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The extended dominating set in graphs

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Let $G = (V(G), E(G))$ be a graph and let k be an integer. A vertex subset $S \subseteq V(G)$ is called a k-extended dominating set if every vertex u of G satisfies one of the following conditions: the distance between u and S is at most one or there are at least k different vertices $s_1, s_2, \ldots, s_k \in S$ such that the distance between u and s_i ($i \in [k]$) is two. The k-extended domination number $\gamma_e^k(G)$ of G is the minimum size over all k-extended dominating sets in G. When $k = 2$, they are called the extended dominating set and the extended domination number of G, respectively. In this paper, we mainly study the bounds of the extended domination numbers of graphs. Firstly, we obtain the exact values of the extended domination numbers for paths and cycles. And then the Nordhaus-Gaddum bounds for the extended domination number are provided. Additionally, we give some bounds of the extended domination numbers for planar graphs with small diameters. Finally, we consider the behavior of the k-extended domination number of the Random graph $G(n, p)$.

Keywords: Domination; extended domination; planar graph; random graph.

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1. Introduction

In this paper, only finite, undirected and simple graphs are considered. Let $G = (V(G), E(G))$ be a graph. Suppose u and v are two vertices in G. The *distance* between u, v denoted by $d(u, v)$ is the length of the shortest path connecting u and v. A vertex v is called a *quasi-neighbour* of u if $d(u, v) = 2$. The distance between v and S is defined as $d(v, S) = \min\{d(v, u)|u \in S\}$, where S is a vertex subset of G. Given an integer k, a vertex set S is called as an k*-extended dominating set* (abbreviated k-EDS) of G if every vertex u in G satisfies that $d(u, S) \leq 1$ or u has k different quasi-neighbours in S. The smallest cardinality of a k-extended dominating set of G is the k*-extended domination number* of G, denoted by $\gamma_e^k(G)$. Expressly, when $k = 2$ we call them the extended dominating set (abbreviated EDS) and the extended domination number of G, respectively.

The extended dominating set was introduced by Wu (2002), which has many applications in ad hoc networks. A wireless ad hoc network is a decentralized type of wireless network. The network is ad hoc because it does not rely on a pre-existing infrastructure, such as routers in wired networks or access points in wireless networks. Alternatively, each vertex participates in routing by forwarding data to other vertices. We call a vertex *source vertex* if it has the whole data, and can transfer all data to its neighbours, but only partial data to its quasi-neighbors. Therefore, the determination of which vertices are the source vertices, so that all vertices can collect all data, is essential. This can be converted as follows. Given a network (graph) G, how to select the source vertices in G is became how to find an extended dominating set in G. Wu et al. (2006) showed that the determination of the extended domination number is NP-hard and they also gave several heuristic algorithms to compute the extended domination number of a network. Inspired by this, it is significant to determine the quantum of the extended domination number of graphs. And there are few results about the bounds of the extended domination number of graphs. In this paper, we are first to study the bounds of the extended domination number of graphs.

Actually, the k-extended dominating set is a generalization of the classical dominating set. Given a graph G, a vertex set S is a *dominating set* if each vertex $u \in V(G)$ such that $d(u, S) \leq 1$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set in G. Let t be an integer. The k-extended dominating set is also closed to the distance-t dominating set, which is a domination-type variable and was introduced by Meir and Moon (1975). A *distance-t dominating set* S is a vertex set of $V(G)$ such that for each vertex u outside S satisfying $d(u, S) \leq t$. The minimum size of distance-t dominating set is the distance-t domination number, denoted by $\gamma_t(G)$. From the definition of the k-extended dominating set, we know that if $k = 1$, an 1-EDS of G is exactly a distance-2 dominating set; if $k \geq n - 1$, a k-EDS is equivalent to a dominating set of G. It is trivial that any complete graph and complete multipartite graph have extended domination numbers equal to their domination numbers. While not every graph has this property, for example, Petersen graph has domination number 4, but the extended domination number is between 2 and 4 depending on k.

Domination and its variations in graphs had been well studied, the more details refer to the excellent books (Haynes et al. (1998)-1; Haynes et al. (1998)-2; Haynes et al. (2020); Haynes et al. (2021)). One of the core problems in graph domination is how to discover the good bounds for the domination numbers of graphs. There are many results about the bounds of the domination-type numbers (Zverovich and Poghosyan (2011); Henning and Lichiardopol (2017); Li et al. (2018); Cai et al. (2019); Henning (2020); Bujtás (2021); Deshpande et al. (2022)). In particular, the bounds of domination-type numbers of planar graph are rich, the more details refer to (MacGillivray and Seyffarth (1996); Goddard and Henning (2002); Araki and Yumoto (2018); Borg and Kaemawichanurat (2020); Zhuang (2021)).

To end this section, we give the structure of this paper. In Section 2, we give some notations and the exact values of the extended domination number of paths and cycles. In Section 3, we give the Nordhaus-Gaddum bounds for the extended domination number of graphs. The bounds of the extended domination number for planar graphs with small diameters are provided in Section 4. Finally, we study the behavior of the k-extended domination number in Random graph $G(n, p)$.

2. EDS on cycles and paths

The notations we use are as follows. Let $G = (V(G), E(G))$ be a graph with order $n = |V(G)|$ and let u and v be two vertices in G. The *eccentricity* of a vertex v in G is defined as $ecc(v) = \max\{d(u, v) : u \in V(G)\}$. The radius of G is defined as $rad(G) = \min\{e(v) : v \in V(G)\}$, and the diameter of G is defined as $diam(G) = \max\{e(v) : v \in V(G)\}$. The distance from a vertex v to a vertex set S is defined as $d(v, S) = \min\{d(v, s) : s \in S\}$. The *private neighbours* of a vertex v for a given vertex subset S are the vertices only adjacent to v but not to the other vertices in S. Let $u, v \in V(G)$. The neighbour of v is denoted by $N(v)$. Let $N_2(v)$ be the set of vertex which has distance 2 from v in G. A vertex u is *quasi-neighbour* of v if the vertex u belongs to $N_2(v)$. Suppose A and B are two vertex subsets in G. We say A dominates B if for each vertex v in B has a neighbour in A. A vertex u is *distance-k dominated* by v if $d(u, v) \leq k$. We follow Chartrand and Lesniak (1996) for notation and terminology not defined here.

Next, we show the extended domination numbers of cycles and paths. Before that, we need the following observation.

Observation 1. Let $G = (V, E)$ be a connected graph and G' be its spanning subgraph, then $\gamma_e^k(G) \leq \gamma_e^k(G')$.

Given a cycle C, we know that for each vertex v satisfies $|N_2(v)| \leq 2$. If $k \geq 3$, then a k-extended dominating set becomes a dominating set in C. And consequently we only consider the extended domination number and give the following theorem.

Theorem 2.1. *Let C_n be a cycle. Then*

$$\gamma_e^2(C_n) = \begin{cases} 1, & \text{if } n = 3, \\ 2, & \text{if } n = 4, \\ \lceil \frac{n}{4} \rceil, & \text{if } n \geq 5. \end{cases}$$

Proof. The conclusion holds when $n = 3$ or 4 clearly. Now we assume that $n \geq 5$. Labeling all the vertices of C_n from any vertex as $v_0, v_1, v_2, v_3, \dots, v_{n-1}$ clockwise. Then we chose a vertex subset $S = \{v_{4k+1} | k \in \{0, 1, 2, \dots, \lceil \frac{n}{4} \rceil - 1\}\}$, and it is easy to check that S is an EDS. Thus $\gamma_e^2(C_n) \leq \lceil \frac{n}{4} \rceil$. On the other hand, we will show that $\gamma_e^2(G) \geq \lceil \frac{n}{4} \rceil$. Obviously, each EDS contains at least one vertex in every P_4 in C_n. If not, there exists a P_4 that contains no vertex for any EDS in C_n, then the two central vertices of this P_4 should not be extended dominated by any vertex in C_n, a contradiction. So we get that $\gamma_e^2(C_n) \geq \lfloor \frac{n}{4} \rfloor$ since there are at least $\lfloor \frac{n}{4} \rfloor$ vertex-disjoint P_4s in C_n.

It is clear that the result holds when $n \equiv 0(\text{mod}4)$. Now we only consider $n \equiv 1(\text{mod}4)$ in the following proof. Since the others are similar to analysis, we omit the details. Set $n = 4k + 1$ for some integer k. It is clear that there exist $k + 1$ consecutive segments A_i $(i \in [k+1])$ in C_n such that A_i is a copy of P_4 for $1 \leq i \leq k$ and A_{k+1} is an isolated vertex. Without loss of generality, we can assume that $A_i = v_{4i-4}v_{4i-3}v_{4i-2}v_{4i-1}$ for $1 \leq i \leq k$ and $A_{k+1} = v_{4k}$ by relabeling the indices of vertices in C_n. Suppose that the consequence does not hold. That is, $\gamma_e^2(C_n) = k$ by combining the above analysis. Suppose that S is a minimum EDS of C_n. Then $|S| = k$ and thus each A_i contains only one vertex in S for $1 \leq i \leq k$. By symmetry, we know that only one of v_0 and v_1 is contained in S. If $v_0 \in S$, then we get that $S = \{v_0, v_4, \dots, v_{4k-4}\}$ by the definition of EDS. Moreover, the vertex v_{4k-2} is not extended dominated by S, a contradiction. If $v_1 \in S$, then $S = \{v_1, v_5, \dots, v_{4k-3}\}$ by the same reason. And thus the vertex v_{4k-1} is not extended dominated by S, again a contradiction. So we get that $\gamma_e^2(G) = \lceil \frac{n}{4} \rceil$. □

For paths, we obtain the exact values of the extended domination number, which is stated as follows.

Theorem 2.2. *Let P_n be a path. Then*

$$\gamma_e^2(P_n) = \begin{cases} \frac{n}{4} + 1, & \text{if } n \equiv 0(\text{mod}4), \\ \lceil \frac{n}{4} \rceil, & \text{otherwise.} \end{cases}$$

Proof. The conclusion holds when $n = 1$ or 2 clearly. We label the vertices of P_n as $P_n = v_0v_1 \dots v_{n-1}$. In order to prove the upper bounds, we construct a vertex subset S as follows.

$$S = \begin{cases} \{v_{4k+1} | k \in \{0,1,2,\ldots,\lceil \frac{n}{4} \rceil - 1\} \cup \{v_{n-2}\}, & \text{if } n \equiv 0(\text{mod}4), \\ \{v_{4k+1} | k \in \{0,1,2,\ldots,\lfloor \frac{n}{4} \rfloor - 1\} \cup \{v_{n-2}\}, & \text{if } n \equiv 1(\text{mod}4), \\ \{v_{4k+1} | k \in \{0,1,2,\ldots,\lceil \frac{n}{4} \rceil - 1\}, & \text{otherwise.} \end{cases}$$

Clearly, S is an EDS of P_n. Then the upper bounds hold.

On the other hand, for the lower bound, we know that $\gamma_e^2(P_n) \geq \gamma_e^2(C_n) \geq \lceil \frac{n}{4} \rceil$ by Observation 1 and Theorem 2.1. Then the theorem holds for $n \not\equiv 0(\text{mod}4)$. Now we consider $n = 4k$ for some integer k. As the above theorem, we can take k consecutive segments $A_i = \{v_{4i-4}, v_{4i-3}, v_{4i-2}, v_{4i-1}\}$ for $1 \leq i \leq k$. Conversely, suppose that $\gamma_e^2(P_n) = k$ and S is a minimum EDS in P_n. Since v_0 is an end vertex and the definition of the EDS, then only one of v_0 or v_1 must be contained in S. If $v_0 \in S$, then $S = \{v_0, v_4, \ldots, v_{4k-4}\}$ and it can not extended dominate the vertex v_n, a contradiction. Otherwise, if $v_1 \in S$, then $S = \{v_1, v_5, \ldots, v_{4k-3}\}$ and the vertex v_n can not be dominated by S, a contradiction. The result holds. $\square$

3. Nordhaus-Gaddum bounds

Many studies have been done on the Nordhaus-Gaddum bounds for the original domination number, including the sum and product forms. In this section, we give some such results for the extended domination number of a graph G by using the fact $\gamma_e^2(G) \leq \gamma(G)$ and the analysis of the structure of G and its complement. Firstly, we list some known results which will be used in the following proof.

Lemma 3.1. *Let G be a graph with n vertices and $\overline{G}$ be its complement graph. Then*

(1) $\gamma(G) + \gamma(\overline{G}) \leq n+1$ and $\gamma(G)\gamma(\overline{G}) \leq n$. Moreover, the equalities hold if and only if $G = K_n$. (Jaeger and Payan (1972))
(2) If neither G nor $\overline{G}$ contains an isolated vertex, then $\gamma(G)+\gamma(\overline{G}) \leq \frac{n}{2}+2$.(Joseph and Arumugam (1995))
(3) If $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$, then $\gamma(G) + \gamma(\overline{G}) \leq \frac{2n}{5} + 3$, apart some small exceptions.(Dunbar et al. (2005))
(4) If $\delta(G) \geq 3$ and $\delta(\overline{G}) \geq 3$, then $\gamma(G) + \gamma(\overline{G}) \leq \frac{3n}{8} + 2$, apart some small exceptions.(Dunbar et al. (2005))

Lemma 3.2. *Let G be a graph with n vertices. Then*

(1) If $\delta(G) = 1$, then $\gamma(G) \leq \frac{n}{2}$.(Ore (1962))
(2) If $\delta(G) = 2$, then $\gamma(G) \leq \frac{2n}{5}$, except the graphs as defined in Figure 1.(Ore (1962))
(3) If $\delta(G) = 3$, then $\gamma(G) \leq \frac{3n}{8}$.(Reed (1996))
(4) If $\delta(G) = 4$, then $\gamma(G) \leq \frac{4n}{11}$.(Sohn and Yuan (2009))
(5) If $\delta(G) = 5$, then $\gamma(G) \leq \frac{n}{3}$.(Bujtás (2021))

The following corollary is obtained by doing some checks on the extended numbers of the graphs in Figure 1.

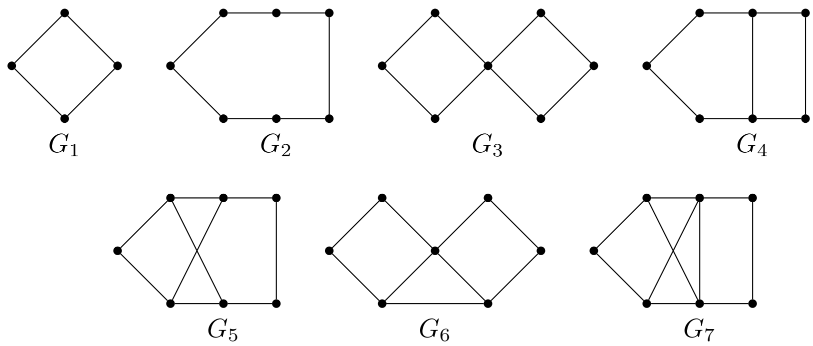


Fig. 1. Exceptions for $\gamma(G) \leq \frac{2n}{5}$.

Corollary 3.1. *Let $G \neq C_4$ be a graph with n vertices and $\delta(G) = 2$. Then $\gamma_e^2(G) \leq \frac{2n}{5}$.*

In the following statement, we let $\delta^*(G) = min\{\delta(G), \delta(\overline{G})\}$. Now we give the first theorem as follows.

Theorem 3.1. *Let G be a graph with n vertices. Then*

(1) $\gamma_e^2(G) + \gamma_e^2(\overline{G}) \leq n + 1$ and $\gamma_e^2(G)\gamma_e^2(\overline{G}) \leq n$. Moreover, the equalities hold if and only if $G = K_n$.
(2) If neither G nor $\overline{G}$ contains an isolated vertex, then $\gamma_e^2(G) + \gamma_e^2(\overline{G}) \leq \frac{n}{2} + 2$ and $\gamma_e^2(G)\gamma_e^2(\overline{G}) \leq n$.
(3) If $\delta^*(G) = 2$, then $\gamma_e^2(G) + \gamma_e^2(\overline{G}) \leq \frac{2}{5}n + 2$ and $\gamma_e^2(G)\gamma_e^2(\overline{G}) \leq \frac{4}{5}n$.
(4) If $\delta^*(G) = 3$, then $\gamma_e^2(G) + \gamma_e^2(\overline{G}) \leq \frac{3}{8}n + 2$ and $\gamma_e^2(G)\gamma_e^2(\overline{G}) \leq \frac{3}{4}n$.
(5) If $\delta^*(G) = 4$, then $\gamma_e^2(G) + \gamma_e^2(\overline{G}) \leq \frac{4}{11}n + 2$ and $\gamma_e^2(G)\gamma_e^2(\overline{G}) \leq \frac{8}{11}n$.
(6) If $\delta^*(G) = 5$, then $\gamma_e^2(G) + \gamma_e^2(\overline{G}) \leq \frac{n}{3} + 2$ and $\gamma_e^2(G)\gamma_e^2(\overline{G}) \leq \frac{2}{3}n$.

Proof.

(1) Using the fact $\gamma_e^2(G) \leq \gamma(G)$, we know that the results hold as an immediate consequence of Lemma 3.1 (1).
(2) The first inequality holds by using Lemma 3.1 (2) and the fact $\gamma_e^2(G) \leq \gamma(G)$. Now we consider the second inequality. Clearly, $n \geq 4$. We claim that both $\gamma_e^2(G) \geq 2$ and $\gamma_e^2(\overline{G}) \geq 2$ hold. Indeed, without loss of generality, suppose that $\gamma_e^2(G) = 1$. Then we know that G has a vertex v dominating all other vertices in G, this is, $d_G(v) = n - 1$. However, v is an isolated vertex in $\overline{G}$, a contradiction. By symmetry, assume $\gamma_e^2(G) \geq \gamma_e(\overline{G}) \geq 2$. If $\gamma_e^2(\overline{G}) = 2$, then

the result holds following directly Lemma 3.2 (1). Now we want to prove that $\gamma_e^2(\overline{G}) \geq 3$ does not exist. Without loss of generality, suppose G is disconnected or $diam(G) \geq 3$, then $\gamma_e^2(\overline{G}) = 2$, and we are done in the above. So G and $\overline{G}$ are connected and $diam(G) = diam(\overline{G}) = 2$. This is in contradiction to the fact that $\gamma_e^2(G) \leq 2$ for any connected graph G with diameter 2. The result holds.

(3) Since $\delta^*(G) = 2$, we note $n \geq 5$. By symmetry, assume $\gamma_e^2(G) \geq \gamma_e^2(\overline{G}) \geq 2$. If $\gamma_e^2(\overline{G}) = 2$, then the two inequalities hold directly from Lemma 3.2(2). Now we consider the case $\gamma_e^2(\overline{G}) \geq 3$. For the same reason in the above case, we know that G and $\overline{G}$ are connected and $diam(G) = diam(\overline{G}) = 2$, a contradiction. Hence the results hold.

(4) Since $\delta^*(G) = 3$, we note $n \geq 7$. And further, it is easy to check that there is no graph with $n = 7$ such that $\delta^*(G) = 3$, thus $n \geq 8$. By symmetry, assume $\gamma_e^2(G) \geq \gamma_e^2(\overline{G}) \geq 2$. If $\gamma_e^2(\overline{G}) = 2$, then two inequalities hold directly from Lemma 3.2(3). Thus we only consider $\gamma_e^2(\overline{G}) \geq 3$. As mentioned earlier, we can derive a contradiction to show that this case does not exist. The result holds.

(5) Since $\delta^*(G) = 4$, we note $n \geq 9$. By symmetry, assume $\gamma_e^2(G) \geq \gamma_e^2(\overline{G}) \geq 2$. If $\gamma_e^2(\overline{G}) = 2$, then the two inequalities hold directly from Lemma 3.2 (4). Thus we assume $\gamma_e^2(\overline{G}) \geq 3$, for the same reason as previously stated, we prove that this is impossible.

(6) Since $\delta^*(G) = 5$, we note $n \geq 11$. By symmetry, assume $\gamma_e^2(G) \geq \gamma_e^2(\overline{G}) \geq 2$. If $\gamma_e^2(\overline{G}) = 2$, then the two inequalities hold directly from Lemma 3.2 (5). As the same reason, $\gamma_e^2(G) \geq \gamma_e^2(\overline{G}) \geq 3$ does not exist. □

As we note that the bounds in Theorem 3.1 are sharp as $G = \frac{n}{2}K_2$ for (2) and $G = C_5$ for (3), there is a natural question to show whether the bounds in (4-6) are sharp.

And we further consider the bounds in terms of the maximum degree and give the following theorem.

Theorem 3.2. *Let G be a graph with maximum degree Δ. If $\gamma_e^2(G) = 2$, then $\gamma_e^2(G) + \gamma_e^2(\overline{G}) \leq \Delta + 2$ and $\gamma_e^2(G)\gamma_e^2(\overline{G}) \leq 2\Delta$.*

Proof. Since $\gamma_e^2(G) = 2$, then $\Delta \geq 2$ and there is an extended dominating set $S = \{x, y\}$ in G. Now we wish to prove the theorem by examining the following two cases.

Case 1. $N(x) \cap N(y) = \emptyset$.

Since $N(x) \cap N(y) = \emptyset$, then in $\overline{G}$, we know that the vertices in $N(x)$ are adjacent to y, and the vertices in $N(y)$ are adjacent to x. Moreover, the vertices in $N_2(x) \cup N_2(y)$ are adjacent to both x and y in $\overline{G}$. Then $\{x, y\}$ is also an extended dominating set, and thus $\gamma_e^2(\overline{G}) \leq 2$. So $\gamma_e^2(G) + \gamma_e^2(\overline{G}) \leq 4$ and $\gamma_e^2(G)\gamma_e^2(\overline{G}) \leq 4$ hold.

Case 2. $|N(x) \cap N(y)| = k$ for some integer $k \geq 1$.

The same as Case 1, we know that every vertices except $N(x) \cap N(y) \cup \{x, y\}$ are adjacent to x or y in $\overline{G}$. It is easy to check that $N(x) \cap N(y) \cup \{x, y\}$ is an extended dominating set of $\overline{G}$. Then $\gamma_e^2(\overline{G}) \leq 2 + k \leq 2 + \Delta$. Moreover, both $\gamma_e^2(G) + \gamma_e^2(\overline{G}) \leq \Delta + 2$ and $\gamma_e^2(G)\gamma_e^2(\overline{G}) \leq 2\Delta$ hold. □

As remark that the bound is sharp by taking $G \in \{P_4, P_5\}$. In the next theorem, we consider the graphs with the extended domination numbers at least 3.

Theorem 3.3. *Let G be a graph with $\gamma_e^2(G) \geq 3$. Then $\gamma_e^2(\overline{G}) \leq 2$ with equality holds when G contains no isolated vertex.*

Proof. Let S be a γ_e^2-set in G. Then $|S| \geq 3$ and $diam(G) \geq 3$. Then there exist two vertices x, y such that $d_G(x, y) \geq 3$. And thus the following results hold in $\overline{G}$: (1)x is adjacent to every vertex w in $V(G) \setminus \{N_G(x) \cup \{x\}\}$; (2) y is adjacent to each vertex in $N_G(x) \cup \{x\}$. It is easy to check that $\{x, y\}$ is an extended dominating set in $\overline{G}$. Thus $\gamma_e^2(\overline{G}) \leq 2$. It is obvious that the equality holds when G contains no isolated vertex. □

Employing the same idea, we can prove a similar result as follows, in which we omit the details.

Corollary 3.2. *Let G be a graph with $\gamma_e^k(G) \geq k + 1$ for some integer k. Then $\gamma_e^k(\overline{G}) \leq k$ with equality holds when G contains no isolated vertex.*

4. EDS on planar graphs with small diameter

In this section, we consider the extended domination numbers of planar graphs with small diameters. One may wonder how larger are the extended domination numbers of planar graphs with diameter at least 4. However, it is unbounded as follows. Consider a star graph G' consist of the central vertex x and leaves $\{v_1, v_2, \ldots, v_t\}$. Let G be a graph obtained from G' by subdividing each edge exactly once, and denote the middle vertices to be $\{v_1', v_2', \ldots, v_t'\}$. Then it is easy to verify that G is a planar graph with diameter 4, but $\gamma_e^2(G) = t$. Indeed, let S be a minimum extended dominating set of G. Then S contains either v_i or v_i', and $\gamma_e^2(G) \geq t$. However, $\{v_1', v_2', \ldots, v_t'\}$ is a dominating set of G' and thus an EDS of G', so $\gamma_e^2(G) = t$.

It is trivial that any graph with diameter 2 has a k-extended dominating set. So the interesting question is what are the extended domination numbers for planar graphs with diameter 3. Technically, we utilize the methods proposed by Goddard and Henning (2002). For convenience, we say a vertex u can reach v, if there is a path of length at most 3 from u to v.

Now we state the main theorem in this section.

Theorem 4.1. *Let G be a planar graph with diameter 3 and radius 2. Then $\gamma_e^2(G) \leq 3$.*

In order to prove the theorem, we require the following definitions. Let G be an embedding of a planar graph. Let $x \in V(G)$ be the vertex whose eccentricity is 2, and defined as the *central vertex*. A *fundamental cycle* $C := xv_1v_2\dots v_rx$ is a cycle such that on both sides of the cycle there is a vertex whose neighbour on the cycle is a subset of two vertices except x, see Figure 2 for example. Note that this vertex is not adjacent to x. A *cut cycle* is a cycle that partition the space into two regions which are called *inside* and *outside* respectively. It's straightforward that the vertices of such a cycle dominate either inside or outside. Otherwise, if there are two vertices on both sides which have a distance of 2 from the cycle, then the distance between them is at least 4.

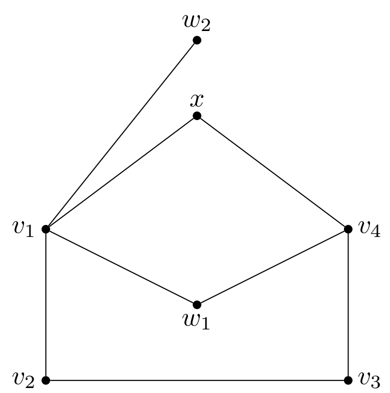


Fig. 2. w_1, w_2's neighbors on the cycle is a subset of $\{v_1, v_4\}$, thus it's a fundamental cycle of length 5.

Lemma 4.1. *Let G be a planar graph of diameter 3 and radius 2. If G contains a fundamental cycle of length at most 5 other than the cycles defined in Figure 3, then $\gamma_e^2(G) \leq 3$.*

Proof. Let $C = xv_1v_2\dots v_rx$ be the fundamental cycle for $2 \leq r \leq 4$. By the definition, there exist two vertices w_1 and w_2 which are inside and outside of C respectively. Without loss of generality, set v_i and v_j are the neighbours of $w_1 \cup w_2$ in C for some $1 \leq i < j \leq r$. We claim that $\{v_i, v_j, x\}$ is an EDS of G. Indeed, we need only to show that $d(v', \{v_i, v_j\}) \leq 2$ for any vertex $v' \in V(G)$, since x is the central vertex. Suppose to the contrary, there is a vertex u inside the cycle such that $d(u, v_i) = d(u, v_j) = 3$. In order to prove the result, we will show that $d(w_2, u) \geq 4$ contradicting to $diam(G) = 3$. In the following, we classify the situations based on the value of r.

Case 1. $r = 2$.

Now $C = xv_1v_2x$ and let $i = 1, j = 2$. Let P_{w_2u} be the shortest path connecting w_2 to u. Since C separates w_2 and u, then P_{w_2u} must use one vertex of $\{x, v_1, v_2\}$. If v_1 or v_2 is contained in P_{w_2u}, then $d(w_2, u) \geq 4$ by $d(u, v_1) = d(u, v_2) = 3$,

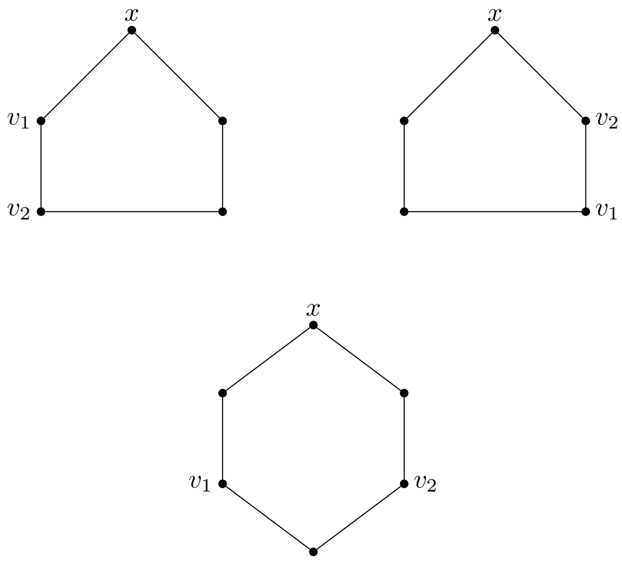


Fig. 3. Exceptions of fundamental cycle of length 5 and a fundamental cycle of length 6, where v_1, v_2 are the vertices that have private neighbor in both sides.

this is a contradiction. Otherwise, since w_2 and u are not dominated by x, then $d(w_2, u) \geq d(w_2, x) + d(x, u) \geq 4$, a contradiction.

Case 2. $r = 3$.

Now $C = xv_1v_2v_3x$. Let the neighbours of $\{w_1, w_2\}$ on C is $\{v_i, v_j\}$, where $1 \leq i \leq j \leq 3$ and the left vertex $\{v_1, v_2, v_3\} \setminus \{v_i, v_j\}$ is denoted as v_k. Since $d(u, v_i) = d(u, v_j) = 3$, so u is not dominated by x or v_k, this is, $d(u, \{x, v_k\}) \geq 2$. Again consider the shortest path from w_2 to u, then v_i and v_j can't belong to $V(P_{w_2u})$. However, by the definition of w_2, $d(w_2, \{x, v_k\}) \geq 2$. So we get $d(w_2, u) \geq 4$, a contradiction.

Case 3. $r = 4$.

Now $C = xv_1v_2v_3v_4x$. If v_i, v_j are not consecutive vertices as depicted in Figure 3, then the remaining two vertices of $\{v_1, v_2, v_3, v_4\} \setminus \{v_i, v_j\}$ can't dominate u because $d(u, v_i) = d(u, v_j) = 3$. And the remaining proof is the same as in Case 2.

From the above analysis, the proof is complete. Furthermore, a special fundamental cycle of length 6 is also held for the conclusion see Figure 3. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. We prove the theorem by contradiction, that is, $\gamma_e^2(G) \geq 4$. Assume there is no fundamental cycle as stated in Lemma 4.1 in the sequel. Since the radius is 2, there is a central vertex, denoted as x. Let $Y = V(G) - N[x]$ and M be a minimal subset of $N(x)$ that dominates Y. Let $m := |M|$. We claim that $m \geq 4$. Indeed, if $2 \leq m \leq 3$, then M is an EDS of G; if $m = 1$, then $M \cup \{x\}$ is an

EDS of G. In the above two cases, we can get an EDS with a size at most 3 in G, a contradiction.

Let the vertices of M be $n_0, n_1, \ldots, n_{m-1}$ in the cyclic order (clockwise) around x in G and $Y = Y_0 \cup Y_1 \cup \cdots \cup Y_{m-1}$ be a partition of Y, such that $Y_i \subset N(n_i)$ for each i where $0 \leq i \leq m-1$. And further, let $Y_i' \subset Y_i$ be the private neighbours of n_i for each i. Note that Y_i' is not empty according to the minimality of M for each i. If there is a vertex of Y_i' adjacent to both a vertex of Y_{i-1} and a vertex of Y_{i+1} (where addition is taken modulo m), then this vertex is unique by the planarity of G and is denoted by y_i. Otherwise, we let y_i be any vertex of Y_i'. For each i, if $y_i y_{i+1} \in E(G)$, then we define L_i as the region inside the 5-cycle $x n_i y_i y_{i+1} n_{i+1} x$. Now we first prove the following claims.

Claim 1.

(1) If there is an edge from Y_i to Y_j, then i and j are consecutive.
(2) If $y_i y_{i+1} \in E(G)$, then for each vertex $a \in L_i \cap Y$, either n_i dominates a or n_{i+1} dominates a.

Proof.

(i) Let $u \in Y_i$ and $v \in Y_j$, where $j \neq i \pm 1$ and $j > i$. Then there exist numbers $i < k < j$ and $j < l < n-1$ such that n_i, n_k, n_j, n_l appear in order as shown in Figure 4. Assume $uv \in E(G)$, then we will show that $C = x n_i u v n_j x$ is a fundamental cycle of length 5. Indeed, we take a vertex $w_k' \in Y_k'$ and a vertex $w_l' \in Y_l'$, then w_k' is in L_i and w_l' is out in L_i by the planarity of G, and further, their neighbours on C are a subset of $\{u, v\}$. It is clear that C is a fundamental cycle, a contradiction.

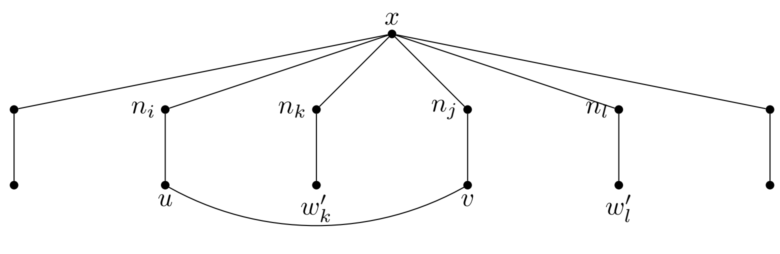


Fig. 4. $C = x n_i u v n_j x$ is a fundamental cycle.

(ii) Since M is a minimal vertex set that dominates Y and the definition of L_i, the conclusion holds. □

Claim 2. *Let $i, j \in [m-1]$ and $j \neq i \pm 1$.*

(1) If $n_i, n_j \in M$, then they have no common neighbours in Y.
(2) If $y_i \in Y_i$ and $y_j \in Y_j$, then y_i and y_j have no common neighbours in $N(x)$.

Proof. As proof in Claim 1, since i and j are not consecutive, there exist numbers $i < k < j$ and $j < l < n-1$ such that n_i, n_k, n_j, n_l appear in order and w'_k and w'_l are private neighbours as defined above, respectively.

(i) Let w_4 be a common neighbour of n_i and n_j in Y. Then the neighbours of w'_l and w'_k is only w_4, thus $xn_iw_4n_jx$ is a fundamental cycle of length 4, a contradiction.

(ii) Let w_5 be a common neighbour of y_i and y_j in $N(x)$. By the definition of y_i, we know that $w_5 \notin M$. Then either $xn_iy_iw_5x$ or $xn_jy_jw_5x$ forms a fundamental cycle of length 4. This is a contradiction. □

Claim 3. *Let $y_i \in Y_i$ for each $i \in [m]$. If $m \geq 6$, the following conclusions hold.*

(1) The shortest y_i-y_{i+3}-path is y_i, y_{i+1}, y_{i+2}, y_{i+3}, or if $m = 6$ possibly y_i, y_{i-1}, y_{i-2}, y_{i+3}.

(2) Let $P = y_iy_{i+1}y_{i+2}y_{i+3}$ be a path. Then y_{i+1} dominates $(L_i \cap Y) \cup (L_{i+1} \cap Y)$, and y_{i+2} dominates $(L_{i+1} \cap Y) \cup (L_{i+2} \cap Y)$.

Proof. *(i)* By Claim 2(ii), the vertices y_i and y_{i+3} have no common neighbors in $N(x)$. Thus the shortest path between y_i and y_{i+3} has length 3 by the planarity of G. Suppose that the shortest path between y_i and y_{i+3} is y_i, a, b, y_{i+3}. Again by Claim 2 (i) and (ii), we know that a and b must be in Y. Otherwise, we can find a path of length at least four by planarity of G, a contradiction. Then Claim 1 implies that $a = y_{i+1}$ and $b = y_{i+2}$, this is the path is $y_i, y_{i+1}, y_{i+2}, y_{i+3}$. Moreover, if $m = 6$, then it is clear that the path may be $y_i, y_{i-1}, y_{i-2}, y_{y+3}$. The conclusion holds.

(ii) Take any vertex $y \in L_i \cap Y$. Since $diam(G) = 3$, we know that there exists a path P of length 3 between y and y_{i+3}. By Claim 2 (i) and the planarity of G, P must contains y_{i+1}, since the distance between n_{i+1} and y_{i+3} is 3. Thus y_{i+1} dominates $L_i \cap Y$. By the symmetry, we get that y_{i+1} dominates $L_{i+1} \cap Y$. So y_{i+1} dominates $(L_i \cap Y) \cup (L_{i+1} \cap Y)$ holds. Similarly, we get that similarly y_{i+2} dominates $(L_{i+1} \cap Y) \cup (L_{i+2} \cap Y)$. □

If $m \geq 8$, then, from the proof of Claim 3, we know that the distance between y_i and y_{i+4} is 4, a contradiction to $diam(G) = 3$. Thus we will complete the proof by discussing the cases according to the value of m.

Case 1. $m = 7$.

Consider the shortest path from y_i to y_{i+3} for $i \in [7]$. By Claim 3 *(i)*, we have $y_0y_1y_2y_3y_4y_5y_6y_0$ is a cycle. We claim that $\{x, y_0, y_3\}$ is an EDS of G. Indeed, for any vertex $v \in (L_5 \cup L_6 \cup L_0 \cup L_1) \cap Y$, the distance between v and y_0 is at most 2 by By Claim 3 *(ii)*, and similarly any vertex $u \in (L_1 \cup L_2 \cup L_3 \cup L_4) \cap Y$ has distance at most 2 from y_3. And further, since x is a central vertex, then $\{x, y_0, y_3\}$ is an EDS of G, a contradiction.

Case 2. $m = 6$.

Consider the shortest path from y_i to y_{i+3}, where $i \in \{0, 1, 2\}$. From Claim 3 *(i)*, we know that there exists at least five consecutive edges $y_i y_{i+1}$. Without loss of generality, assume that $y_0y_1y_2y_3y_4y_5$. Using the similar analysis as in Case 1, again we get that $\{x, y_1, y_4\}$ is an EDS of G, a contradiction.

Case 3. $m = 5$.

Let G_y be the induced graph $G[y_0, y_1, y_2, y_3, y_4]$. And we will complete the proof by examining the following subcases.

Subcase 3.1. G_y contains a path of length 4.

Without loss of generality, we assume the path is $P = y_0y_1y_2y_3y_4$. We will prove that $S = \{x, y_1, y_3\}$ is an EDS of G, which contradicts to $\gamma_e(G) \geq 4$ and thus the result holds. Contrarily, suppose that there is a vertex $v \in V(G)$ is not extended dominated by S. This is, $d(v, y_1) = d(v, y_3) = 3$ and $d(v, x) = 2$ since x is the central vertex. By symmetry between L_0,L_3 and between L_1, L_2, we only consider the conditions that $v \in L_0$ or $v \in L_1$. Now if $v \in L_0$, then we know that n_0 dominates v or n_1 dominates v from Claim 1 (ii). Because $d(v, y_1) = 3$, it means that $d(v, n_1) = 2$ and $d(v, n_0) = 1$. Besides, $d(y_2, n_1) = 2$, $d(y_2, y_0) = 2$ and $d(y_2, n_0) = 3$ hold because y_2 is a private neighbour of n_2. And further, by considering the condition that the cycle $xn_0y_0y_1n_1$ is a cut cycle, then we get that $d(v, y_2) \geq 4$, a contradiction. Otherwise, again by Claim 1(ii) and $d(v, y_1) = 3$, then we get that $d(v, n_2) = 1$, $d(v, n_1) \geq 2$ and $d(v, y_2) = 2$. Since y_0 is private neighbour of n_0, then $d(y_0, n_1) \geq 2$, $d(y_0, n_2) = 3$, and $d(y_0, y_2) = 2$. From the cycle $xn_1y_1y_2n_2$ is cut cycle, we get that $d(v, y_0) \geq 4$, a contradiction.

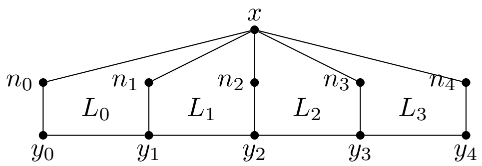
Subcase 3.2. G_y contains a path of length 3 and no paths of length 4.

Without loss of generality, assume that $P = y_0y_1y_2y_3$ is a path of length 3. And then $y_0y_4 \notin E(G)$ and $y_3y_4 \notin E(G)$. We claim that there exists a path of length 2 between y_3 and y_4, say y_3ay_4. Indeed, if $d(y_3, y_3) = 3$, then from y_2 is a private neighbour of n_2, we know that $d(y_2, y_4) \geq 4$, this is impossible. If $a \in N(x)\backslash M$, then $xn_2y_2y_3ax$ is fundamental. By Lemma 4.1 we know that $\gamma_e(G) \leq 3$, a contradiction. So $a \in Y_3$ or $a \in Y_4$. Similarly, we get that there exits a path y_0by_4, where $b \in Y_0$ or $b \in Y_4$. Since G_y contains no paths of length 4, then $a = y'_3 \in Y_3$ and $b = y'_0 \in Y_0$.

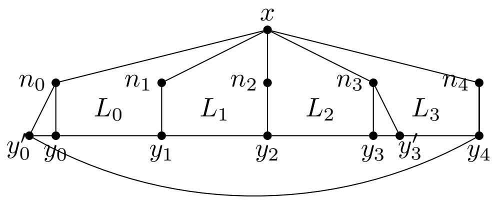
Now we want to show that $S = \{x, y_1, y_4\}$ is an EDS of G. Indeed, suppose that $v \in V(G)$ is not extended dominated by S. This is, $d(v, y_1) = d(v, y_4) = 3$ and $d(v, x) = 2$. Now we complete the proof utilizing the same idea as in Subcase 3.1. Now if $v \in L_0$, then we get that $d(v, n_0) = 1$ and $d(y_3, \{n_1, y_0, n_0\}) = 3$ from Claim 1(ii) and y_3 is a private neighbor of n_3. Thus $d(v, y_3) = 4$, this is impossible. If $v \in L_1$, then $d(v, n_2) = 1$, $d(v, n_2) \geq 2$ $d(y'_0, \{y_2, n_2\}) = 3$, and $d(y'_0, y_1) = 2$ hold by applying Claim 1(ii) and y_i is the private neighbour of n_i for $i \in \{0, 1, 2, 3, 4\}$. And thus $d(v, n_2) = 4$, this is impossible. The other case is that $v \in L_2$. Here we only consider $v \in Y_2$, because $v \in Y_3$ is similar. So using the same reason we get that $d(y'_0, \{y_2, n_2, y_3, n_3\}) = 3$, and thus there is no vertex inside of L_2, a contradiction to $v \in L_2$. The last is that $v \in L_3$, through using the same analysis as $v \in L_3$, we get that $d(v, y'_0) = 4$, this is impossible.

Subcase 3.3: G_y contains a path of length 2 and no paths of length at least 3.

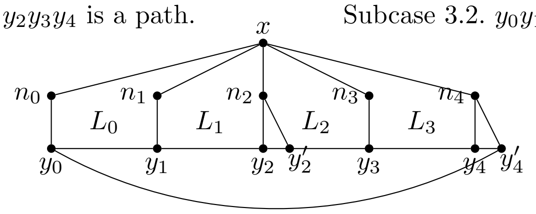
Without loss of generality, let $P = y_0y_1y_2$ be the path (see Figure 5). Indeed, $y_0y_4 \notin E(G)$, $y_2y_3 \notin E(G)$. We get that $d(y_2, y_3) = 2$, otherwise, it is easy to check that $d(y_0, y_3) \geq 4$, a contradiction. Now we assume that y_2ay_3 is a path of length 2. Now if $a \in N(x) \setminus M$, then the cycle $xn_1y_1y_2ax$ is a fundamental, this is impossible following Lemma 4.1. Now by the choice of y_i, it follows that $a \in Y_2$, say $a = y_2' \in Y_2$. Moreover, we know that y_3y_4 is an edge. Otherwise, it is easy to check that $d(y_0, y_3) \geq 3$, a contradiction. Now there exits a vertex $y_4' \in Y_4$ such that $y_0y_4'y_4$ is a path of length 2 with the same reason as in the above. Using the same analysis as in Subcase 3.2, we get that $\{x, y_1, y_4\}$ is an EDS of G, a contradiction.



Subcase 3.1. $y_0y_1y_2y_3y_4$ is a path.



Subcase 3.2. $y_0y_1y_2y_3$ is a path.



Subcase 3.3. $y_0y_1y_2$ is a path.

Fig. 5. Case 3. $m = 5$.

Subcase 3.4: G_y contains no paths of length of at least 2.

If there is an edge in G_y, say $y_0y_1 \in E(G)$, then y_0y_4 and y_1y_2 are not edges in G, since G_y contains no paths of length 2. If $y_2y_3 \in E(G)$, then $y_3y_4 \notin E(G)$. By the choice of y_i and $y_1y_2, y_3y_4 \notin E(G)$, then it is easy to check that $d(y_0, y_3) \geq 4$, a contradiction. If $y_3y_4 \in E(G)$, then $y_2y_3 \notin E(G)$. For the same reason, we check that $d(y_1, y_3) \geq 4$, a contradiction. Otherwise, it is easy to check that $d(y_0, y_3) \geq 4$, a contradiction.

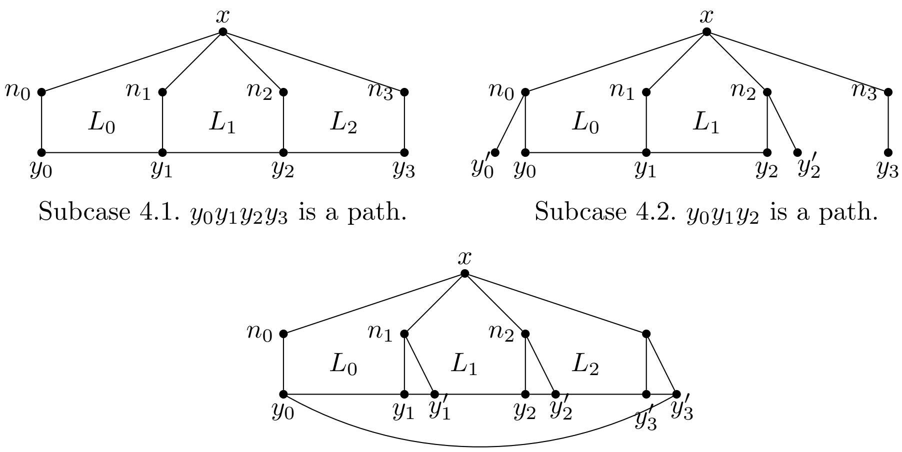
Case 4. $m = 4$.

Subcase 4.1: G_y contains a path of length 3.

Without loss of generality, assume that $P = y_0y_1y_2y_3$ is a path of length 3. Then we show that $\{x, y_1, y_2\}$ is an EDS of G. Every vertex in $L_0 \cap Y$ has a distance at most 2 from y_1. Otherwise, assume $v \in L_0 \cap Y$ and $d(v, y_1) = 3$, then v can't reach y_2. By a similar proof as before, we obtain $\forall u \in L_2 \cap Y$, $d(u, y_2) \leq 2$. It's easy to check $\{y_1, y_2, x\}$ extended dominates L_2. Thus $\{x, y_1, y_3\}$ is an EDS of G.

Subcase 4.2: G_y contains a path of length 2 and no paths of length 3.

Without loss of generality, assume that $P = y_0y_1y_2y_3$ is a path of length 3 as shown in Figure 6.



Subcase 4.1. $y_0y_1y_2y_3$ is a path.

Subcase 4.2. $y_0y_1y_2$ is a path.

Subcase 4.3. y_0y_1 is an edge.

Fig. 6. Case 4. $m = 4$.

So $y_2y_3 \notin E(G)$ and $y_0y_3 \notin E(G)$, because y_1 and y_3 have no common neighbor by Claim 2 (ii), $d(y_1y_3) = 3$. For any vertex v in $L_0 \cap Y$, v is dominated by y_0, or otherwise, v can't reach y_3. And by the similar proof, y_2 dominates all vertices in $L_1 \cap Y$, so we have $\forall v \in L_0 \cup L_1 \cap Y$, $d(v, y_1) \leq 2$. Thus x and y_1 2-extended dominate $L_0 \cup L_1$. If there is $y_2' \in N(n_2)$, then either $d(y_2', y_3) \leq 2$ or $d(y_2', y_1) \leq 2$, or otherwise, $d(y_2', y_0) \geq 4$. In a word, $\{y_1, y_3, x\}$ is an EDS of G.

Subcase 4.3: G_y contains no paths of length at least 2, and contains an edge.

Assume that $y_0y_1 \in E(G)$. The only possible structure is as shown in Figure 6. Indeed, Claim 1 (i) and the fact $d(y_i, y_{i+2}) \leq 3$ together guarantee there has to be a vertex a adjacent to both y_1 and y_2, if $a \in N(x) \setminus M$, then $xn_0y_0y_1ax$ form a fundamental cycle. So $a \in Y_1$ or $a \in Y_2$, we may assume $a = y_1' \in Y_1$, and another case has a similar proof. So the other edges appear for the same reason. We deduce that $\{x, y_1, y_2\}$ is an EDS. Or otherwise, there is a vertex $v \in V(G)$ such that $d(v, y_1) = d(v, y_2) = 3$ and v can't be dominated by x. We subsequently demonstrate that there can't be such a vertex. First, if $v \in L_1$, v is of distance 2 from x, then v is dominated by either n_1 or n_2, by the choice of M, thus $d(v, \{y_1, y_2\}) = 2$, a contradiction. Second, if $v \in L_0$, then v is dominated by n_0 and $d(v, n_1) \geq 2$, moreover, $d(y_2, \{n_0, y_0\}) = 3$, and $d(y_2, n_1) \geq 2$, which together imply that $d(v, y_2) \geq 4$, disobeying the assumption. Third, if $v \in L_2$, then v is dominated by n_3 and $d(v, n_2) \geq 2$ for the same reason, $d(y_1, n_2) \geq 2$, and $d(y_1, \{n_3, y_2', y_3\}) = 3$

together imply that $d(v, y_1) \geq 4$, disobeying the assumption.

Subcase 4.4: G_y contains no edges between different Y_i and $Y_{i\pm1}$.

Consider the distance between y_0 and y_2. Claim 1 (i) implies $y_0y_2 \notin E(G)$. If $d(y_0, y_2) = 2$, let the path be y_0ay_2. So $a \in N(x) \setminus M$, which induces a fundamental cycle $xn_0y_0ay_2x$, a contradiction. Thus $d(y_0, y_2) = 3$, and let the path is y_0aby_2, by a similar proof, either a or b belongs to $N(x) \setminus M$, and a fundamental cycle xn_0y_0abx appears, a contradiction.

In all cases, we can get an EDS of size 3 including x, hence the conclusion holds.

When the radius of the graph is 3, we utilize the lemma proposed by Goddard and Henning (2002), where they showed a method to describe such graphs. Let us recall a family of graphs known as *theta graph.* For $s \geq 3$, an s-*theta* graph (s-TG) is a graph obtained by joining two vertices by s internally disjoint paths, we call these paths *axes* for convenience see Figure 7 for example. A region of a theta graph is a portion of the plane bounded by two consecutive axes in the theta graph. In Goddard and Henning (2002), they gave the following useful lemma.

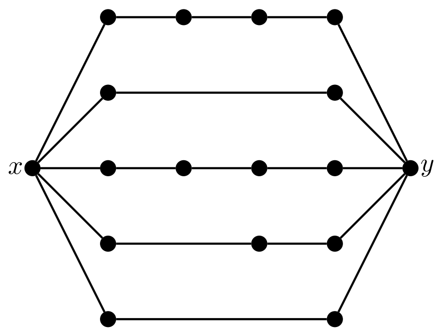


Fig. 7. A 5-theta graph with endvertices x and y.

Lemma 4.2. *(*Goddard and Henning (2002)*) Let d be a positive integer. There exists an arbitrarily large theta graph in a sufficiently large graph of diameter d and radius d.*

Theorem 4.2. *Let G be a sufficiently large planar graph with diameter* 3 *and radius d. Then*

$$\gamma_e^2(G) \leq 4.$$

Proof. Suppose that there is an 8-theta graph H as the subgraph of G from Lemma 4.2. Fix a planar embedding of H, then there exists 8 axis as $A_1, \ldots, A_8$ and 7 regions, named by L_i formed by A_i and A_{i+1} for $i \in \{1, \ldots, 7\}$. Let x and y be the two end vertices of H, respectively. Let S be a subset of vertices which are not extended dominated by $\{x, y\}$. Take any vertex $v \in S$. Then v lies in some region L_{i_0} of H. Set x' is a internal vertex in the farthermost axis A from v. Note that if $A = xy$, we take x' in the farthermost axis except A. From this it follows that the distance between v and x' not using $\{x, y\}$ is at least 4. If $d(v, x) = d(v, y) = 3$, then we get that the distance between v and x' is at least $d(v, x) + d(x, x') \geq 4$ or $d(v, y) + d(y, x') \geq 4$, a contradiction. Now the following two cases hold: $d(v, x) = 2,\ d(v, y) = 3$ or $d(v, y) = 3, d(v, x) = 2$. In the following proof, we only consider that $d(v, x) = 2,\ d(v, y) = 3$, since $d(v, y) = 3, d(v, x) = 2$ is similar. Since G is a planar a graph with diameter of 3, then there exists a path between x and y of length 2, say xay. Let G' be the graph obtained by contracting xay (including the vertices x and y) to a single vertex v_{xy}. Clearly G' is planar with diameter at most 3. Moreover, every vertex is within distance 2 from v_{xy} in G'. Thus, G' has radius at most 2 and v_{xy} is a center vertex. And thus $\gamma_e^2(G') \leq 3$ from Theorem 4.1, and further every extended dominating set contains v_{xy}. So $\gamma_e^2(G) \leq 4$. □

5. *k*-EDS on Random graphs

Several authors studied domination parameters in Random graphs (Joel (1992); Bollobás (1998); Zverovich and Poghosyan (2011)). In this section, we study the k-extended domination number on Random graphs. The *Random graph* $G(n, p)$ is a probability space over the set of graphs on the vertex set $\{v_1, v_2, \ldots, v_n\}$ determined by $Pr(v_iv_j \in E(G)) = p$ with these events mutually independent. We say that an event holds asymptotically almost surely (a.a.s.) if the probability that it holds tends to 1 as n tends to infinity. So we wonder how large p should be to guarantee the k-extended domination number is k for given integer k. The following theorem provides a specific answer.

Theorem 5.1. *Let $G \in G(n, p)$, where $p > c\sqrt{\frac{\ln n}{2n}}$ with a constant $c > \sqrt{2}$. Then a.a.s. $\gamma_e^k(G(n, p)) = k$.*

Proof. Let S be a vertex set of size k, and label the vertices as $\{v_1, v_2, \ldots, v_k\}$. Fix a vertex v_i for $i \in [k]$. The probability that a vertex w is in $N_2(v_i)$ is given by $Pr(w \in N_2(v_i)) = (1-p)[1-(1-p)(1-p^2)^{n-2}]$, denoted by p_1. While the probability that a vertex w is of distance at least 3 from v_i is given by $Pr(w \notin (N_2(v_i) \cup N(v_i))) = (1-p)(1-p^2)^{n-2}$, denoted by p_2. Let X be a random variable that denotes the number of vertices that are not extended dominated by S. We want to show that $Pr(X > 0) \to 0$ as $n \to 0$. We say a vertex u *bad* if it is not extended dominated by S. This means that u either has distance at least 3 from S, or belongs to $N_2(S)$ and has less than k quasi-neighbours in S. So we give the following expression of $Pr(u \text{ is bad})$ for some u.

$$\begin{aligned} Pr(u \text{ is bad}) &\leq \sum_{i=1}^{k} \binom{k}{i} p_2^i p_1^{k-i} \\ &= (p_1 + p_2)^k - p_1^k \\ &= (A - p_1)(A^{k-1} + A^{k-2} p_1 + \cdots + p_1^{k-1}) \\ &\leq p_2 k A^{k-1} \\ &= p_2 k [(1-p)\left(p\left(1-p^2\right)^{n-2} + 1\right)]^{k-1} \\ &\leq (1-p)\left(1-p^2\right)^{n-2} k (1-p^2)^{k-1} \\ &\leq k e^{-p^2(n+k-3)}. \end{aligned}$$

Let $A := p_1 + p_2$ for the sake of calculation and the last inequality holds because $1 - x \leq e^{-x}$. Then by the linearity of the expectation we have

$$\begin{aligned} E(X) &= (n-k) Pr(\text{fixed } u \text{ is bad}) \\ &\leq (n-k) k e^{-p^2(n+k-3)}. \end{aligned}$$

Now we need to determine the probability of p to satisfy $e^{-p^2(n+k-3)} > nk > (n-k)k$, which implies that $p > \sqrt{\frac{\ln nk}{n+k-3}} > \sqrt{\frac{\ln nk}{2n}}$. We take $p > c\sqrt{\frac{\ln n}{2n}}$ is enough, since k is a constant, and n is sufficiently large, where $c > \sqrt{2}$ is a constant. So by Markov's inequality, when n is sufficiently large,

$$\begin{aligned} Pr(X > 0) \leq E(x) &\leq (n-k) k e^{-p^2(n+k-3)} \\ &< \frac{nk}{e^{(c\sqrt{\frac{\ln n}{2n}})^2 (n+k-3)}} \\ &= \frac{nk}{n^{c^2(\frac{1}{2} + \frac{k-3}{2n})}} \to 0. \end{aligned}$$

Obviously, it tends to 0 when n tends to infinity. □

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