$) over all vertices u in G, called the *minimum out-degree* (*minimum in-degree*) of a digraph D.

During the past decades, a great deal of research have been done on the existence of rainbow cycles in an edge-colored graph. For more details, we refer the reader to the literatures [10, 11, 12, 15, 16]. Recently, Han et al. [13] showed that every edge-colored complete graph G of order n with $\Delta^{mon}(G) \le n - 2k$ contains k properly colored cycles of different lengths. Motivated by this, we begin to devote ourselves to studying the existence of vertex-disjoint rainbow cycles (of different lengths) in an edge-colored graph, and as we all know that this question is closely related to the existence of vertex-disjoint directed cycles (of different lengths) in a digraph. The reader can find some results about vertex-disjoint directed cycles (of different lengths) in [2, 7, 14, 17, 18, 20, 21].

To illustrate the relationship between rainbow cycles in edge-colored graphs and directed cycles in digraphs, we present the following two famous conjectures.

Conjecture 1.1 ([8]). *Let D be a digraph of order n and r be a positive integer. If $\delta^+(D) \ge r$, then D contains a directed cycle of length at most $\lceil \frac{n}{r} \rceil$.*

Conjecture 1.2 ([1]). *Let G be an edge-colored graph of order n and c be an edge-coloring of G with n colors, and let r be a positive integer. If every color-class has a size at least*

r, then G contains a rainbow cycle of length at most $\lceil \frac{n}{r} \rceil$.

In fact, we can see that Conjecture 1.2 is a generalization of Conjecture 1.1 by the following construction: Let D be a digraph such that $\delta^+(D) \geq r$ and G be the underlying graph of D. Suppose $V(D) = \{v_1, v_2, ..., v_n\}$. For any arc v_iv_j in D, we color the edge v_iv_j of G with color i. In this way, we get an edge-coloring c of G with n colors such that every color-class has a size at least r. Note that D contains a directed cycle of length at most $\lceil \frac{n}{r} \rceil$ if and only if G contains a rainbow cycle of length at most $\lceil \frac{n}{r} \rceil$. The above construction is also the main approach in this paper.

There are a lot of literatures about the existence of rainbow cycles in an edge-colored graph G, using the minimum color-degree $\delta^c(G)$ in [11, 16] and color-degree sum $d^c(u) + d^c(v)$ in [19]. However, one can find that there are very few literatures using the color-neighborhood union $CN(u) \cup CN(v)$. Broersma, Li, Woeginger and Zhang first used the condition $CN(u) \cup CN(v)$ and proved the following result in [6].

Theorem 1.3 ([6]). *Let G be an edge-colored graph of order n. If $|CN(u) \cup CN(v)| \geq n-1$ for every pair of vertices u, v of G, then G contains a rainbow cycle of length at most four.*

In 2019, Fujita et al. strengthened Theorem 1.3 in [12].

Theorem 1.4 ([12]). *Let G be an edge-colored graph of order $n \geq 6$. If $|CN(u) \cup CN(v)| \geq n-1$ for every pair of vertices u, v of G, then G contains a rainbow triangle unless G is a rainbow $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.*

In this paper, we propose the following problem on the existence of rainbow cycles using the color-neighborhood unions of pairs of vertices in an edge-colored graph.

Problem 1.5. *Let G be an edge-colored graph of order n. Determine an integer valued function $f(n)$ with value as small as possible, such that if $|CN(u) \cup CN(v)| \geq f(n)$ for every pair of vertices u, v of G, then G contains a rainbow cycle.*

We investigate the above problem and get that $f(n) \leq \lfloor \frac{n}{2} \rfloor + 2$.

Theorem 1.6. *Let G be an edge-colored graph of order n. If $|CN(u) \cup CN(v)| \geq \lfloor \frac{n}{2} \rfloor + 2$ for every pair of vertices u, v of G, then G contains a rainbow cycle.*

In 1981, Bermond and Thomassen proposed the following conjecture and conjectured the bound is sharp.

Conjecture 1.7 ([4]). *Every digraph with minimum out-degree at least $2k-1$ contains k vertex-disjoint directed cycles.*

Bermond and Thomassen in [4] observed that complete symmetrical directed graphs on $2k-1$ vertices have out-degrees $2k-2$ and they contain at most $k-1$ vertex-disjoint directed cycles. Hence, they also conjectured that this bound is in fact sharp. However, this conjecture is so difficult that it has not yet been fully resolved. So far, the best result was given by Bucić in [7].

Lemma 1.8 ([7]). *Every digraph with minimum out-degree at least $18k$ contains k vertex-disjoint directed cycles.*

Furthermore, in [12], Fujita et al. also gave a sufficient condition for the existence of k vertex-disjoint rainbow cycles in edge-colored graphs.

Theorem 1.9 ([12]). *Let G be an edge-colored graph of order n satisfying that $|CN(u) \cup CN(v)| \geq \frac{n}{2}+64k+1$ for every pair of vertices u, v of G. Then G contains k vertex-disjoint rainbow cycles.*

Using Lemma 1.8 and doing the same discussion as in [12], we can improve the lower bound of Theorem 1.9. Even so, this bound is still not sharp.

Theorem 1.10. *Let G be an edge-colored graph of order n. If $|CN(u) \cup CN(v)| \geq \frac{n}{2} + 18k + 1$ for every pair of vertices u, v of G then G contains k vertex-disjoint rainbow cycles.*

While for the cases $k = 2, 3$, we get better lower bounds than Theorem 1.10.

Theorem 1.11. *Let G be an edge-colored graph of order n. If $k \in \{2, 3\}$ and $|CN(u) \cup CN(v)| \geq \frac{n}{2} + 5k - 4$ for every pair of vertices u, v of G, then G contains k vertex-disjoint rainbow cycles.*

Inspired by the above results, we propose the following problem on the existence of rainbow cycles of different lengths in an edge-colored graph.

Problem 1.12. *Suppose that k and n are integers and n is sufficiently large. Let G be an edge-colored graph of order n. Determine an integer valued function $f(k, n)$ with value as small as possible, such that if $|CN(u) \cup CN(v)| \geq f(k, n)$ for every pair of vertices u, v of G, then G contains at least k (vertex-disjoint) rainbow cycles of different lengths.*

To support the existence of $f(k, n)$, we prove the following result.

Theorem 1.13. *Let G be an edge-colored graph of order n. Each one of the following two conditions can guarantee the existence of two vertex-disjoint rainbow cycles of different lengths in G:*

(1) $|CN(u) \cup CN(v)| \geq \frac{2n}{3} + 6$ for every pair of vertices u, v of G;

(2) G contains a rainbow triangle and $|CN(u) \cup CN(v)| \geq \frac{n}{2} + 11$ for every pair of vertices u, v of G.

The rest of the paper is organized as follows: In Section 2, we give some basic notations and useful lemmas for the proofs of our main results. In Section 3, we are devoted to studying the existence of vertex-disjoint rainbow cycles in an edge-colored graph and giving proof for Theorem 1.6. Then we prove Theorem 1.11 by generalizing the main idea of Theorem 1.6. In the last section of the paper, we consider the existence of two vertex-disjoint rainbow cycles of different lengths in an edge-colored graph and prove Theorem 1.13.

2 Terminology and lemmas

Let G be an edge-colored graph. Choose an edge $xy \in E(G)$, and let $X = \{x_1, x_2, ..., x_s\} \subset N_G(x)\setminus\{y\}$ and $Y = \{y_1, y_2, ..., y_t\} \subset N_G(y)\setminus\{x\}$ such that the following conditions hold:

(a) $c(x_ix) \neq c(x_jx)$ for all $1 \leq i < j \leq s$ and $c(y_iy) \neq c(y_jy)$ for all $1 \leq i < j \leq t$;

(b) $c(xy) \notin \{c(x_ix), c(y_jy)\}$ and $c(x_ix) \neq c(y_jy)$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$;

(c) subject to (a) and (b), $s + t$ is maximized.

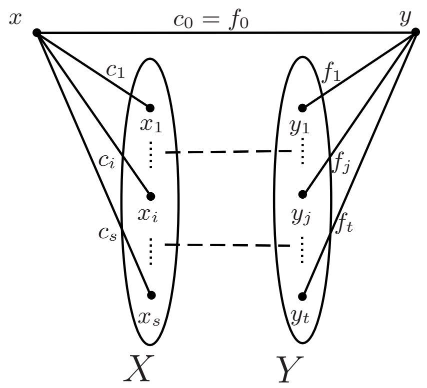

Figure 1: A subgraph $G[X \cup Y \cup \{x, y\}]$ of an edge-colored graph G without rainbow C_3 and C_4

For convenience, as shown in Figure 1, we assume that $c(xy) = c_0 = f_0$, $c(xx_i) = c_i$ and $c(yy_j) = f_j$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$. Note that the conditions (a), (b) and (c) guarantee that any two colors in the color-set $C^* = \{c_0, c_1, ..., c_s, f_1, f_2, ..., f_t\}$ are different.

If G contains no rainbow C_3 and C_4, then we have the following properties:

(i) $X \cap Y = \emptyset$ and $c(x_iy_j) \in \{c_0, c_i, f_j\}$ for $x_iy_j \in E(G)$;

(ii) $c(x_ix_j) \in \{c_i, c_j\}$ for $x_ix_j \in E(G)$ and $c(y_iy_j) \in \{f_i, f_j\}$ for $y_iy_j \in E(G)$.

Next, we define the *local associated digraph* $D = D_{xy}(X, Y)$ of G as follows: $V(D) = X \cup Y$ and $A(D) = \cup_{l=1}^{4} A_l(D)$, where

$$A_1(D) = \{x_ix_j | c(x_ix_j) = c_j\},\ A_2(D) = \{y_iy_j | c(y_iy_j) = f_j\},$$

$$A_3(D) = \{x_iy_j | c(x_iy_j) = f_j\},\ A_4(D) = \{y_ix_j | c(y_ix_j) = c_j\}.$$

Lemma 2.1. *Suppose that G is an edge-colored graph without rainbow cycles of length at most four, $D = D_{xy}(X, Y)$ is a local associated digraph of G and $H = G[X \cup Y \cup \{x, y\}]$. Then*

(1) D contains a directed cycle if and only if H contains a rainbow cycle;

(2) $d^+_D(u) \leq d^c_H(u) \leq d^+_D(u) + 2$;

(3) $|CN_H(u) \cup CN_H(v)| \leq d^+_D(u) + d^+_D(v) + 3$.

Proof. From the definition of the local associated digraph, statement (1) follows. Choose an arbitrary vertex $y_j \in Y$ for $1 \leq j \leq t$. If $xy_j \in E(G)$, to avoid that xyy_jx ia a rainbow triangle, we have $c(xy_j) \in \{c_0, f_j\}$. By a similar argument, we have $c(yx_i) \in \{c_0, c_i\}$ if $yx_i \in E(G)$ for all $1 \leq i \leq s$. From properties (i), (ii) and (iii), we can easily show that $d^+_D(u) \leq d^c_H(u) \leq d^+_D(u) + 2$, where the term 2 comes from the fact that one of ux and uy is an edge of G, and possibly there is an edge incident to u with color c_0. Statement (2) thus follows.

Choose two arbitrary vertices $u, v \in X \cup Y$. If u and v belong to different sets of X and Y, without loss of generality, set $u \in X$ and $v \in Y$. Note that the out-degree $d^+_D(u)$ ($d^+_D(v)$) of u (v) implies that there are at least $d^+_D(u)$ ($d^+_D(v)$) colors different from $c(xu)$ ($c(yv)$). Consider the possibly existing edge incident to u or v with the color c_0 in H. Then, $|CN_H(u) \cup CN_H(v)| \leq d^+_D(u) + d^+_D(v) + 3$. Similarly, we can show the case that u and v belong to the same set of X and Y. The lemma thus follows. □

For any two vertices x, y of an edge-colored graph G, we say that an edge subset S *contributes* k colors to $CN(x) \cup CN(y)$ if S has k edges incident to x or y with distinct colors. Now we consider the number of colors between a short rainbow cycle and other vertices in an edge-colored graph.

Lemma 2.2. *Let G be an edge-colored graph of order n and let $G_1, G_2, ..., G_r$ be r vertex-disjoint rainbow cycles such that $|G_i| \leq 4$ and $| \cup_{i=1}^r G_i|$ is the minimum in G. If G_i is a rainbow C_4, for any two vertices $u, v \in V(G) \setminus \cup_{i=1}^r V(G_i)$, we have the following two statements:*

(1) $E(\{u, v\}, G_i)$ contributes at most 6 colors to $CN(u) \cup CN(v)$ if $uv \notin E(G)$;

(2) $E(\{u, v\}, G_i)$ contributes at most 5 colors to $CN(u) \cup CN(v)$ if $uv \in E(G)$.

Proof. Suppose to the contrary that $E(\{u, u\}, G_i)$ contributes at least 7 colors to $CN(u) \cup CN(v)$ when $uv \notin E(G)$. Then there are two successive vertices x and y of G_i such that $E(\{u, v\}, \{x, y\})$ contributes 4 colors to $CN(u) \cup CN(v)$, which implies that $c(xy) \notin \{c(ux), c(uy)\}$ or $c(uv) \notin \{c(vx), c(vy)\}$. Thus, $uxyu$ or $vxyv$ is a rainbow triangle vertex-disjoint from G_j for all $j \neq i$ in G. Without loss of generality, suppose $uxyu$ is a rainbow triangle in G. Set $H_i = uxyu$ and $H_j = G_j$ for all $j \neq i$. Then we find other r vertex-disjoint rainbow cycles $H_1, H_2, ..., H_r$ such that $|H_i| \le 4$ and $|\cup_{i=1}^{r} H_i|$ is smaller than $|\cup_{i=1}^{r} G_i|$ in G, a contradiction. Statement (1) thus holds.

We show statement (2) by contradiction. Assume that $E(\{u,v\}, G_i)$ contributes at least 6 colors to $CN(u) \cup CN(v)$ when G_i is a rainbow C_4, where $1 \le i \le r$. Note that there are two successive or diagonal vertices x and y of G_i such that $E(\{x,y\}, G_i)$ contributes 4 colors to $CN(u) \cup CN(v)$.

In the former case, we can obverse that $c(xy) \notin \{c(ux), c(uy)\}$ or $c(xy) \notin \{c(vx), c(vy)\}$. This implies that $uxyu$ or $vxyv$ is a rainbow triangle vertex-disjoint from G_j for all $j \neq i$ in G. Without loss of generality, suppose $uxyu$ is a rainbow triangle in G. Set $H_i = uxyu$ and $H_j = G_j$ for all $j \neq i$. Then we find other r vertex-disjoint rainbow cycles $H_1, H_2, ..., H_r$ such that $|H_i| \le 4$ and $|\cup_{i=1}^{r} H_i|$ is smaller than $|\cup_{i=1}^{r} G_i|$ in G, a contradiction. In the latter case, we have $c(uv) \notin \{c(ux), c(vx)\}$ or $c(uv) \notin \{c(vx), c(vy)\}$. This implies that $uvxu$ or $uyvu$ is a rainbow triangle vertex-disjoint from G_j for all $j \neq i$ in G. By a similar argument, we can get a contradiction. The lemma thus follows. □

In [9], Čada et al. showed a lemma (statement (a) of Lemma 2.3) about an edge-colored graph containing no rainbow 4-cycles. It is easy to see that the proof of this lemma contains the result of statement (b) of Lemma 2.3.

Lemma 2.3. *[9] Let G be an edge-colored graph containing no rainbow 4-cycles and let $\{xy_iz\}_{i=1}^{p}$ be a set of rainbow (x,z)-paths of length two in G.*

(a) If $\{xy_i\}_{i=1}^{p}$ is rainbow, then $|\{C(y_iz)_{i=1}^{p}\}| \le 3$.

(b) If $\{xy_i\}_{i=1}^{p}$ is rainbow and $|\{C(y_iz)_{i=1}^{p}\}| = 3$, then $\{C(xy_i)_{i=1}^{p}\} = \{C(y_iz)_{i=1}^{p}\}$.

3 Vertex-disjoint rainbow cycles

As rainbow cycles in edge-colored graphs and directed cycles in digraphs are closely related, we first state a useful lemma about the existence of vertex-disjoint directed cycles in digraphs before our proofs.

Lemma 3.1 ([18, 21])**.** *Every digraph with minimum out-degree at least $2k-1$ contains k vertex-disjoint directed cycles for $k = 1, 2, 3$.*

Proof of Theorem 1.6: Suppose to the contrary that G contains no rainbow cycles. Let $D = D_{xy}(X, Y)$ be a local associated digraph of G, $H_1 = G[X \cup Y \cup \{x, y\}]$ and $H_2 = G - H_1$.

Note that $|V(H_1)| - 1 \ge |CN(x) \cup CN(y)| \ge \lfloor \frac{n}{2} \rfloor + 2$. Then $|V(H_1)| \ge \lfloor \frac{n}{2} \rfloor + 3$, which implies that

$$|V(H_2)| = |V(G)| - |V(H_1)| \le n - (\lfloor \frac{n}{2} \rfloor + 3) = \lceil \frac{n}{2} \rceil - 3.$$

By Lemma 2.1, the condition that G contains no rainbow cycles implies that D contains no directed cycles. Then D contains a vertex w_1 such that $d^+_D(w_1) = 0$. From the fact that $D \setminus \{w_1\}$ contains no directed cycles, it follows that $D \setminus \{w_1\}$ contains a vertex w_2 such that $d^+_{D\setminus\{w_1\}}(w_1) = 0$. Then $d^+_D(w_2) \leq 1$. Clearly, $w_2w_1 \in A(D)$ if $d^+_D(w_2) = 1$, which means that $c(w_2w_1) = c(xw_1)$ if $d^+_D(w_2) = 1$. Hence, $|CN_{H_1}(w_1) \cup CN_{H_1}(w_2)| \leq 3$.

Choose an arbitrary vertex $u \in H_2$. If both w_1 and w_2 belong to X or Y, say X, then, to avoid that $w_1uw_2xw_1$ is a rainbow cycle, either uw_1 or uw_2 does not exist or $c(uw_1) = c(uw_2)$ or $\{c(uw_1), c(uw_2)\} \cap \{c(xw_1), c(xw_2)\} \neq \emptyset$. If $w_1 \in X$ and $w_2 \in Y$, then, to avoid that xw_1uw_2yx is a rainbow cycle, either uw_1 or uw_2 does not exist or $c(uw_1) = c(uw_2)$ or $\{c(uw_1), c(uw_2)\} \cap \{c(xw_1), c(xw_2), c(xy)\} \neq \emptyset$. Thus, $\{uw_1, uw_2\}$ contributes at most one color to $CN(w_1) \cup CN(w_2)$ distinct from $c(\{w_1, w_2\}, V(H_1)) \cup \{c(xy)\}$.

Consequently, $|CN(w_1) \cup CN(w_2)| \leq |CN_{H_1}(w_1) \cup CN_{H_1}(w_2)| + |V(H_2)| \leq \lceil \frac{n}{2} \rceil$, which contradicts the assumption that $|CN(w_1) \cup CN(w_2)| \geq \lfloor \frac{n}{2} \rfloor + 2$. The result thus follows.□

Proof of Theorem 1.11: At first, assume that $k = 2$ and G does not contain two vertex-disjoint rainbow cycles.

Claim 1. G contains a rainbow cycle of length at most four.

Proof. Suppose to the contrary that G contains no rainbow cycles of length at most four. Let $D = D_{xy}(X, Y)$ be a local associated digraph of G, $H_1 = G[X \cup Y \cup \{x, y\}]$ and $H_2 = G - H_1$. Note that $|V(H_1)| - 1 \geq |CN(x) \cup CN(y)| \geq \frac{n}{2} + 6$. Then $|V(H_1)| \geq \frac{n}{2} + 7$, which implies $|V(H_2)| = |V(G)| - |V(H_1)| \leq n - (\frac{n}{2} + 7) = \frac{n}{2} - 7$.

We assert that there are two vertices $w_1, w_2 \in V(H_1)$ such that $|CN_{H_1}(w_1) \cup CN_{H_1}(w_2)| \leq 7$. Similarly, by Lemmas 2.1 and 3.1, there are two vertices w_1 and w_2 in D such that $d^+_D(w_1) \leq 2$ and $d^+_D(w_2) \leq 3$. Note that if $d^+_D(w_2) = 3$, we have $w_2w_1 \in D$, and then $c(w_2w_1) = c(xw_1)$. Using statements (2) and (3) of Lemma 2.1, if $d^+_D(w_2) \leq 2$, then

$$|CN_{H_1}(w_1) \cup CN_{H_1}(w_2)| \leq d^+_D(w_1) + d^+_D(w_2) + 3 \leq 7.$$

If $d^+_D(w_2) = 3$, then

$$|CN_{H_1}(w_1) \cup CN_{H_1}(w_2)| \leq d^+_D(w_1) + d^+_D(w_2) - 1 + 3 \leq 7.$$

We assert that $E(\{w_1, w_2\}, V(H_2))$ contributes at most $|V(H_2)| + 1$ colors to $CN(w_1) \cup CN(w_2)$ distinct from $c(\{w_1, w_2\}, V(H_1))$. If not, then there are two vertices $u_1, u_2 \in V(H_2)$ such that $u_1w_1u_2w_2u_1$ is a rainbow C_4 in H^*, a contradiction. Consequently,

$$\begin{aligned}|CN(w_1) \cup CN(w_2)| &\leq |CN_{H_1}(w_1) \cup CN_{H_1}(w_2)| + |V(H_2)| + 1\\ &\leq 7 + \frac{n}{2} - 7 + 1 = \frac{n}{2} + 1,\end{aligned}$$

which contradicts the assumption that $|CN(w_1) \cup CN(w_2)| \geq \frac{n}{2} + 6$. The claim thus follows. □

Next, assume that G_1 is a minimum rainbow cycle in G and $H^* = G - G_1$. Clearly, $3 \leq |V(G_1)| \leq 4$ and H^* contains no rainbow cycles. Let $D = D_{xy}(X, Y)$ be a local associated digraph of H^*, $H_1 = G[X \cup Y \cup \{x, y\}]$ and $H_2 = H^* - H_1$. From Lemma 2.2, we have $|V(H_1)| - 1 + 6 \geq |CN(x) \cup CN(y)| \geq \frac{n}{2} + 6$. Then $|V(H_1)| \geq \frac{n}{2} + 1$, which implies that

$$|V(H_2)| = |V(H^*)| - |V(H_1)| = n - |V(G_1)| - |V(H_1)| \leq n - 3 - |V(H_1)| \leq \frac{n}{2} - 4.$$

Claim 2. There are two vertices $w_1, w_2 \in H_1$ such that

(1) $|CN_{H_1}(w_1) \cup CN_{H_1}(w_2)| \leq 3$;

(2) $c(xy) \in c(\{w_1, w_2\}, V(H_1))$ if $|CN_{H_1}(w_1) \cup CN_{H_1}(w_2)| = 3$.

Proof. We can easily deduce that $|CN_{H_1}(w_1) \cup CN_{H_1}(w_2)| \leq 3$ by a similar discussion to Theorem 1.6. Furthermore, from the proof of Lemma 2.1, we can see that $c(xy) \in c(\{w_1, w_2\}, V(H_1))$ if $|CN_{H_1}(w_1) \cup CN_{H_1}(w_2)| = 3$. □

Now we choose two vertices $w_1, w_2 \in V(H_1)$ such that w_1 and w_2 satisfy Claim 2. By a similar argument to Claim 1, we can get that $E(\{w_1, w_2\}, V(H_2))$ contributes at most $|V(H_2)| + 1$ colors to $CN(w_1) \cup CN(w_2)$ distinct from $c(\{w_1, w_2\}, V(H_1))$.

Claim 3. If $E(\{w_1, w_2\}, V(H_2))$ contributes exactly $|V(H_2)| + 1$ colors to $CN(w_1) \cup CN(w_2)$ distinct from $c(\{w_1, w_2\}, V(H_1))$, then $c(xy) \in c(\{w_1, w_2\}, V(H_2))$.

Proof. If $E(\{w_1, w_2\}, V(H_2))$ contributes exactly $|V(H_2)| + 1$ colors to $CN(w_1) \cup CN(w_2)$ distinct from $c(\{w_1, w_2\}, V(H_1))$, then there is a vertex $u \in H_2$ such that $c(uw_1) \neq c(uw_2)$ and $c(uw_i) \notin c(\{w_1, w_2\}, V(H_1))$ for $i = 1, 2$. If $w_1, w_2 \in X$ or $w_1, w_2 \in Y$, then xw_1uw_2x or yw_1uw_2y is a rainbow C_4 in H^*, a contradiction. If w_1 and w_2 belong to different sets of X and Y, without loss of generality, set $w_1 \in X$ and $w_2 \in Y$. If $c(xy) \notin \{c(w_1u), c(w_2u)\}$, then xyw_2uw_1x is a rainbow C_5, a contradiction. The claim thus follows. □

From Lemma 2.2 and Claims 2 and 3, we have

$$\begin{aligned}|CN(w_1) \cup CN(w_2)| &\leq |CN_{H_1}(w_1) \cup CN_{H_1}(w_2)| + |V(H_2)| + 1 + 6 - |\{c(xy)\}| \\ &\leq 3 + \frac{n}{2} - 4 + 6 + 1 - 1 = \frac{n}{2} + 5,\end{aligned}$$

which contradicts the condition $|CN(w_1) \cup CN(w_2)| \geq \frac{n}{2} + 6$. Hence, there are two vertex-disjoint rainbow cycles in G when $k = 2$. By a similar argument, we can prove the case $k = 3$. The proof is now complete. □

4 Vertex-disjoint rainbow cycles of different lengths

In this section, we first give a crucial lemma to the proof of Theorem 1.13.

Lemma 4.1. *[17] Every digraph of minimum out-degree at least 4 contains two vertex-disjoint directed cycles of different lengths.*

Proof of Theorem 1.13: At first, we consider the condition (2). Assume that $H_0 = x_0y_0z_0x_0$ is a rainbow triangle in G and $G_1 = G - \{x_0, y_0, z_0\}$.

Note that $|CN_{G_1}(u) \cup CN_{G_1}(v)| \geq \frac{n}{2} + 5$ for any two vertices u, v of G_1. Then G_1 contains a rainbow cycle by Theorem 1.6. If G_1 contains a rainbow cycle of length at least four, then the result follows. Next, we suppose that G_1 only contains rainbow triangles. Assume that $H^* = x^*y^*z^*x^*$ is a rainbow triangle in G_1 with $c(x^*y^*) = c'$, $c(y^*z^*) = f'$ and $c(x^*z^*) = g'$.

For each edge $xy \in H^*$, we choose $X = \{x_1, x_2, ..., x_s\} \subset N_{G_1}(x) \setminus V(H^*)$ and $Y = \{y_1, y_2, ..., y_t\} \subset N_{G_1}(y) \setminus V(H^*)$, such that the following four conditions hold:

(a) $c(xx_i) \neq c(xx_j)$ for all $1 \leq i < j \leq s$ and $c(yy_i) \neq c(yy_j)$ for all $1 \leq i < j \leq t$;

(b) $c(xx_i) \neq c(yy_j)$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$;

(c) $c(xx_i) \notin \{c', f', g'\}$ and $c(yy_j) \notin \{c', f', g'\}$ all $1 \leq i \leq s$ and $1 \leq j \leq t$.

(d) subject to (a), (b) and (c), $s + t$ is maximized.

Without loss of generality, we assume that x^*y^* satisfies the above four conditions, $c(x^*x_i) = c_i$ and $c(y^*y_j) = f_j$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$; see Figure 2. Note that the conditions (a), (b) and (c) guarantee that $CN_{G_1}(x^*) \cup CN_{G_1}(y^*) = \{c', f', g', c_1, ..., c_s, f_1, f_2, ..., f_t\}$. Since G contains no rainbow cycles of length at least

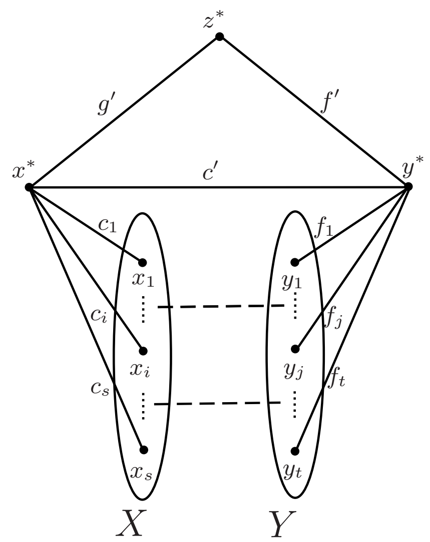

Figure 2: $H^* = x^*y^*z^*x^*$ and $G_2 = G[X \cup Y \cup V(H^*)]$

four, we have the following claim:

Claim 1. (1) $X \cap Y = \emptyset$;

(2) $c(x_iy_j) \in \{c_i, f_j\}$ for all all $1 \leq i \leq s$ and $1 \leq j \leq t$ when $x_iy_j \in E(X,Y)$.

Proof. At first, we consider statement (1) of the claim. If $X \cap Y \neq \emptyset$, then we choose a vertex $u^* \in X \cap Y$ such that $\{c(u^*x^*), c(u^*y^*)\} \cap \{c', f', g'\} = \emptyset$ and $c(u^*x^*) \neq c(u^*y^*)$. Consequently, $x^*y^*u^*z^*x^*$ is a rainbow cycle of length four in G_1, a contradiction. Next, we prove statement (2) of the claim by contradiction. Assume that there are two integer $1 \leq i \leq s$ and $1 \leq j \leq t$ such that $c(x_iy_j) \notin \{c_i, f_j\}$. If $c(x_iy_j) \neq c'$, then $x^*y^*y_jx_ix^*$ is a rainbow C_4 vertex-disjoint from H_0 in G, a contradiction. If $c(x_iy_j) = c'$, then $x^*z^*y^*y_jx_ix^*$ is a rainbow C_5 vertex-disjoint from H_0 in G, a contradiction. The claim thus follows. □

Let $G_2 = G[X \cup Y \cup V(H^*)]$. Note that $|V(G_2)| + 2|V(H_0)| \geq |CN(x^*) \cup CN(y^*)| \geq \frac{n}{2} + 11$. Then

$$|V(G_2| \geq |CN(x^*) \cup CN(y^*)| - 6 \geq \frac{n}{2} + 5.$$

We obtain a subgraph G_2^* from G_2 by deleting the following two types of edges:

- x_ix_j if $c(x_ix_j) \notin \{c_i, c_j\}$ for all $1 \leq i < j \leq s$;
- y_iy_j if $c(y_iy_j) \notin \{f_i, f_j\}$ for all $1 \leq i < j \leq t$.

Let $D = D[X \cup Y]$ be a local associated digraph of G_2^*. Note that every directed cycle in D corresponds to a rainbow cycle in G_2^*. Since G_2^* does not contain two vertex-disjoint rainbow cycles of different lengths, this means that D does not contain two directed cycles of different lengths. By Lemma 4.1, there are two vertices w_1 and w_2 in D such that $d_D^+(w_1) \leq 3$ and $d_D^+(w_2) \leq 4$. Furthermore, we can see that $w_2w_1 \in A(D)$ if $d_D^+(w_1) = 4$.

Claim 2. $|CN_{G_2}(w_1) \cup CN_{G_2}(w_2)| \leq 14$.

Proof. Without loss of generality, suppose $w_1 \in X$. From Claim 1 and the definition of D, we have $|CN_{G_2^*}(w_1)| \leq d_D^+(w_1) + |\{c(xw_1), z^*\}| \leq 5$. From Lemma 2.3 and the fact that G_2 contains no rainbow C_4, at most three edges incident to w_1 are deleted in the operation of constructing G_2^*, say $w_1x_1, ..., w_1x_i$, $1 \leq i \leq 3$.

We assert that w_1z^* does not exist in G_2 when exactly three distinct colored edges incident to w_1 are deleted. If not, to avoid that $z^*y^*x^*w_1z^*$ is a rainbow C_4, we have $c(w_1z^*) \neq g'$. From Lemma 2.3, we have $\{c(w_1x_1), c(w_1x_2), c(w_1x_3)\} = \{c_1, c_2, c_3\}$. Without loss of generality, assume that $c(w_1x_1) = c_2$, $c(w_1x_2) = c_3$ and $c(w_1x_3) = c(c_1)$. Note that there are at least two colors, say c_1 and c_2, such that $c(w_1z^*) \notin \{c_1, c_2\}$. Recall that $g' \neq c_i$ for $1 \leq i \leq s$, which means that $z^*x^*x_1w_1z^*$ is a rainbow C_4, a contradiction. Consequently, $|CN_{G_2}(w_1)| \leq 7$. Recall that $w_2w_1 \in A(D)$ if $d_D^+(w_1) = 4$, which implies that $c(w_2w_1) = c(x^*w_1)$. Similarly, we have $|CN_{G_2}(w_2)| \leq 7$. Then $|CN_{G_2}(w_1) \cup CN_{G_2}(w_2)| \leq 14$. □

Note that $E(\{w_1, w_2\}, V(G) - V(G_2))$ contributes at most $n - |V(G_2)| + 1$ colors to $CN(w_1) \cup CN(w_2)$ distinct from $c(\{w_1, w_2\}, V(G_2))$. Then

$$\begin{aligned}|CN(w_1) \cup CN(w_2)| &\leq |CN_{G_2}(w_1) \cup CN_{G_2}(w_2)| + |V(G) - V(G_2)| + 1 \\ &\leq 14 + n - (\frac{n}{2} + 5) + 1 = \frac{n}{2} + 10,\end{aligned}$$

which contradicts the assumption that $|CN(w_1) \cup CN(w_2)| \geq \frac{n}{2} + 11$. Hence, the condition (2) is right.

Finally, we consider the condition (1) by contradiction. From the above discussion, we know that G contains no rainbow triangles, and then G has the following property:

Property A: For each edge $uv \in G$ and each vertex $w \in G \setminus \{u, v\}$, $E(w, \{u, v\})$ contributes at most one color to $CN(u) \cup CN(v)$ distinct from $c(uv)$.

Claim 4. *G contains a rainbow C_4.*

Proof. Suppose to the contrary that G contains no rainbow C_4. Let $D = D_{xy}(X, Y)$ be a local associated digraph of G, $H_1 = G[X \cup Y \cup \{x, y\}]$ and $H_2 = G - H_1$. Note that $|V(H_1)| - 1 = |CN(x) \cup CN(y)| \geq \frac{2n}{3} + 6$. Then

$$|V(H_1)| \geq \frac{2n}{3} + 7.$$

Since G does not contain two vertex-disjoint rainbow cycles of different lengths, from statement (1) of Lemma 2.1 and Lemma 4.1, D contains two vertices w_1 and w_2, say $w_1 \in X$, such that $d^+_D(w_1) \leq 3$ and $d^+_D(w_2) \leq 4$. Furthermore, if $d^+_D(w_2) = 4$, we have $w_2 w_1 \in A(D)$, and then $c(w_2 w_1) = c(x w_1)$. From statement (3) of Lemma 2.1, it follows that $|CN_{H_1}(w_1) \cup CN_{H_2}(w_2)| \leq 6 + 3$.

We can see that $E(\{w_1, w_2\}, V(H_2))$ contributes at most $|V(H_2)|+1$ colors to $CN(w_1) \cup CN(w_2)$ distinct from $c(\{w_1, w_2\}, V(H_1))$. Consequently,

$$\begin{aligned}|CN(w_1) \cup CN(w_2)| &\leq |CN_{H_1}(w_1) \cup CN_{H_1}(w_2)| + |V(H_2)| + 1 \\ &\leq 9 + |V(G)| - |V(H_1)| + 1 \\ &\leq 10 + n - (\frac{2n}{3} + 7) = \frac{n}{3} + 3,\end{aligned}$$

which contradicts the assumption that $|CN(w_1) \cup CN(w_2)| \geq \frac{2n}{3} + 6$. □

Next, assume that $H_0 = x_0 y_0 z_0 w_0 x_0$ is a rainbow C_4 in G and $G_1 = G - H_0$. Repeating the following argument, we can deduce that G_1 contains a rainbow C_4. Hence, assume that $H^* = x^* y^* z^* w^* x^*$ is a rainbow C_4 in G_1 with $c(x^* y^*) = c'$, $c(y^* z^*) = f'$, $c(z^* w^*) = g'$ and $c(x^* w^*) = h'$.

For each edge $xy \in H^*$, we choose $X = \{x_1, x_2, ..., x_s\} \subset N_{G_1}(x) \setminus V(H^*)$ and $Y = \{y_1, y_2, ..., y_t\} \subset N_{G_1}(y) \setminus V(H^*)$, such that the following four conditions hold:

(*a*) $c(xx_i) \neq c(xx_j)$ for all $1 \leq i < j \leq s$ and $c(yy_i) \neq c(yy_j)$ for all $1 \leq i < j \leq t$;

(*b*) $c(xx_i) \neq c(yy_j)$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$;

(*c*) $c(xx_i) \notin \{c', f', g', h'\}$ and $c(yy_j) \notin \{c', f', g', h'\}$ all $1 \leq i \leq s$ and $1 \leq j \leq t$;

(*d*) subject to (*a*), (*b*) and (*c*), $s + t$ is maximized.

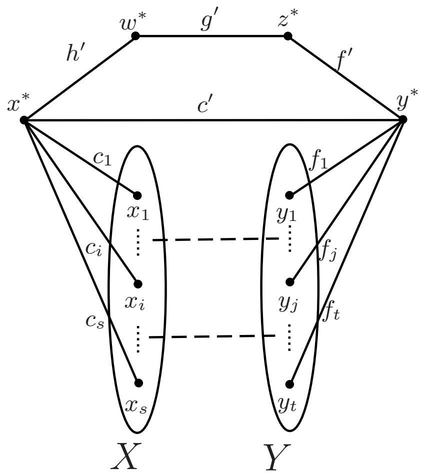

Figure 3: $H^* = x^*y^*z^*w^*x^*$ and $G_2 = G[X \cup Y \cup V(H^*)]$

Without loss of generality, assume that x^*y^* satisfies the above four conditions, $c(x^*x_i) = c_i$ and $c(y^*y_j) = f_j$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$; see Figure 3. Since G_1 only contains rainbow C_4, we have $c(x_iy_j) \neq c'$ for all $1 \leq i \leq s$ and $1 \leq j < t$. Otherwise, $x^*w^*z^*y^*y_jx_ix^*$ is a rainbow C_6. Then we have the following three properties:

(1) $c(x_ix_j) \in \{c_i, c_j\}$ for all $1 \leq i < j < s$;

(2) $c(y_iy_j) \in \{f_i, f_j\}$ for all $1 \leq i < j < t$;

(3) $c(x_iy_j) \in \{c_i, f_j, f', g', h'\}$ for all $1 \leq i \leq s$ and $1 \leq j < t$.

Let $G_2 = G[X \cup Y \cup V(H^*)]$. The Property A implies that $|V(G_2)| - 1 + |H_0| + |\{g'\}| \geq |CN(x^*) \cup CN(y^*)| \geq \frac{2n}{3} + 6$. Then

$$|V(G_2| \geq |CN(x^*) \cup CN(y^*)| - 4 \geq \frac{2n}{3} + 2.$$

We obtain a subgraph G_2^* from G_2 by deleting the following type of edges:

- x_iy_j if $x_iy_j \in E(G_2)$ and $c(x_iy_j) \in \{f', g', h'\}$ for all $1 \leq i < j \leq s$.

Let $D = D[X \cup Y]$ be a local associated digraph of G_2^*. Recall that every directed cycle of D corresponds to a rainbow cycle of G_2^*. By a similar discussion and Lemma 4.1, there are two vertices $w_1, w_2 \in D$ such that $d_D^+(w_1) \leq 3$ and $d_D^+(w_1) \leq 4$. Furthermore, if $d_D^+(w_1) = 4$, we have $w_2w_1 \in A(D)$, and then $c(w_2w_1) = c(x^*w_1)$.

Claim 5. $|CN_{G_2}(w_1) \cup CN_{G_2}(w_2)| \leq 11$.

Proof. If w_1 and w_2 belong to different sets of X and Y, say $w_1 \in X$ and $w_2 \in Y$, we assert that $c(w^*w_1) = c(x^*w_1)$ if w^*w_1 exists. Otherwise, $w^*w_1x^*w^*$ or $w_1x^*y^*z^*w^*w_1$ is a rainbow cycle. Similarly, we have $c(z^*w_2) = c(y^*w_2)$ if z^*w_2 exists. Recall that $d_D^+(w_1) \leq 3$, $d_D^+(w_1) \leq 4$ and $c(w_2w_1) = c(x^*w_1)$ if $d_D^+(w_1) = 4$. Thus, $|CN_{G_2^*}(w_1) \cup CN_{G_2^*}(w_2)| \leq 6 + |\{c(w_1z^*), c(w_2w^*)\}|$. From the construction of G_2^* from G_2, at most three distinct colored edges are deleted from G_2. Then $|CN_{G_2}(w_1) \cup CN_{G_2}(w_2)| \leq |CN_{G_2^*}(w_1) \cup CN_{G_2^*}(w_2)| + 3 \leq 11$. Similarly, we can prove the case that both w_1 and w_2 belong to X or Y. □

From Lemma 2.2, we have

$$\begin{aligned}|CN(w_1) \cup CN(w_2)| &\leq |CN_{H_0}(w_1) \cup CN_{H_0}(w_2)| + |CN_{G_2}(w_1) \cup CN_{G_2}(w_2)| \\ &\quad + 2|V(G_1) - V(G_2)| \\ &\leq 6 + 11 + 2(n - 4 - \frac{2n}{3} - 2) = \frac{2n}{3} + 5,\end{aligned}$$

which contradicts the assumption that $|CN(w_1) \cup CN(w_2)| \geq \frac{2n}{3} + 6$. The proof is finally complete. □

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