Perfect models for finite Coxeter groups

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Abstract

A model for a finite group is a set of linear characters of subgroups that can be induced to obtain every irreducible character exactly once. A perfect model for a finite Coxeter group is a model in which the relevant subgroups are the quasiparabolic centralizers of perfect involutions. In prior work, we showed that perfect models give rise to interesting examples of W-graphs. Here, we classify which finite Coxeter groups have perfect models. Specifically, we prove that the irreducible finite Coxeter groups with perfect models are those of types A_n, B_n, D_{2n+1}, H_3, or $\mathsf{I}_2(n)$. We also show that up to a natural form of equivalence, outside types A_3, B_n, and H_3, each irreducible finite Coxeter group has at most one perfect model. Along the way, we also prove a technical result about representations of finite Coxeter groups, namely, that induction from standard parabolic subgroups of corank at least two is never multiplicity-free.

1 Introduction

A *model* for a finite group G is a set of linear characters $\sigma_i : H_i \to \mathbb{C}$ of subgroups such that adding up the induced characters $\sum_i \mathrm{Ind}_{H_i}^G(\sigma_i)$ gives the multiplicity-free sum $\sum_{\psi\in\mathrm{Irr}(G)} \psi$ of all complex irreducible characters of G. A model for G lets one construct an explicit G-representation, with a natural basis relative to which the elements of G act as monomial matrices, containing each irreducible G-representation exactly once.

Example 1.1. For a positive integer n let S_n be the symmetric group of permutations of $[n] := \{1, 2, \dots, n\}$. Embed $S_i \times S_{n-i} \subseteq S_n$ as the subgroup of elements preserving $\{1, 2, \dots, i\}$. This subgroup acts on itself by $(g, h) : (x, y) \mapsto (g^* x g^{-1}, hyh^{-1})$ where $g^* \in S_i$ is the permutation mapping $a \mapsto i + 1 - g(i + 1 - a)$. Let H_i denote the stabilizer subgroup of $1 \in S_i \times S_{n-i}$ and define $\sigma_i(x, y) = \mathrm{sgn}(y)$. Then $\{\sigma_i : H_i \to \{\pm 1\} : i = 0, 2, 4, \dots 2\lfloor \frac{n}{2} \rfloor\}$ is a model for S_n [14].

In our previous work [20] we introduced the notion of a *perfect model* for a finite Coxeter group. The preceding construction gives an example of such a model for the symmetric group. For more general Coxeter groups the precise definition of a perfect model goes as follows.

Let (W, S) be a finite Coxeter system. Define $\mathrm{Aut}(W, S)$ to be the set of automorphisms $\theta \in \mathrm{Aut}(W)$ with $\theta(S) = S$. Let W^+ be the set of pairs $(w, \theta) \in W \times \mathrm{Aut}(W, S)$, viewed as a group with multiplication $(u, \alpha)(v, \beta) = (u\alpha(v), \alpha\beta)$. We view W as a subgroup of W^+ by identifying $w \in W$ with $(w, \mathrm{id}) \in W^+$.

An element $z \in W^+$ is a *perfect involution* if $z^2 = (zt)^4 = 1$ for all $t \in \{wsw^{-1} : (w, s) \in W \times S\}$. For example, every fixed-point-free involution in S_n when n is even is perfect. Rains and Vazirani introduced the notion of perfect involutions in [23] as an example of a *quasiparabolic set*; see Section 2.3 for further discussion of this background.

Let $\mathscr{I} = \mathscr{I}(W,S)$ denote the set of perfect involutions in W^+. The group W acts on $\mathscr{I}$ by conjugation. Given a subset $J \subseteq S$, write $W_J := \langle s \in J \rangle$ and let

$$\mathscr{I}_J := \mathscr{I}(W_J, J) \subseteq W_J^+ := (W_J)^+.$$

A *(perfect) model triple* $\mathbb{T} = (J, \mathcal{K}, \sigma)$ for (W,S) consists of a subset $J \subseteq S$, a W_J-conjugacy class $\mathcal{K} \subseteq \mathscr{I}_J$, and a linear character $\sigma : W_J \to \{\pm 1\}$.[1] The *character* of $\mathbb{T}$ is

$$\chi^{\mathbb{T}} := \mathrm{Ind}_{C_J(z)}^{W} \mathrm{Res}_{C_J(z)}^{W_J}(\sigma) \tag{1.1}$$

where $z \in \mathcal{K}$ is arbitrary and $C_J(z) := \{w \in W_J : wz = zw\}$. A *perfect model* for W, finally, is a set of model triples $\mathcal{P}$ such that $\sum_{\mathbb{T}\in\mathcal{P}} \chi^{\mathbb{T}} = \sum_{\psi\in\mathrm{Irr}(W)} \psi$.

Example 1.2. If $s_1, s_2, \dots, s_{n-1}$ are the usual simple generators of S_n, then the perfect model corresponding to Example 1.1 comes from taking $J = \{s_1, s_2, \dots, s_{i-1}, s_{i+1}, s_{i+2}, \dots, s_{n-1}\}$ and $\mathcal{K} = \{(g^* g^{-1}, *) \in (S_i \times S_{n-i})^+ : g \in S_i\}$ for $i = 0, 2, 4, \dots, 2\lfloor \frac{n}{2} \rfloor$.

Our goal in this article is to classify which finite Coxeter groups have perfect models. Before stating our main results, we briefly explain why such models are interesting to consider.

A *Gelfand model* for a group or algebra is a semisimple module containing exactly one constituent in each isomorphism class of irreducible representations. Each perfect model gives rise to a pair of Gelfand models for the Iwahori-Hecke algebra $\mathcal{H}(W)$ of (W,S) with some nice properties. The perfect model for S_n described above leads in this way to the $\mathcal{H}(S_n)$-representation previously studied in [1], for example.

There are simple formulas for the action of the standard generators of $\mathcal{H}(W)$ in these Gelfand models. Each module also has a unique *bar operator* that is compatible with the usual bar operator of $\mathcal{H}(W)$, and a unique bar invariant *canonical basis* [19, 20] analogous to the Kazhdan-Lusztig basis of $\mathcal{H}(W)$. The action of the standard generators of $\mathcal{H}(W)$ on these canonical bases may be encoded as *W-graphs* in the sense of [15]. These objects then provide examples of *Gelfand W-graphs*: W-graphs whose corresponding Iwahori-Hecke algebra representations are Gelfand models. Our results about perfect models will precisely classify the Gelfand W-graphs that can arise in this way.

Another reason to be interested in perfect models is for their connection to models of finite groups of Lie type. One can view the perfect model for S_n in Example 1.1 as the "$q \to 1$ limit" of the so-called *Klyachko model* for the finite general linear group $\mathsf{GL}(n,q)$ [13, 16]. We do not know of much related work on Klyachko models for the other classical finite groups of Lie type, but we expect that such models should be similarly related to perfect models for classical Weyl groups.

We now summarize our results. The following combines Theorem 2.3 and the main theorems in Sections 3, 4, 5, and 6.

Theorem 1.3. A finite Coxeter group has a perfect model if and only if each of its irreducible factors has a perfect model. An irreducible finite Coxeter group has a perfect model if and only if it is of type A_{n-1}, B_n, D_{2n+1}, H_3, or $\mathsf{I}_2(n+1)$ for an integer $n \geq 2$.

An *involution model* for a finite group G is a model $\{\lambda_i : H_i \to \mathbb{C}\}$ in which the subgroups H_i range over the centralizers of the distinct conjugacy classes of involutions $g = g^{-1} \in G$. Such models are natural to consider when G has all real representations, since then the Frobenius-Schur involution counting theorem asserts that $\sum_{\psi\in\mathrm{Irr}(G)} \psi(1) = |\{g \in G : g = g^{-1}\}|$.

Involution models for finite Coxeter groups were studied and classified in [4, 5, 14, 25]. Comparing Theorem 1.3 with the main result in [25] gives the following corollary.

[1] Since Coxeter groups are generated by involutions, any linear character $\sigma : W_J \to \mathbb{C}$ takes values in $\{\pm 1\}$. More generally, every character $\chi \in \mathrm{Irr}(W)$ takes values in a subfield of $\mathbb{R} \subsetneq \mathbb{C}$ [12, Thm. 5.3.8].

Corollary 1.4. A finite Coxeter group has a perfect model if and only if it has an involution model.

We do not know of an explanation for this phenomenon that avoids appealing to the case-by-case classification of both kinds of models. More general kinds of involution models for complex reflection groups have been studied and classified in [7, 8, 9, 17, 18]. It would be interesting to know if the notion of a perfect model can be extended to that context.

Besides settling existence questions, our results also establish some uniqueness properties of perfect models. A finite Coxeter group may have many different perfect models, each producing different Gelfand W-graphs. We can show, however, that these W-graphs are all isomorphic after possibly ignoring edge labels and reversing edge orientations. We do this by studying a form of equivalence for perfect models introduced in [20]; see Section 2.5 for the definition. Perfect models that are equivalent give rise to essentially the same W-graphs, in a way that will be made precise below.

In [20] we described the Gelfand W-graphs associated to a specific perfect model for each classical Weyl group, excluding type D_{2n}. The following result (combining Theorems 3.3, 4.5 and 5.8 and Proposition 6.1) gives a sense in which these Gelfand W-graphs are canonical.

Theorem 1.5. If W is an irreducible finite Coxeter group not of type A_3, B_n, or H_3 then W has at most one equivalence class of perfect models.

If W is of type B_n for $n \neq 3$ then there are exactly two equivalence classes of perfect models, one of which is a trivial "refinement" of the other; see Theorem 4.5. There are a few additional models when W is of type A_3, B_3, or H_3; see Examples 3.4 and 4.6 and Proposition 6.2.

Our proofs rely on the following property which may be of independent interest. This theorem extends results in [2, 3] which address the case when $\chi = \mathbb{1}$ is the trivial character:

Theorem 1.6. If (W, S) is an irreducible finite Coxeter system and $J \subseteq S$ has $|S \setminus J| \geq 2$, then $\mathrm{Ind}_{W_J}^W(\chi)$ is not multiplicity-free for any irreducible character $\chi \in \mathrm{Irr}(W_J)$.

Section 2 contains some background and general results about perfect models. Sections 3, 4, and 5 classify the perfect models up to equivalence for each classical Weyl group. Section 6 briefly explains the perfect model classification for the remaining exceptional finite Coxeter groups. Appendix A, finally contains the proofs of Theorem 1.6 and another technical result.

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2 Preliminaries

2.1 Restriction and induction

Suppose $H \subseteq G$ are finite groups. Write $\mathrm{Irr}(H)$ and $\mathrm{Irr}(G)$ for the corresponding sets of complex irreducible characters. If $\psi : H \to \mathbb{C}$ and $\chi : G \to \mathbb{C}$ are class functions (that is, maps constant on conjugacy classes), then we denote the restriction of χ to H by $\mathrm{Res}_H^G(\chi)$ and the class function of G induced from ψ by $\mathrm{Ind}_H^G(\psi)$. An explicit formula for the induced function is

$$\mathrm{Ind}_H^G(\chi)(x) = \frac{1}{|H|} \sum_{\substack{g \in G \\ gxg^{-1} \in H}} \chi(gxg^{-1}) \quad \text{for } x \in G. \tag{2.1}$$

Induction from H to G is the unique linear operation such that $\langle \chi, \mathrm{Ind}_H^G(\psi) \rangle_G = \langle \mathrm{Res}_H^G(\chi), \psi \rangle_H$ for all $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(H)$, where $\langle \cdot, \cdot \rangle_G$ is the bilinear form on class functions of G relative to which $\mathrm{Irr}(G)$ is an orthonormal basis.

2.2 Extended Coxeter groups

Let (W, S) be a finite Coxeter system with length function $\ell : W \to \mathbb{N} = \{0, 1, 2, \dots\}$. The group W always has at least two linear characters, given by the trivial character $\mathbb{1} : w \mapsto 1$ and the sign character $\mathrm{sgn} : w \mapsto (-1)^{\ell(w)}$.

Define $W^+ = W \rtimes \mathrm{Aut}(W, S)$ as in the introduction. We extend ℓ to W^+ by setting $\ell(w, \theta) = \ell(w)$. Besides identifying $w \in W$ with $(w, \mathrm{id}) \in W^+$, we also identify each $\theta \in \mathrm{Aut}(W, S)$ with the element $(1, \theta) \in W^+$ and view $\mathrm{Aut}(W, S) \subseteq W^+$ as a subgroup in this way. Every $\alpha \in \mathrm{Aut}(W, S)$ extends to an automorphism of W^+ by the formula

$$\alpha : (w, \theta) \mapsto (\alpha(w), \alpha\theta\alpha^{-1}) = (1, \alpha)(w, \theta)(1, \alpha)^{-1}.$$

Suppose $z = (w, \theta) \in W^+$. Then $z^{-1} = (\theta(w)^{-1}, \theta^{-1})$, so $z^2 = 1$ if and only if $\theta = \theta^{-1}$ and $w^{-1} = \theta(w)$. The conjugation action of $g \in W$ on W^+ is $gzg^{-1} = (g \cdot w \cdot \theta(g)^{-1}, \theta)$. We refer to the orbits of this W-action in the set $\mathscr{I}$ of perfect involutions as *perfect conjugacy classes*.

2.3 Quasi-parabolic sets

Introduced by Rains and Vazirani in [23], a *quasi-parabolic W-set* is a set X with a height function $\mathsf{ht} : X \to \mathbb{Z}$ and a left W-action satisfying a short list of technical axioms. The motivating example is the set of distinguished coset representatives $W^J := \{w \in W : \ell(sw) > \ell(w) \text{ for all } s \in J\}$ where $J \subseteq S$ and $\mathsf{ht} = \ell$. The quasi-parabolic axioms ensure that there are simple formulas for a module of the Iwahori-Hecke algebra of W deforming the permutation representation of W on X.

The set of perfect involutions $\mathscr{I}$ in W^+ is an example of a quasi-parabolic W-set, relative to the conjugation action of W and the height function $\mathsf{ht}(z) := \lfloor \frac{\ell(z)}{2} \rfloor$ [23, §4]. This is perhaps the most interesting general construction of a quasiparabolic set that is not isomorphic to one of the "parabolic" examples W^J. This fact has many consequences; we note here just one technical property. An element $z \in W^+$ is *W-minimal* if $\ell(szs) \geq \ell(z)$ for all $s \in S$. Because $\mathscr{I}$ is quasi-parabolic, each perfect conjugacy class contains a unique W-minimal element [23, Cor. 2.10]. This element is also the unique minimal-length element in its class.

2.4 Dual model triples

Recall the notion of a *(perfect) model triple* for (W, S) from the introduction. We do not distinguish between model triples $(J, \mathcal{K}, \sigma_1)$ and $(J, \mathcal{K}, \sigma_2)$ when $\mathrm{Res}^{W_J}_{C_J(z)}(\sigma_1) = \mathrm{Res}^{W_J}_{C_J(z)}(\sigma_2)$, as these give rise to the same character via (1.1).

Given a subset $J \subseteq S$ let w_J denote the longest element of W_J and define $w_0 := w_S$. For $w \in W$ let $\mathrm{Ad}(w) \in \mathrm{Aut}(W)$ denote the inner automorphism $x \mapsto wxw^{-1}$. Then $\mathrm{Ad}(w_J) \in \mathrm{Aut}(W_J, J)$ and the element $w_J^+ := (w_J, \mathrm{Ad}(w_J))$ is a central involution in W_J^+, so $w_J^+ \in \mathscr{I}_J$. Let $w_0^+ := w_S^+$. The *dual* of a model triple $\mathbb{T} = (J, \mathcal{K}, \sigma)$ is $\mathbb{T}^\vee := (J^\vee, \mathcal{K}^\vee, \sigma^\vee)$ where

$$\begin{aligned} J^\vee &:= \mathrm{Ad}(w_0)(J) = w_0 J w_0, \\ \mathcal{K}^\vee &:= \mathrm{Ad}(w_0) \cdot w_J^+ \cdot \mathcal{K} \cdot \mathrm{Ad}(w_0) = \{(w_0 x w_J w_0, \mathrm{Ad}(w_0)\mathrm{Ad}(w_J)\theta\mathrm{Ad}(w_0)) : (x, \theta) \in \mathcal{K}\}, \\ \sigma^\vee &:= \sigma \circ \mathrm{Ad}(w_0). \end{aligned}$$

Since w_0 and w_J^+ are involutions, it is easy to see that $(\mathbb{T}^\vee)^\vee = \mathbb{T}$. It holds by [20, Prop. 3.33] that if $\mathbb{T}$ is a model triple for (W, S) then so is $\mathbb{T}^\vee$ and $\chi^{\mathbb{T}} = \chi^{\mathbb{T}^\vee}$.

2.5 Model equivalence

Let $\mathbb{T} = (J, \mathcal{K}, \sigma)$ be a model triple for (W, S). Given $\alpha \in \mathrm{Aut}(W, S)$, define

$$\mathbb{T}^\alpha := (\alpha^{-1}(J), \alpha^{-1}(\mathcal{K}), \sigma \circ \alpha).$$

As explained in [20, §3.5], this is also a model triple of (W, S) with $\chi^{\mathbb{T}^\alpha} = \chi^{\mathbb{T}} \circ \alpha$.

Suppose $\mathbb{T}' = (J', \mathcal{K}', \sigma')$ is another model triple for W. We write $\mathbb{T} \equiv \mathbb{T}'$ if $J = J'$ and it holds that $C_J(z) = C_{J'}(z')$ and $\mathrm{Res}^{W_J}_{C_J(z)}(\sigma) = \mathrm{Res}^{W_J}_{C_J(z)}(\sigma')$ where $z \in \mathcal{K}$ and $z' \in \mathcal{K}'$ are the unique minimal-length elements in each W_J-conjugacy class. In this case $\chi^{\mathbb{T}} = \chi^{\mathbb{T}'}$.

Let $\sim$ denote the transitive closure of the relation on model triples that has $\mathbb{T} \sim \mathbb{T}'$ when $\mathbb{T} \equiv \mathbb{T}'$ or $\mathbb{T}^\vee = \mathbb{T}'$ or $\mathbb{T}^\alpha = \mathbb{T}'$ for an inner automorphism $\alpha \in \mathrm{Aut}(W, S) \cap \{\mathrm{Ad}(w) : w \in W\}$. When $\mathbb{T} \sim \mathbb{T}'$ we say that the model triples are *strongly equivalent*. The following is clear:

Proposition 2.1. Strongly equivalent model triples for (W, S) have the same characters.

Finally write $\mathrm{sgn} : w \mapsto (-1)^{\ell(w)}$ for the sign character of W and define $\overline{\mathbb{T}} := (J, \mathcal{K}, \sigma\,\mathrm{sgn})$. This is another model triple for (W, S) with $\chi^{\overline{\mathbb{T}}} = \chi^{\mathbb{T}}\,\mathrm{sgn}$.

We define $\approx$ to be the transitive closure of the relation on model triples for (W, S) that has $\mathbb{T} \approx \mathbb{T}'$ whenever $\mathbb{T} \sim \mathbb{T}'$, $\overline{\mathbb{T}} = \mathbb{T}'$, or $\mathbb{T}^\alpha = \mathbb{T}'$ for an outer automorphism $\alpha \in \mathrm{Aut}(W, S)$. When $\mathbb{T} \approx \mathbb{T}'$ we say that the two model triples are *equivalent*. When $\mathcal{P}$ and $\mathcal{P}'$ are sets of model triples, we write $\mathcal{P} \approx \mathcal{P}'$ and say that $\mathcal{P}$ and $\mathcal{P}'$ are *equivalent* if there is a bijection $\mathcal{P} \to \mathcal{P}'$ such that if $\mathbb{T} \mapsto \mathbb{T}'$ then $\mathbb{T} \approx \mathbb{T}'$.

Here is why this is an appropriate notion of equivalence. To each perfect model there is a pair of associated W-graphs $\Upsilon^{\mathbf{m}}(\mathcal{P})$ and $\Upsilon^{\mathbf{n}}(\mathcal{P})$ [20]. Suppose $\mathcal{P}$ and $\mathcal{P}'$ are equivalent perfect models for W. Then there is a canonical bijection between the sets of vertices in the disjoint unions $\Upsilon^{\mathbf{m}}(\mathcal{P}) \sqcup \Upsilon^{\mathbf{n}}(\mathcal{P})$ and $\Upsilon^{\mathbf{m}}(\mathcal{P}') \sqcup \Upsilon^{\mathbf{n}}(\mathcal{P}')$. This bijection restricts on each weakly-connected component of the underlying directed graphs to a map that is either an isomorphism or an anti-isomorphism onto its image [20, Cor. 3.35]. Thus, when we ignore edge labels and orientations, the graphs $\Upsilon^{\mathbf{m}}(\mathcal{P}) \sqcup \Upsilon^{\mathbf{n}}(\mathcal{P})$ and $\Upsilon^{\mathbf{m}}(\mathcal{P}') \sqcup \Upsilon^{\mathbf{n}}(\mathcal{P}')$ are isomorphic.

2.6 Factorizable model triples

A character of a finite group is *multiplicity-free* if it is a sum of distinct irreducible characters. We say that a model triple $\mathbb{T}$ is *multiplicity-free* if its character $\chi^{\mathbb{T}}$ is multiplicity-free. All model triples appearing in a perfect model must have this property.

The *Coxeter diagram* of (W, S) is the graph with vertex set S that has an edge between two elements $s, t \in S$ whenever $st \neq ts$; this edge is labeled by the order of the product $st \in W$. The *irreducible components* of (W, S) are the subsystems (W_J, J) where $J \subseteq S$ is the set of vertices in a connected component of the Coxeter diagram. A Coxeter system (W, S) is *irreducible* if it has exactly one irreducible component.

For $z = (w, \theta) \in W^+$ let $\mathsf{aut}(z) := \theta$. Suppose $\mathbb{T} = (J, \mathcal{K}, \sigma)$ is a model triple. Then the set $\{\mathsf{aut}(z) : z \in \mathcal{K}\}$ has just one element, which we denote by $\mathsf{aut}(\mathcal{K}) \in \mathrm{Aut}(W_J, J)$. We say that $\mathbb{T}$ is *factorizable* if $\mathsf{aut}(\mathcal{K})$ preserves each irreducible component of (W_J, J).

Theorem 2.2. If W is irreducible then every multiplicity-free model triple for W is factorizable.

We prove this result in Section A.

2.7 Models for reducible groups

Let (W, S) be a finite Coxeter system. Suppose $L_1, L_2, \ldots, L_k$ are disjoint, nonempty sets such that $S = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_k$ and every $s \in L_i$ commutes with every $t \in L_j$ for all $1 \leq i < j \leq k$. Let $W_i = W_{L_i}$ for $i \in [k]$. The subsystems (W_i, L_i) might be the irreducible factors of (W, S), for example, or they might be larger subgroups.

Each automorphism of W_i extends to an automorphism of W fixing all elements of W_j for $i \neq j$, so we may view $W_i^+ \subseteq W^+$ and $\mathscr{I}_i := \mathscr{I}(W_i, L_i) \subseteq \mathscr{I} = \mathscr{I}(W, S)$. Each $w \in W$ can be written uniquely as $w = w_1 w_2 \cdots w_k$ with $w_i \in W_i$, so given functions $f_i : W_i \to \mathbb{C}$ for $i \in [k]$ we may define $f_1 \otimes f_2 \otimes \cdots \otimes f_k : W \to \mathbb{C}$ by $w \mapsto f_1(w_1)f_2(w_2)\cdots f_k(w_k)$. This gives a bijection

$$\mathrm{Irr}(W_1) \times \mathrm{Irr}(W_2) \times \cdots \times \mathrm{Irr}(W_k) \to \mathrm{Irr}(W).$$

If $\mathbb{T}_i = (J_i, \mathcal{K}_i, \sigma_i)$ is a model triple for (W_i, L_i) for each $i \in [k]$, then we define

$$\mathbb{T}_1 \otimes \mathbb{T}_2 \otimes \cdots \otimes \mathbb{T}_k := (J, \mathcal{K}, \sigma)$$

where $J := J_1 \sqcup J_2 \sqcup \cdots \sqcup J_k$, $\mathcal{K} := \mathcal{K}_1\mathcal{K}_2\cdots\mathcal{K}_k$, and $\sigma := \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_k$. Note that $\mathcal{K}$ is a well-defined subset of $\mathscr{I}(W_J, J)$, although the latter might not be a subset of $\mathscr{I} \subseteq W^+$ if there are Coxeter automorphisms of W_J that do not extend to W.

It is straightforward to see that if $\mathbb{T}_i$ is a factorizable model triple for (W_i, L_i) for each $i \in [k]$ then $\mathbb{T}_1 \otimes \mathbb{T}_2 \otimes \cdots \otimes \mathbb{T}_k$ is a factorizable model triple for (W, S) with $\chi^{\mathbb{T}_1 \otimes \mathbb{T}_2 \otimes \cdots \otimes \mathbb{T}_k} = \chi^{\mathbb{T}_1} \otimes \chi^{\mathbb{T}_2} \otimes \cdots \otimes \chi^{\mathbb{T}_k}$. Every factorizable model triple for (W, S) arises in this way.

Theorem 2.3. A finite Coxeter system (W, S) has a perfect model if and only if each of its irreducible factors has a perfect model.

Proof. If each irreducible factor of (W, S) has a perfect model then a perfect model for W is obtained by tensoring together the corresponding model triples.

Suppose instead that (W, S) is a reducible Coxeter system with a perfect model. Then there exists a nonempty subset $S' \subsetneq S$ such that $(W_{S'}, S')$ is irreducible. Let $S'' := S \setminus S'$. We will show that $W_{S'}$ also has a perfect model. This is nontrivial primarily because although $W = W_{S'} \times W_{S''}$, the extended group W^+ is not always isomorphic to $W^+_{S'} \times W^+_{S''}$.

Suppose $\mathbb{T} = (J, \mathcal{K}, \sigma)$ is a model triple for W. Let $\theta := \mathsf{aut}(\mathcal{K}) \in \mathrm{Aut}(W_J, J)$. Then we can express $J = A \sqcup B \sqcup C \sqcup D$ for disjoint subsets $A, B \subseteq S'$ and $C, D \subseteq S''$ with $\theta(A) = A$, $\theta(B) = C$, $\theta(C) = B$, and $\theta(D) = D$.

In this setup $(W_B, B) \cong (W_C, C)$ and all elements $a \in A$, $b \in B$, $c \in C$, and $d \in D$ must pairwise commute. Additionally, the minimal-length element of $\mathcal{K}$ must have the form $z_{\min} := (z_A z_D, \theta)$ for some $z_A \in W_A$ and $z_D \in W_D$. Let $\theta_A = \theta|_{W_A}$ and $\theta_D = \theta|_{W_D}$. Then the centralizer of $z_{\min}$ in $W_J = W_A \times W_B \times W_C \times W_D$ is

$$H := C_{W_A}((z_A, \theta_A)) \times \Delta_\theta(W_B \times W_C) \times C_{W_D}((z_D, \tau_D))$$

where $\Delta_\theta(W_B \times W_C) := \{b \cdot \theta(b) : b \in W_B\} \subseteq W_B \times W_C$. Define

$$\sigma_A := \mathrm{Res}^{W_J}_{W_A}(\sigma), \quad \sigma_D := \mathrm{Res}^{W_J}_{W_D}(\sigma), \quad \text{and} \quad \sigma_B(b) = \sigma(b \cdot \theta(b)) \text{ for } b \in W_B.$$

Since all characters of finite Coxeter groups are real-valued, Frobenius reciprocity implies that

$$\chi^{\mathbb{T}} = \mathrm{Ind}^W_H \mathrm{Res}^{W_J}_H(\sigma) = \mathrm{Ind}^W_{W_J}\Big(\chi^{\mathbb{T}}_A \otimes \Big(\sum_{\psi \in \mathrm{Irr}(W_B)} \sigma_B \psi \otimes \psi \circ \theta\Big) \otimes \chi^{\mathbb{T}}_D\Big)$$

where $\chi^{\mathbb{T}}_A := \mathrm{Ind}^{W_A}_{C_{W_A}((z_A,\theta_A))}\mathrm{Res}^{W_A}_{C_{W_A}((z_A,\theta_A))}(\sigma_A)$ and $\chi^{\mathbb{T}}_D := \mathrm{Ind}^{W_D}_{C_{W_D}((z_D,\theta_D))}\mathrm{Res}^{W_D}_{C_{W_D}((z_D,\theta_D))}(\sigma_D)$. Since $W = W_{S' \sqcup S''} = W_{S'} \times W_{S''}$ we can rewrite this as

$$\chi^{\mathbb{T}} = \sum_{\psi \in \mathrm{Irr}(W_B)} \mathrm{Ind}^{W_{S'}}_{W_A \times W_B}\left(\chi^{\mathbb{T}}_A \otimes \sigma_B \psi\right) \otimes \mathrm{Ind}^{W_{S''}}_{W_C \times W_D}\left(\psi \circ \theta \otimes \chi^{\mathbb{T}}_D\right).$$

A basis for the class functions on W is given by the irreducible characters $\chi \otimes \psi$ for $\chi \in \mathrm{Irr}(W_{S'})$ and $\psi \in \mathrm{Irr}(W_{S''})$. Let $\mathcal{L}$ be the linear map from class functions on W to class functions on $W_{S'}$ that sends $\chi \otimes \psi \mapsto \chi$ if $\psi = \mathbb{1}$ and to zero otherwise. Since for $\psi \in \mathrm{Irr}(B)$ we have

$$\Big\langle \mathbb{1}, \mathrm{Ind}^{W_{S''}}_{W_C \times W_D}\left(\psi \circ \theta \otimes \chi^{\mathbb{T}}_D\right)\Big\rangle_{W_{S''}} = \left\langle \mathbb{1}, \psi \circ \theta \otimes \chi^{\mathbb{T}}_D \right\rangle_{W_C \times W_D} = \begin{cases} \langle \mathbb{1}, \chi^{\mathbb{T}}_D \rangle_{W_D} & \text{if } \psi = \mathbb{1} \\ 0 & \text{if } \psi \neq \mathbb{1}, \end{cases}$$

it follows that $\mathcal{L}(\chi^{\mathbb{T}}) = \langle \mathbb{1}, \chi^{\mathbb{T}}_D \rangle_{W_D} \mathrm{Ind}_{W_A \times W_B}^{W_J} \left(\chi^{\mathbb{T}}_A \otimes \sigma_B \right)$. Define $\mathcal{K}'$ to be the $W_{A \sqcup B}$-conjugacy class of (z_A, θ'_A) where $\theta'_A : ab \mapsto \theta(a)b$ for $a \in W_A$ and $b \in W_B$. Then $\mathrm{Ind}_{W_A \times W_B}^{W_J} \left(\chi^{\mathbb{T}_A} \otimes \sigma_B \right)$ is just the character of the model triple $\mathbb{T}' := (A \sqcup B, \mathcal{K}', \sigma_A \otimes \sigma_B)$ for $(W_{S'}, S')$.

Let $\mathcal{P}$ be a perfect model for W. Then $\langle \mathbb{1}_{W_D}, \chi^{\mathbb{T}}_D \rangle_{W_D} \in \{0, 1\}$ for all $\mathbb{T} \in \mathcal{P}$ and $\sum_{\mathbb{T} \in \mathcal{P}} \chi^{\mathbb{T}} = \sum_{\chi \in \mathrm{Irr}(W_{S'})} \sum_{\psi \in \mathrm{Irr}(W_{S''})} \chi \otimes \psi$. Define $\mathcal{P}' = \{\mathbb{T}' : \mathbb{T} \in \mathcal{P} \text{ has } \langle \mathbb{1}_{W_D}, \chi^{\mathbb{T}}_D \rangle_{W_D} = 1\}$. Then $\sum_{\mathbb{S} \in \mathcal{P}'} \chi^{\mathbb{S}} = \sum_{\mathbb{T} \in \mathcal{P}} \mathcal{L}(\chi^{\mathbb{T}}) = \mathcal{L}\left(\sum_{\mathbb{T} \in \mathcal{P}} \chi^{\mathbb{T}} \right) = \sum_{\chi \in \mathrm{Irr}(W_{S'})} \chi$ so $\mathcal{P}'$ is a perfect model for $W_{S'}$. □

2.8 Classical Weyl groups

By Theorem 2.3 our main classification problem reduces to understanding which irreducible finite Coxeter groups have perfect models—in particular those groups in the three infinite families of classical Weyl groups.

Let $(i, i+1)$ for $i \in \mathbb{Z}$ denote the permutation of $\mathbb{Z}$ interchanging i and $i+1$ while fixing all other points. The group of permutations of $\mathbb{Z}$ with finite support is $S_{\mathbb{Z}} := \langle (i, i+1) : i \in \mathbb{Z} \rangle$. We realize the classical Weyl groups as subgroups of $S_{\mathbb{Z}}$ in the following way. Define $s_0 := (-1, 1)$. For integers $i > 0$ let $s_i := (i, i+1)(-i, -i-1)$ and $s_{-i} := (i, -i-1)(-i, i+1)$. For $n \geq 1$ set

$$S_{n+1} := \langle s_1, s_2, \ldots, s_n \rangle \quad \text{and} \quad W^{\mathsf{B}}_n := \langle s_0, s_1, s_2 \ldots, s_{n-1} \rangle.$$

For each $n \geq 2$ set

$$W^{\mathsf{D}}_n := \langle s_{-1}, s_1, s_2, \ldots, s_{n-1} \rangle \quad \text{where} \quad S_1 = W^{\mathsf{D}}_1 := \{1\} \subseteq S_{\mathbb{Z}}.$$

These are the finite Coxeter groups of types A_n (for $n \geq 1$), B_n (for $n \geq 2$), and D_n (for $n \geq 4$) relative to the given simple generators. Note that

$$W^{\mathsf{B}}_1 = \langle s_0 \rangle \cong S_2, \quad W^{\mathsf{D}}_2 = \langle s_{-1}, s_1 \rangle \cong S_2 \times S_2, \quad \text{and} \quad W^{\mathsf{D}}_3 = \langle s_{-1}, s_2, s_1 \rangle \cong S_4.$$

The elements of W^{B}_n are the permutations $w \in S_{\mathbb{Z}}$ with $w(-i) = -w(i)$ for all $i \in [n] := \{1, 2, \ldots, n\}$ and with $w(i) = i$ if $|i| > n$. The group S_n is the subgroup of such permutations which preserve $[n]$, and the group W^{D}_n is the subgroup of elements $w \in W_n$ with an even number of sign changes, that is, with $|\{i \in [n] : w(i) < 0\}| \equiv 0 \pmod 2$.

Assume $W \in \{S_n, W^{\mathsf{B}}_n, W^{\mathsf{D}}_n\}$ is one of these classical Weyl groups. Each element $w \in W$ is uniquely determined by its *one-line representation*, which is the word $w_1 w_2 \cdots w_n$ where $w_i = w(i)$ and where we write negative numbers $-1, -2, -3, \ldots$ as $\bar{1}, \bar{2}, \bar{3}, \ldots$, respectively. When $i \geq 0$ and $s_i \in W$, one has $\ell(ws_i) < \ell(w)$ for $w \in W$ if and only if $w_i > w_{i+1}$; when $w \in W = W^{\mathsf{D}}_n$ one has $\ell(ws_{-1}) < \ell(w)$ if and only if $-w_1 > w_2$ [6, Props. 8.1.2 and 8.2.2].

3 Model classification in type A

In this section we fix a positive integer n and consider the Coxeter group $W = S_n$ with generating set $S = \{s_1, s_2, \ldots, s_{n-1}\}$. Our main result here is Theorem 3.3.

3.1 Perfect conjugacy classes in type A

The longest element in S_n is the reverse permutation $w_0 = n \cdots 321$ and the only nontrivial Coxeter automorphism is $\mathrm{Ad}(w_0)$. Let $\mathcal{K}^{S_n}_{\mathrm{id}} := \{1\}$ and when n is even define $\mathcal{K}^{S_n}_{\mathrm{fpf}}$ to be the set of fixed-point-free involutions in S_n. Let $\mathcal{K}^{S_n}_{\mathrm{id}^+} := \{w_0^+\}$ and $\mathcal{K}^{S_n}_{\mathrm{fpf}^+} := \mathcal{K}^{S_n}_{\mathrm{fpf}} \cdot w_0^+$. The unique minimal-length elements in $\mathcal{K}^{S_n}_{\mathrm{fpf}}$ and $\mathcal{K}^{S_n}_{\mathrm{fpf}^+}$ are then

$$s_1 s_3 s_5 \cdots s_{n-1} \in S_n \quad \text{and} \quad (1, \mathrm{Ad}(w_0)) \in S^+_n.$$

The perfect conjugacy classes in S^+_n are $\mathcal{K}^{S_n}_{\mathrm{id}}$ and $\mathcal{K}^{S_n}_{\mathrm{id}^+}$, together with $\mathcal{K}^{S_n}_{\mathrm{fpf}}$ and $\mathcal{K}^{S_n}_{\mathrm{fpf}^+}$ when n is even [23, Ex. 9.2]. One has $\mathcal{K}^{S_1}_{\mathrm{id}} = \mathcal{K}^{S_1}_{\mathrm{id}^+}$ and $\mathcal{K}^{S_2}_{\mathrm{id}} = \mathcal{K}^{S_2}_{\mathrm{fpf}^+}$ and $\mathcal{K}^{S_2}_{\mathrm{id}^+} = \mathcal{K}^{S_2}_{\mathrm{fpf}}$.

3.2 Model indices in type A

Let $\mathsf{Index}(S_n)$ denote the set of 3-line arrays of the form

$$\Theta = \left[\begin{smallmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_l \\ \beta_1 & \beta_2 & \cdots & \beta_l \\ \gamma_1 & \gamma_2 & \cdots & \gamma_l \end{smallmatrix}\right]$$

where $\alpha_1, \alpha_2, \ldots, \alpha_l$ are positive integers summing to n, each β_i is a symbol in $\{\mathrm{id}, \mathrm{id}^+, \mathrm{fpf}, \mathrm{fpf}^+\}$, and each $\gamma_i \in \{\mathbb{1}, \mathrm{sgn}\}$ subject to the requirement that $\beta_i \in \{\mathrm{id}, \mathrm{id}^+\}$ when $\alpha_i \in \{1, 3, 5, 7 \ldots\}$. We refer to Θ as a *model index* for S_n. Let $\Theta = \left[\begin{smallmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_l \\ \beta_1 & \beta_2 & \cdots & \beta_l \\ \gamma_1 & \gamma_2 & \cdots & \gamma_l \end{smallmatrix}\right] \in \mathsf{Index}(S_n)$. For $i \in [l]$ define

$$J_i := \{s_j \in S_n : \alpha_1 + \cdots + \alpha_{i-1} < j < \alpha_1 + \cdots + \alpha_i\}$$

and let φ_i be the isomorphism $S_{\alpha_i} \to \langle J_i \rangle$ mapping $s_j \mapsto s_{\alpha_1+\alpha_2+\cdots+\alpha_{i-1}+j}$ for $j \in [\alpha_i - 1]$. This extends to an isomorphism $S^+_{\alpha_i} \cong \langle J_i \rangle^+$ via $(w, \theta) \mapsto (\varphi_i(w), \varphi_i \theta \varphi_i^{-1})$. Let $\mathcal{K}_i$ be the image of $\mathcal{K}^{S_{\alpha_i}}_{\beta_i}$ under φ_i. Using the notation in Section 2.7, we define a model triple

$$\mathbb{T}^\Theta := (J_1, \mathcal{K}_1, \gamma_1) \otimes (J_2, \mathcal{K}_2, \gamma_2) \otimes \cdots \otimes (J_l, \mathcal{K}_l, \gamma_l).$$

Every factorizable model triple (and therefore every multiplicity-free model triple by Theorem 2.2) for S_n arises as $\mathbb{T}^\Theta$ for some $\Theta \in \mathsf{Index}(S_n)$. This representation is almost unique. However, $\mathbb{T}^\Theta$ is unaltered by the following modifications to Θ:

- when $\alpha_i = 1$, changing $\beta_i = \mathrm{id}$ to id^+ (or vice versa) or $\gamma_i = \mathbb{1}$ to sgn (or vice versa);
- when $\alpha_i = 2$, changing $\beta_i = \mathrm{id}$ to fpf^+ (or vice versa) or $\beta_i = \mathrm{id}^+$ to fpf (or vice versa).

In view of Theorem 2.2, the character

$$\chi^\Theta_{\mathsf{A}} := \chi^{\mathbb{T}^\Theta}$$

is never multiplicity-free if Θ has more than two columns. However, it will be useful later to allow these more general indices.

Define $\Theta^* := \left[\begin{smallmatrix} \alpha_l & \cdots & \alpha_2 & \alpha_1 \\ \beta_l & \cdots & \beta_2 & \beta_1 \\ \gamma_l & \cdots & \gamma_2 & \gamma_1 \end{smallmatrix}\right]$, $\Theta^\vee := \left[\begin{smallmatrix} \alpha_l & \cdots & \alpha_2 & \alpha_1 \\ \beta_l^\vee & \cdots & \beta_2^\vee & \beta_1^\vee \\ \gamma_l & \cdots & \gamma_2 & \gamma_1 \end{smallmatrix}\right]$, and $\overline{\Theta} := \left[\begin{smallmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_l \\ \beta_1 & \beta_2 & \cdots & \beta_l \\ \bar\gamma_1 & \bar\gamma_2 & \cdots & \bar\gamma_l \end{smallmatrix}\right]$ where

$$\beta_i^\vee := \begin{cases} \mathrm{fpf} & \text{if } \beta_i = \mathrm{fpf}^+ \\ \mathrm{fpf}^+ & \text{if } \beta_i = \mathrm{fpf} \\ \mathrm{id} & \text{if } \beta_i = \mathrm{id}^+ \\ \mathrm{id}^+ & \text{if } \beta_i = \mathrm{id} \end{cases} \quad \text{and} \quad \bar\gamma_i := \gamma_i \,\mathrm{sgn} = \begin{cases} \mathbb{1} & \text{if } \gamma_i = \mathrm{sgn} \\ \mathrm{sgn} & \text{if } \gamma_i = \mathbb{1}. \end{cases}$$

It is straightforward to check that $\mathbb{T}^{\Theta^*} = (\mathbb{T}^\Theta)^{\mathrm{Ad}(w_0)}$, $\mathbb{T}^{\Theta^\vee} = (\mathbb{T}^\Theta)^\vee$, and $\mathbb{T}^{\overline{\Theta}} = \overline{\mathbb{T}^\Theta}$. Likewise, if Θ' is formed from Θ by changing any entries id^+ in the second row to id, then $\mathbb{T}^\Theta \equiv \mathbb{T}^{\Theta'}$ since $\mathcal{K}^{S_{\alpha_i}}_{\mathrm{id}}$ and $\mathcal{K}^{S_{\alpha_i}}_{\mathrm{id}^+}$ are singleton sets with the same centralizers in S_{α_i}.[2]

If $\Theta, \Psi \in \mathsf{Index}(S_n)$ then we write $\Theta \equiv \Psi$ if $\mathbb{T}^\Theta \equiv \mathbb{T}^\Psi$, $\Theta \sim \Psi$ if $\mathbb{T}^\Theta \sim \mathbb{T}^\Psi$, and $\Theta \approx \Psi$ if $\mathbb{T}^\Theta \approx \mathbb{T}^\Psi$. In the second two cases we say that Θ and Ψ are *strongly equivalent* and *equivalent*. If $\mathcal{M}$ and $\mathcal{M}'$ are sets of model indices then we write $\mathcal{M} \equiv \mathcal{M}'$, $\mathcal{M} \sim \mathcal{M}'$, or $\mathcal{M} \approx \mathcal{M}'$ if there is a bijection $\mathcal{M} \to \mathcal{M}'$ with $\Theta \equiv \Theta'$, $\Theta \sim \Theta'$, or $\Theta \approx \Theta'$, respectively, whenever $\Theta \mapsto \Theta'$.

[2] If Θ'' is formed from Θ by changing any fpf^+ entries to fpf, then $\chi^\Theta_{\mathsf{A}} = \chi^{\Theta''}_{\mathsf{A}}$ always holds but we could have $\mathbb{T}^\Theta \not\equiv \mathbb{T}^{\Theta''}$. If $(J, \mathcal{K}, \sigma) \equiv (J', \mathcal{K}', \sigma')$ then $J = J'$ and the W_J-centralizers of the minimal-length elements of $\mathcal{K}$ and $\mathcal{K}'$ must be equal. For $\mathcal{K} = \mathcal{K}^{S_{\alpha_i}}_{\mathrm{fpf}}$ and $\mathcal{K}' = \mathcal{K}^{S_{\alpha_i}}_{\mathrm{fpf}^+}$ these centralizers are conjugate but not equal.

3.3 Littlewood-Richardson coefficients in type A

The irreducible characters of S_n are indexed by partitions of n, that is, by weakly decreasing sequences of positive integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. The diagram of a partition λ is the set $\mathsf{D}_\lambda := \{(i,j) : i > 0 \text{ and } 1 \leq j \leq \lambda_i\}$. The *transpose* of λ is the unique partition $\lambda^\top$ with $\mathsf{D}_{\lambda^\top} = \{(j,i) : (i,j) \in \mathsf{D}_\lambda\}$.

We write $\lambda \vdash n$ to indicate that λ is a partition of n, and χ^λ for the irreducible character of S_n indexed by $\lambda \vdash n$ following the standard construction explained in [12, §5.4]. The linear characters of S_n are $\mathbb{1} = \chi^{(n)}$ and $\mathrm{sgn} = \chi^{(1^n)}$ where $(1^n) := (1,1,\ldots,1) \vdash n$. It is well-known that $\chi^\lambda \,\mathrm{sgn} = \chi^{\lambda^\top}$ for any $\lambda \vdash n$.

Let $p, q \in \mathbb{N}$ with $n = p+q$. We identify $S_p \times S_q$ with the subgroup $\langle s_i : p \neq i \in [n-1]\rangle \subseteq S_n$ and write $u \times v \in S_{p+q}$ for the image $(u,v) \in S_p \times S_q$ under this inclusion. Given functions $f : S_p \to \mathbb{C}$ and $g : S_q \to \mathbb{C}$ define $f \boxtimes g : S_p \times S_q \to \mathbb{C}$ to be the map sending $u \times v \mapsto f(u)g(v)$ for all $u \in S_p$ and $v \in S_q$. If f and g are class functions, then we further define

$$f \bullet_{\mathsf{A}} g = \mathrm{Ind}_{S_p \times S_q}^{S_{p+q}}(f \boxtimes g). \tag{3.1}$$

This is a commutative, associative, and bilinear operation. If $\Theta = \left[\begin{smallmatrix} \alpha_1 & \cdots & \alpha_l \\ \beta_1 & \cdots & \beta_l \\ \gamma_1 & \cdots & \gamma_l \end{smallmatrix}\right] \in \mathsf{Index}(S_n)$ then

$$\chi_{\mathsf{A}}^{\Theta} = \chi_{\mathsf{A}}^{\left[\begin{smallmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{smallmatrix}\right]} \bullet_{\mathsf{A}} \chi_{\mathsf{A}}^{\left[\begin{smallmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{smallmatrix}\right]} \bullet_{\mathsf{A}} \cdots \bullet_{\mathsf{A}} \chi_{\mathsf{A}}^{\left[\begin{smallmatrix} \alpha_l \\ \beta_l \\ \gamma_l \end{smallmatrix}\right]}. \tag{3.2}$$

Let $c^\nu_{\lambda\mu} \in \mathbb{N}$ denote the *Littlewood-Richardson coefficients* satisfying $\chi^\lambda \bullet_{\mathsf{A}} \chi^\mu = \sum_\nu c^\nu_{\lambda\mu} \chi^\nu$. The *Pieri rules* [12, Ex. 6.3] state that if $p \in \mathbb{N}$ and $\mu = (p)$ (respectively, $\mu = (1^p)$) then $c^\nu_{\lambda\mu} = 1$ if $\mathsf{D}_\nu \setminus \mathsf{D}_\lambda$ consists of p cells in distinct columns (respectively, rows) and otherwise $c^\nu_{\lambda\mu} = 0$.

3.4 Perfect models in type A

When n is even, let $\mathsf{ERows}(n)$ denote the set of partitions $\lambda \vdash n$ whose parts λ_i are all even, and let $\mathsf{ECols}(n) = \{\lambda^\top : \lambda \in \mathsf{ERows}(n)\}$ where $\lambda^\top$ is the transpose of a partition λ. Then

$$\chi_{\mathsf{A}}^{\left[\begin{smallmatrix} n \\ \mathrm{fpf} \\ \mathbb{1} \end{smallmatrix}\right]} = \chi_{\mathsf{A}}^{\left[\begin{smallmatrix} n \\ \mathrm{fpf}^+ \\ \mathbb{1} \end{smallmatrix}\right]} = \sum_{\lambda \in \mathsf{ERows}(n)} \chi^\lambda \quad\text{and}\quad \chi_{\mathsf{A}}^{\left[\begin{smallmatrix} n \\ \mathrm{fpf} \\ \mathrm{sgn} \end{smallmatrix}\right]} = \chi_{\mathsf{A}}^{\left[\begin{smallmatrix} n \\ \mathrm{fpf}^+ \\ \mathrm{sgn} \end{smallmatrix}\right]} = \sum_{\lambda \in \mathsf{ECols}(n)} \chi^\lambda \tag{3.3}$$

by [14, Lem. 1]. Fix a model index $\Theta \in \mathsf{Index}(S_n)$ and write $\mathsf{ncols}(\Theta)$ for its number of columns.

Lemma 3.1. The character χ_{A}^Θ is not multiplicity-free if $\mathsf{ncols}(\Theta) > 2$. Suppose $\Theta = \left[\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{smallmatrix}\right]$ has exactly two columns. Then χ_{A}^Θ is not multiplicity-free if any of the following holds:

(a) $\alpha_1 \in \{4,6,8,\ldots\}$, $\beta_1 \in \{\mathrm{fpf}, \mathrm{fpf}^+\}$, $\alpha_2 \geq 2$, and $\gamma_1 = \gamma_2$.

(b) $\alpha_2 \in \{4,6,8,\ldots\}$, $\beta_2 \in \{\mathrm{fpf}, \mathrm{fpf}^+\}$, $\alpha_1 \geq 2$, and $\gamma_1 = \gamma_2$.

(c) $\alpha_1, \alpha_2 \in \{4,6,8,\ldots\}$ and $\beta_1, \beta_2 \in \{\mathrm{fpf}, \mathrm{fpf}^+\}$.

Proof. Suppose $\alpha_1 \in \{4,6,8,\ldots\}$, $\beta_1 \in \{\mathrm{fpf}, \mathrm{fpf}^+\}$, and $\alpha_2 \geq 2$. Then it follows from (3.3) that $\chi_{\mathsf{A}}^{\left[\begin{smallmatrix} \alpha_1 \\ \beta_1 \\ \mathbb{1} \end{smallmatrix}\right]}$ has $\chi^{(\alpha_1)} + \chi^{(\alpha_1-2,2)}$ as a constituent and $\chi_{\mathsf{A}}^{\left[\begin{smallmatrix} \alpha_2 \\ \beta_2 \\ \mathbb{1} \end{smallmatrix}\right]}$ has $\chi^{(\alpha_2)}$ as a constituent, regardless of the value of $\beta_2 \in \{\mathrm{id}, \mathrm{id}^+, \mathrm{fpf}, \mathrm{fpf}^+\}$. But $\chi^{(\alpha_1)} \bullet_A \chi^{(\alpha_2)}$ and $\chi^{(\alpha_1-2,2)} \bullet_A \chi^{(\alpha_2)}$ both have $\chi^{(\alpha_1+\alpha_2-2,2)}$ as a constituent, so χ_{A}^Θ is not multiplicity-free when $\gamma_1 = \gamma_2 = \mathbb{1}$ by (3.2). Since $\chi_{\mathsf{A}}^\Theta = \chi_{\mathsf{A}}^{\Theta^*} = \chi_{\mathsf{A}}^{\overline{\Theta}}$, it follows that this character is not multiplicity-free in cases (a) and (b).

It remains to show that χ_{A}^Θ is not multiplicity-free when $\alpha_1, \alpha_2 \in \{4,6,8,\ldots\}$, $\beta_1, \beta_2 \in \{\mathrm{fpf}, \mathrm{fpf}^+\}$, and $\gamma_1 \neq \gamma_2$. In this case $\chi_{\mathsf{A}}^\Theta = \sum_{\lambda \in \mathsf{ERows}(p)} \sum_{\mu \in \mathsf{ECols}(q)} \sum_{\nu \vdash p+q} c^\nu_{\lambda\mu} \chi^\nu$ for some

$\{p,q\} = \{\alpha_1,\alpha_2\}$ by (3.2) and (3.3). If $p \geq q$ then for $\nu = (p, q/2, q/2)$ and $\mu = (q/2, q/2)$ we have $c^{\nu}_{\lambda\mu} = 1$ for both $\lambda = (p)$ and $\lambda = (p-2,2)$ since $\chi^{(p-2,2)} = \chi^{(p-2)} \bullet_{\mathsf{A}} \chi^{(2)} - \chi^{(p-1)} \bullet_{\mathsf{A}} \chi^{(1)}$, so $\chi^{\Theta}_{\mathsf{A}}$ is not multiplicity-free. If $p \leq q$ then we reach the same conclusion by considering $\nu = (q, p/2, p/2)^\top$, $\lambda = (p/2, p/2)^\top$, and $\mu = (q)^\top$ or $(q-2,2)^\top$. $\square$

Let $\mathsf{ORows}(n,q)$ be the set of partitions $\lambda \vdash n$ with exactly q odd parts, and define $\mathsf{OCols}(n,q) = \{\lambda^\top : \lambda \in \mathsf{ORows}(n,q)\}$.

Proposition 3.2. Suppose $\Theta \in \mathsf{Index}(S_n)$ and $\chi^{\Theta}_{\mathsf{A}}$ is multiplicity-free. Then Θ is strongly equivalent to a model index of one of the following forms:

(a) $\begin{bmatrix} n \\ \mathrm{id} \\ \sigma \end{bmatrix}$ for either linear character $\sigma \in \{\mathbb{1}, \mathrm{sgn}\}$, in which case $\chi^{\Theta}_{\mathsf{A}} = \sigma$.

(b) $\begin{bmatrix} n \\ \mathrm{fpf} \\ \mathbb{1} \end{bmatrix}$ with $n \in \{4,6,8,\dots\}$, in which case $\chi^{\Theta}_{\mathsf{A}} = \sum_{\lambda \in \mathsf{ERows}(n)} \chi^{\lambda}$.

(c) $\begin{bmatrix} n \\ \mathrm{fpf} \\ \mathrm{sgn} \end{bmatrix}$ with $n \in \{4,6,8,\dots\}$, in which case $\chi^{\Theta}_{\mathsf{A}} = \sum_{\lambda \in \mathsf{ECols}(n)} \chi^{\lambda}$.

(d) $\begin{bmatrix} k & n-k \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix}$ with $2 < k < n-2$, in which case $\chi^{\Theta}_{\mathsf{A}} = \chi^{(k+1,1^{n-k-1})} + \chi^{(k,1^{n-k})}$.

(e) $\begin{bmatrix} k & n-k \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathbb{1} \end{bmatrix}$ with $0 < k < n$, in which case $\chi^{\Theta}_{\mathsf{A}} = \sum_{j=0}^{\min(k,n-k)} \chi^{(n-j,j)}$.

(f) $\begin{bmatrix} k & n-k \\ \mathrm{id} & \mathrm{id} \\ \mathrm{sgn} & \mathrm{sgn} \end{bmatrix}$ with $0 < k < n$, in which case $\chi^{\Theta}_{\mathsf{A}} = \sum_{j=0}^{\min(k,n-k)} \chi^{(2^j,1^{n-2j})}$.

(g) $\begin{bmatrix} 2k & n-2k \\ \mathrm{fpf} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix}$ with $0 < k < n/2$, in which case $\chi^{\Theta}_{\mathsf{A}} = \sum_{\lambda \in \mathsf{ORows}(n,n-2k)} \chi^{\lambda}$.

(h) $\begin{bmatrix} 2k & n-2k \\ \mathrm{fpf} & \mathrm{id} \\ \mathrm{sgn} & \mathbb{1} \end{bmatrix}$ with $0 < k < n/2$, in which case $\chi^{\Theta}_{\mathsf{A}} = \sum_{\lambda \in \mathsf{OCols}(n,n-2k)} \chi^{\lambda}$.

Proof. Lemma 3.1 implies that Θ must have at most two columns and not more than one entry in the second row equal to fpf or fpf^+; moreover, if the second row of Θ has an entry equal to fpf or fpf^+ then the two linear characters in the third row must be distinct. As we can change any entries in the second row of Θ from id^+ to id without altering its strong equivalence class, and since $\Theta \sim \Theta^\vee \sim \Theta^*$, it follows that Θ is strongly equivalent to one of the cases listed. The reason why case (d) has $2 < k < n-2$ rather than $0 < k < n$ is because

$$\begin{bmatrix} 1 & n-1 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix} = \begin{bmatrix} 1 & n-1 \\ \mathrm{id} & \mathrm{id} \\ \mathrm{sgn} & \mathrm{sgn} \end{bmatrix}, \quad \begin{bmatrix} n-1 & 1 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix} = \begin{bmatrix} n-1 & 1 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathbb{1} \end{bmatrix}, \quad \begin{bmatrix} 2 & n-2 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix} \equiv \begin{bmatrix} 2 & n-2 \\ \mathrm{fpf} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix}, \quad \begin{bmatrix} n-2 & 2 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix} \equiv \begin{bmatrix} 2 & n-2 \\ \mathrm{fpf} & \mathrm{id} \\ \mathrm{sgn} & \mathbb{1} \end{bmatrix}.$$

The given character formulas are consequences of the Pieri rules and (3.3). $\square$

Let $\mathbb{T}^{\begin{bmatrix} n & 0 \\ \mathrm{fpf} & \mathrm{id} \\ \gamma_1 & \gamma_2 \end{bmatrix}} := \mathbb{T}^{\begin{bmatrix} n \\ \mathrm{fpf} \\ \gamma_1 \end{bmatrix}}$ when n is even and $\mathbb{T}^{\begin{bmatrix} 0 & n \\ \mathrm{fpf} & \mathrm{id} \\ \gamma_1 & \gamma_2 \end{bmatrix}} := \mathbb{T}^{\begin{bmatrix} n \\ \mathrm{id} \\ \gamma_2 \end{bmatrix}}$. When $\mathcal{X}$ is a set of model triples, let $\overline{\mathcal{X}} := \{\overline{\mathbb{T}} : \mathbb{T} \in \mathcal{X}\}$. Then $\mathcal{X} \approx \overline{\mathcal{X}}$ where $\approx$ is the equivalence relation from Section 2.5.

Theorem 3.3. The following set is a perfect model for S_n:

$$\mathcal{P}^{\mathsf{A}}_{n-1} := \left\{ \mathbb{T}^{\Theta} : \Theta = \begin{bmatrix} 2k & n-2k \\ \mathrm{fpf} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix} \text{ for } 0 \leq k \leq \lfloor \tfrac{n}{2} \rfloor \right\}.$$

If $n \notin \{2,4\}$ then each perfect model for S_n is strongly equivalent to $\mathcal{P}^{\mathsf{A}}_{n-1}$ or $\overline{\mathcal{P}^{\mathsf{A}}_{n-1}}$.

Proof. The claim that $\mathcal{P}^{\mathsf{A}}_{n-1}$ is a perfect model is well-known [14], or can be seen as an immediate consequence of Proposition 3.2. The difficult part of the theorem is showing that every perfect model is strongly equivalent to $\mathcal{P}^{\mathsf{A}}_{n-1}$ or $\overline{\mathcal{P}^{\mathsf{A}}_{n-1}}$ when $n \notin \{2,4\}$.

Suppose $\mathcal{M}$ is a set of model indices $\Theta \in \mathsf{Index}(S_n)$ such that $\{\mathbb{T}^\Theta : \Theta \in \mathcal{M}\}$ is a perfect model for S_n. Every perfect model for S_n arises in this way. After replacing $\mathcal{M}$ by a strongly equivalent set of indices, and updating our notation for model indices to allow zeros in the first row, we may assume by Proposition 3.2 that every $\Theta \in \mathcal{M}$ has the form

$$\Theta^H_k = \left[\begin{smallmatrix} k & n-k \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{smallmatrix}\right], \quad \Theta^{\mathbb{1}}_l = \left[\begin{smallmatrix} l & n-l \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathbb{1} \end{smallmatrix}\right], \quad \Theta^{\mathrm{sgn}}_l = \left[\begin{smallmatrix} l & n-l \\ \mathrm{id} & \mathrm{id} \\ \mathrm{sgn} & \mathrm{sgn} \end{smallmatrix}\right], \quad \Theta^{\mathsf{OR}}_m = \left[\begin{smallmatrix} n-m & m \\ \mathrm{fpf} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{smallmatrix}\right], \text{ or } \Theta^{\mathsf{OC}}_m = \left[\begin{smallmatrix} n-m & m \\ \mathrm{fpf} & \mathrm{id} \\ \mathrm{sgn} & \mathbb{1} \end{smallmatrix}\right]$$

where $2 < k < n-2$, $0 < l < n$, and $m \in \Delta := \{0 \leq j \leq n : j \equiv n \pmod 2\}$.[3] We now argue that the only possibility for $\mathcal{M}$ is $\mathcal{X}^{\mathsf{OR}} := \{\Theta^{\mathsf{OR}}_m : m \in \Delta\}$ or $\mathcal{X}^{\mathsf{OC}} := \{\Theta^{\mathsf{OC}}_m : m \in \Delta\}$. For small values of n this claim can be checked by a short computer calculation using Proposition 3.2. In the following argument we assume $n > 10$.

If μ and λ are partitions with $\mathsf{D}_\mu \subseteq \mathsf{D}_\lambda$ then we write $\mu \subseteq \lambda$. To simplify our notation, we say that "λ is a constituent of Θ" as an abbreviation for "χ^λ is a constituent of χ^Θ." With this convention, every $\lambda \vdash n$ is a constituent of exactly one $\Theta \in \mathcal{M}$, and the constituents of Θ^{OR}_m and Θ^{OC}_m are the partitions whose diagrams have m odd rows or m odd columns, respectively. The formulas in Proposition 3.2 tell us that no constituents λ of Θ^H_k, $\Theta^{\mathbb{1}}_l$, or Θ^{sgn}_l have $(3,2,1) \subseteq \lambda$. Thus, $\mathcal{M}$ must share at least one element with $\mathcal{X}^{\mathsf{OR}}$ or $\mathcal{X}^{\mathsf{OC}}$. We argue below that in fact, exactly one of the intersections $\mathcal{M} \cap \mathcal{X}^{\mathsf{OR}}$ or $\mathcal{M} \cap \mathcal{X}^{\mathsf{OC}}$ is nonempty:

- If $n = 4a+1 \equiv 1 \pmod 4$ then the partition $\lambda := (2a, 2a, 1) \vdash n$ contains $(3,2,1)$ and has one odd row and one odd column, so either $\Theta^{\mathsf{OR}}_1 \in \mathcal{M}$ or $\Theta^{\mathsf{OC}}_1 \in \mathcal{M}$. By considering the partitions of the form $(p,q) \vdash n$ and their transposes, one finds that there are partitions of n with 1 odd row and m odd columns, or with 1 odd column and m odd rows, for every $m \in \Delta$. Thus if $\Theta^{\mathsf{OR}}_1 \in \mathcal{M}$ then $\mathcal{M} \cap \mathcal{X}^{\mathsf{OC}} = \varnothing$ and if $\Theta^{\mathsf{OC}}_1 \in \mathcal{M}$ then $\mathcal{M} \cap \mathcal{X}^{\mathsf{OR}} = \varnothing$.
- If $n = 4a+3 \equiv 3 \pmod 4$ then $\lambda := (2a, 2a, 3) \vdash n$ contains $(3,2,1)$ and has 1 odd row and 3 odd columns, so $\Theta^{\mathsf{OR}}_1 \in \mathcal{M}$ or $\Theta^{\mathsf{OC}}_3 \in \mathcal{M}$. If $\Theta^{\mathsf{OR}}_1 \in \mathcal{M}$ then it follows as in the previous case that $\mathcal{M} \cap \mathcal{X}^{\mathsf{OC}} = \varnothing$. Assume $\Theta^{\mathsf{OC}}_3 \in \mathcal{M}$. Since $(2a, 2a-1, 3, 1) \vdash n$ has 3 odd rows and 3 odd columns, we must have $\Theta^{\mathsf{OR}}_3 \notin \mathcal{M}$. But $\lambda^\top = (3,3,3,2^{2a-3})$ has 3 odd rows and 1 odd column, so $\Theta^{\mathsf{OC}}_1 \in \mathcal{M}$, which implies that $\mathcal{M} \cap \mathcal{X}^{\mathsf{OR}} = \varnothing$ as in the previous case.
- If $n = 2a+2$ is even then $\lambda := (a+1, a, 1) \vdash n$ contains $(3,2,1)$ and has 2 odd rows and 2 odd columns, so either $\Theta^{\mathsf{OR}}_2 \in \mathcal{M}$ or $\Theta^{\mathsf{OC}}_2 \in \mathcal{M}$. By considering the partitions of the form $(p,q,r) \vdash n$ and their transposes, one finds that there are partitions of n with 2 odd rows and m odd columns, or with 2 odd columns and m odd rows, for every $m \in \Delta \setminus \{n\}$. Suppose $\Theta^{\mathsf{OR}}_2 \in \mathcal{M}$. Then $\mathcal{M}$ contains no elements of $\mathcal{X}^{\mathsf{OC}}$ except possibly Θ^{OC}_n, whose unique constituent (1^n) has zero odd rows. The partition $\mu := (n-4, 2, 2) \vdash n$ has zero odd rows and $n-4$ odd columns where $(3,2,1) \subseteq \mu$. Since $\Theta^{\mathsf{OC}}_{n-4} \notin \mathcal{M}$ we must have $\Theta^{\mathsf{OR}}_0 \in \mathcal{M}$, so $\mathcal{M} \cap \mathcal{X}^{\mathsf{OC}} = \varnothing$. A symmetric argument shows that if $\Theta^{\mathsf{OR}}_2 \in \mathcal{M}$ then $\mathcal{M} \cap \mathcal{X}^{\mathsf{OR}}$ is empty.

This completes the proof of the claim.

The claim shows that we are in one of two cases. The first case is that $\mathcal{M} \cap \mathcal{X}^{\mathsf{OR}}$ is nonempty and $\mathcal{M} \cap \mathcal{X}^{\mathsf{OC}} = \varnothing$. The partitions of the form $(3+a, 2, 1, 1^{n-a-6})$ for $0 \leq a \leq n-6$ and $(n-4, 2, 2)$ all contain $(3,2,1)$ so must appear as constituents of elements of $\mathcal{M} \cap \mathcal{X}^{\mathsf{OR}}$. The number of odd rows in these partitions range over all $m \equiv n \pmod 2$ with $0 \leq m \leq n-4$, so $\mathcal{X}^{\mathsf{OR}} \setminus \{\Theta^{\mathsf{OR}}_{n-2}, \Theta^{\mathsf{OR}}_n\}$ must be a subset of $\mathcal{M}$. The only partitions of n not appearing as constituents of this subset are (1^n), $(2, 1^{n-1})$, and $(3, 1^{n-3})$.

The remaining models indices that could be in $\mathcal{M}$ are $\Theta^{\mathsf{OR}}_{n-2}$, Θ^{OR}_n, Θ^H_k for $2 < k < n-2$, $\Theta^{\mathbb{1}}_l$ for $0 < l < n$, or Θ^{sgn}_l for $0 < l < n$. Among these, the only ones containing $(3, 1^{n-3})$ as a constituent are $\Theta^{\mathsf{OR}}_{n-2}$ and Θ^H_3. Since the latter index shares the constituent $(4, 1^{n-4})$ with $\Theta^{\mathsf{OR}}_{n-4} \in \mathcal{M}$, we must have $\Theta^{\mathsf{OR}}_{n-2} \in \mathcal{M}$. But now the only partition of n not accounted for as a constituent of some $\Theta \in \mathcal{M}$ is (1^n). The only index that could be in $\mathcal{M}$ that has

[3] We can restrict $0 < l < n$ since $\Theta^{\mathbb{1}}_0 \equiv \Theta^{\mathbb{1}}_n \equiv \Theta^{\mathsf{OC}}_n$ and $\Theta^{\mathrm{sgn}}_0 \equiv \Theta^{\mathrm{sgn}}_n \equiv \Theta^{\mathsf{OR}}_n$.

(1^n) as its unique constituent is Θ^{OR}_n, so we conclude that $\mathcal{M} \supseteq \{\Theta^{\mathsf{OR}}_m : m \in \Delta\}$ whence $\mathcal{M} = \{\Theta^{\mathsf{OR}}_m : m \in \Delta\} = \mathcal{P}^{\mathsf{A}}_{n-1}$.

The other case that could arise is to have $\mathcal{M} \cap \mathcal{X}^{\mathsf{OC}} \neq \varnothing$ and $\mathcal{M} \cap \mathcal{X}^{\mathsf{OR}} = \varnothing$. But then $\overline{\mathcal{M}}$ belongs to the case just considered so must be equal to $\mathcal{P}^{\mathsf{A}}_{n-1}$, which means $\mathcal{M} = \overline{\mathcal{P}^{\mathsf{A}}_{n-1}}$. □

Example 3.4. When $n = 2$ there is one other perfect model consisting of just $\{(\varnothing, \{1\}, \mathbb{1})\}$. When $n = 4$ there is a single additional equivalence of perfect models for S_n, which contains

$$\left\{\mathbb{T}^\Theta : \Theta = \begin{bmatrix} 1 & 3 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 2 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathbb{1} \end{bmatrix}\right\} \approx \left\{\mathbb{T}^\Theta : \Theta = \begin{bmatrix} 1 & 3 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathbb{1} \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 2 \\ \mathrm{id} & \mathrm{id} \\ \mathrm{sgn} & \mathrm{sgn} \end{bmatrix}\right\}.$$

These models belong to a family of type D constructions, which arise here because $S_4 \cong W^{\mathsf{D}}_3$.

Remark 3.5. Assume $n \geq 5$. Theorem 3.3 tells us that if $\mathcal{M}$ is a perfect model for S_n then $\{\chi^{\mathbb{T}} : \mathbb{T} \in \mathcal{M}\}$ contains exactly one of $\mathbb{1}$ or sgn. We refer to $\mathcal{M}$ as a *$\mathbb{1}$-model* if $\mathbb{1} \in \{\chi^{\mathbb{T}} : \mathbb{T} \in \mathcal{M}\}$ and as a *sgn-model* if $\mathrm{sgn} \in \{\chi^{\mathbb{T}} : \mathbb{T} \in \mathcal{M}\}$. One can determine whether $\mathcal{M}$ is a $\mathbb{1}$-model or a sgn-model by examining whether

$$\sum_{\lambda \in \mathsf{OCols}(n, n-2\lfloor \frac{n}{2} \rfloor)} \chi^\lambda \in \{\chi^{\mathbb{T}} : \mathbb{T} \in \mathcal{M}\} \quad \text{or} \quad \sum_{\lambda \in \mathsf{ORows}(n, n-2\lfloor \frac{n}{2} \rfloor)} \chi^\lambda \in \{\chi^{\mathbb{T}} : \mathbb{T} \in \mathcal{M}\}.$$

It will be useful to enumerate all the ways of specifying a perfect model for S_n, since there is some redundancy in our notation. To construct a sgn-model for S_n, first choose Φ_0 to be one of

$$\begin{bmatrix} n \\ \mathrm{id} \\ \mathrm{sgn} \end{bmatrix} \text{ or } \begin{bmatrix} n \\ \mathrm{id}^+ \\ \mathrm{sgn} \end{bmatrix} \tag{3.4}$$

Then let Φ_1 be one of

$$\begin{bmatrix} 2 & n-2 \\ \star & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix}, \quad \begin{bmatrix} 2 & n-2 \\ \star & \mathrm{id}^+ \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix}, \quad \begin{bmatrix} n-2 & 2 \\ \mathrm{id} & \star \\ \mathrm{sgn} & \mathbb{1} \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} n-2 & 2 \\ \mathrm{id}^+ & \star \\ \mathrm{sgn} & \mathbb{1} \end{bmatrix} \tag{3.5}$$

where each $\star$ is any symbol $\{\mathrm{id}, \mathrm{id}^+, \mathrm{fpf}, \mathrm{fpf}^+\}$. For $2 \leq k \leq \lceil n/2 \rceil - 2$ define Φ_k to be one of

$$\begin{bmatrix} 2k & n-2k \\ \mathrm{fpf}^\star & \mathrm{id}^\star \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix} \text{ or } \begin{bmatrix} n-2k & 2k \\ \mathrm{id}^\star & \mathrm{fpf}^\star \\ \mathrm{sgn} & \mathbb{1} \end{bmatrix} \tag{3.6}$$

where $\mathrm{fpf}^\star \in \{\mathrm{fpf}, \mathrm{fpf}^+\}$ and $\mathrm{id}^\star \in \{\mathrm{id}, \mathrm{id}^+\}$ are arbitrary. When n is even, let $\Phi_{n/2-1}$ be one of

$$\begin{bmatrix} n-2 & 2 \\ \mathrm{fpf} & \star \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix}, \quad \begin{bmatrix} n-2 & 2 \\ \mathrm{fpf}^+ & \star \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix}, \quad \begin{bmatrix} 2 & n-2 \\ \star & \mathrm{fpf} \\ \mathrm{sgn} & \mathbb{1} \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 2 & n-2 \\ \star & \mathrm{fpf}^+ \\ \mathrm{sgn} & \mathbb{1} \end{bmatrix} \tag{3.7}$$

where each $\star$ is any symbol $\{\mathrm{id}, \mathrm{id}^+, \mathrm{fpf}, \mathrm{fpf}^+\}$, and let $\Phi_{n/2}$ be one of

$$\begin{bmatrix} n \\ \mathrm{fpf} \\ \mathbb{1} \end{bmatrix} \text{ or } \begin{bmatrix} n \\ \mathrm{fpf}^+ \\ \mathbb{1} \end{bmatrix}. \tag{3.8}$$

Finally, when n is odd choose $\Phi_{(n-1)/2}$ to be one of

$$\begin{bmatrix} n-1 & 1 \\ \mathrm{fpf}^\star & \mathrm{id}^\star \\ \mathbb{1} & \mathbb{1} \end{bmatrix}, \quad \begin{bmatrix} n-1 & 1 \\ \mathrm{fpf}^\star & \mathrm{id}^\star \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix}, \quad \begin{bmatrix} 1 & n-1 \\ \mathrm{id}^\star & \mathrm{fpf}^\star \\ \mathbb{1} & \mathbb{1} \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 1 & n-1 \\ \mathrm{id}^\star & \mathrm{fpf}^\star \\ \mathrm{sgn} & \mathbb{1} \end{bmatrix} \tag{3.9}$$

where $\mathrm{fpf}^\star \in \{\mathrm{fpf}, \mathrm{fpf}^+\}$ and $\mathrm{id}^\star \in \{\mathrm{id}, \mathrm{id}^+\}$ are arbitrary. Then $\mathcal{M} = \{\mathbb{T}^{\Phi_k} : k = 0, 1, \dots \lfloor n/2 \rfloor\}$ is a sgn-model for S_n and every sgn-model arises in this way. The $\mathbb{1}$-models for S_n are constructed in the same way after interchanging "$\mathbb{1}$" and "sgn" in (3.4)–(3.9).

4 Model classification in type B

In this section we fix an integer $n \geq 2$ and consider the Coxeter group $W = W_n^{\mathsf{B}}$ with generating set $S = \{s_0, s_1, \dots, s_{n-1}\}$. Recall that W_n^{B} is the subgroup of permutations $w \in S_{\mathbb{Z}}$ with $w(-i) = -w(i)$ for all $i \in \mathbb{Z}$ and $w(i) = i$ for all $i > n$. Our main result here is Theorem 4.5.

4.1 Perfect conjugacy classes in type B

The longest element in W_n^{B} is the central element $w_0 = \bar{1}\bar{2}\bar{3}\cdots\bar{n}$. If $n = 2$ then there is a single nontrivial Coxeter automorphism (interchanging s_0 and s_1) and otherwise there are no such automorphisms. Given integers $p, q > 0$ with $p + q = n$, let $\mathcal{K}_{(p,q)}^{W_n^{\mathsf{B}}}$ be the set of $w \in W_n^{\mathsf{B}}$ with

$$|\{i \in [n] : w(i) = i\}| = p \quad \text{and} \quad |\{i \in [n] : w(i) = -i\}| = q.$$

Let $\mathcal{K}_{\mathrm{id}}^{W_n^{\mathsf{B}}} := \{1\}$ and $\mathcal{K}_{\mathrm{id}^+}^{W_n^{\mathsf{B}}} := \{w_0\} = \{w_0^+\}$. When n is even define $\mathcal{K}_{\mathrm{fpf}}^{W_n^{\mathsf{B}}}$ to be the set of involutions $z = z^{-1} \in W_n$ with $|z(i)| \neq i$ for all $i \in [n]$. The perfect conjugacy classes in $(W_n^{\mathsf{B}})^+$ consist of $\mathcal{K}_{\mathrm{id}}^{W_n^{\mathsf{B}}}$, $\mathcal{K}_{\mathrm{id}^+}^{W_n^{\mathsf{B}}}$, and $\mathcal{K}_{(p,q)}^{W_n^{\mathsf{B}}}$ for all $p, q > 0$ with $p + q = n$, along with $\mathcal{K}_{\mathrm{fpf}}^{W_n^{\mathsf{B}}}$ when n is even [23, Ex. 9.2]. The unique minimal-length elements of $\mathcal{K}_{(p,q)}^{W_n^{\mathsf{B}}}$ and $\mathcal{K}_{\mathrm{fpf}}^{W_n^{\mathsf{B}}}$ (when n is even) are

$$\bar{1}\bar{2}\cdots\bar{q}(q+1)(q+2)\cdots n \quad \text{and} \quad s_1 s_3 s_5 \cdots s_{n-1}.$$

4.2 Model indices in type B

The linear characters of W_n^{B} are given as follows. Let $\mathbb{1}_{++} := \mathbb{1}$ be the trivial character and let $\mathbb{1}_{--} := \mathrm{sgn}$ be the sign character. Define $\mathbb{1}_{+-}$ to be the linear character of W_n^{B} mapping $s_0 \mapsto 1$ and $s_i \mapsto -1$ for $i > 0$. Define $\mathbb{1}_{-+} := \mathbb{1}_{+-}\,\mathrm{sgn}$ to be the linear character of W_n^{B} mapping $s_0 \mapsto -1$ and $s_i \mapsto 1$ for $i > 0$. If $n = 1$ then $\mathbb{1}_{+-} = \mathbb{1}$ and $\mathbb{1}_{-+} = \mathrm{sgn}$. If $n \geq 2$ then these four linear characters are distinct. If n is even and $z = s_1 s_3 s_5 \cdots s_{n-1} \in \mathcal{K}_{\mathrm{fpf}}^{W_n^{\mathsf{B}}}$ then the centralizer subgroup $C_{\mathrm{fpf}} := \{w \in W_n^{\mathsf{B}} : wz = zw\}$ does not contain s_0 so

$$\mathrm{Res}_{C_{\mathrm{fpf}}}^{W_n^{\mathsf{B}}}(\mathbb{1}_{-+}) = \mathrm{Res}_{C_{\mathrm{fpf}}}^{W_n^{\mathsf{B}}}(\mathbb{1}) \quad \text{and} \quad \mathrm{Res}_{C_{\mathrm{fpf}}}^{W_n^{\mathsf{B}}}(\mathbb{1}_{+-}) = \mathrm{Res}_{C_{\mathrm{fpf}}}^{W_n^{\mathsf{B}}}(\mathrm{sgn}).$$

Let $\mathsf{Index}(W_n^{\mathsf{B}})$ denote the set of 3×2 arrays, to be called *model indices* for W_n^{B}, of the form

$$\Theta = \left[\begin{smallmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{smallmatrix}\right]$$

where $\alpha_0, \alpha_1 \geq 0$ are integers with $\alpha_0 + \alpha_1 = n$; β_0 is a symbol in $\{\mathrm{id}, \mathrm{id}^+, \mathrm{fpf}\}$ or a pair of positive integers (p, q) with $p + q = n$; β_1 is a symbol in $\{\mathrm{id}, \mathrm{id}^+, \mathrm{fpf}, \mathrm{fpf}^+\}$; and $\gamma_0 \in \{\mathbb{1}, \mathrm{sgn}, \mathbb{1}_{+-}, \mathbb{1}_{-+}\}$ and $\gamma_1 \in \{\mathbb{1}, \mathrm{sgn}\}$ are linear characters of $W_{\alpha_0}^{\mathsf{B}}$ and S_{α_1}. We further require that:

- if $\beta_0 = \mathrm{fpf}$ then $\alpha_0 \in \{0, 2, 4, 6, \dots\}$ and if $\beta_1 \in \{\mathrm{fpf}, \mathrm{fpf}^+\}$ then $\alpha_1 \in \{4, 6, 8 \dots\}$;
- if $\alpha_0 \leq 1$ or $\beta_0 = \mathrm{fpf}$ then $\gamma_0 \in \{\mathbb{1}, \mathrm{sgn}\}$.

Suppose $\Theta = \left[\begin{smallmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{smallmatrix}\right] \in \mathsf{Index}(W_n^{\mathsf{B}})$. Let $J_0 := \{s_{j-1} : j \in [\alpha_0]\}$ and $J_1 := \{s_j : \alpha_0 < j < n\}$. Write $\varphi_1 : S_{\alpha_1}^+ \to \langle J_1 \rangle^+$ for the isomorphism sending $s_j \mapsto s_{\alpha_0 + j}$ for $j \in [\alpha_1 - 1]$. Let $\mathcal{K}_0 := \mathcal{K}_{\beta_0}^{W_{\alpha_0}^{\mathsf{B}}}$ and let $\mathcal{K}_1$ be the image of $\mathcal{K}_{\beta_1}^{S_{\alpha_1}}$ under φ_1. Now set

$$\mathbb{T}^{\Theta} := \begin{cases} (J_0, \mathcal{K}_0, \gamma_0) & \text{if } \alpha_0 = n \\ (J_1, \mathcal{K}_1, \gamma_1) & \text{if } \alpha_1 = n \\ (J_0, \mathcal{K}_0, \gamma_0) \otimes (J_1, \mathcal{K}_1, \gamma_1) & \text{otherwise.} \end{cases}$$

In this way Θ indexes a factorizable model triple for W^{B}_n. Also define

$$\chi^\Theta_{\mathsf{B}} := \chi^{\mathbb{T}^\Theta}.$$

Note that if $\alpha_i = 0$ then $\mathbb{T}^\Theta$ does not depend on β_i or γ_i. Moreover, if $\alpha_1 = 1$ then $\mathbb{T}^\Theta$ is unaffected by changing $\beta_1 = \mathrm{id}$ to id^+ (or vice versa) or $\gamma_1 = 1\!\!1$ to sgn (or vice versa). By Theorems 1.6 and 2.2, every multiplicity-free model triple for W^{B}_n with $n \geq 2$ arises as $\mathbb{T}^\Theta$ for some $\Theta \in \mathsf{Index}(W^{\mathsf{B}}_n)$.[4]

Given $\Theta = \left[\begin{smallmatrix}\alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1\end{smallmatrix}\right] \in \mathsf{Index}(W^{\mathsf{B}}_n)$, define $\Theta^\vee := \left[\begin{smallmatrix}\alpha_0 & \alpha_1 \\ \beta_0^\vee & \beta_1^\vee \\ \gamma_0 & \gamma_1\end{smallmatrix}\right]$ and $\overline{\Theta} := \left[\begin{smallmatrix}\alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \bar\gamma_0 & \bar\gamma_1\end{smallmatrix}\right]$ where

$$\beta_i^\vee := \begin{cases} (q,p) & \text{if } i = 0 \text{ and } \beta_0 = (p,q) \\ \mathrm{fpf} & \text{if } i = 0 \text{ and } \beta_0 = \mathrm{fpf}, \text{ or if } \beta_i = \mathrm{fpf}^+ \\ \mathrm{fpf}^+ & \text{if } i = 1 \text{ and } \beta_i = \mathrm{fpf} \\ \mathrm{id} & \text{if } \beta_i = \mathrm{id}^+ \\ \mathrm{id}^+ & \text{if } \beta_i = \mathrm{id} \end{cases} \qquad \text{and} \quad \bar\gamma_i := \mathrm{sgn}\,\gamma_i.$$

We adapt the relations $\equiv$, $\sim$, and $\approx$ to (sets of) model indices in $\mathsf{Index}(W^{\mathsf{B}}_n)$ just as we did for elements of $\mathsf{Index}(S_n)$. It is straightforward to check that $\mathbb{T}^{\Theta^\vee} = (\mathbb{T}^\Theta)^\vee$ and $\mathbb{T}^{\overline{\Theta}} = \overline{\mathbb{T}^\Theta}$. Likewise, if Θ' is formed from Θ by changing any entries equal to id^+ to id, then $\Theta \equiv \Theta'$.

4.3 Littlewood-Richardson coefficients in type B

The irreducible characters of W^{B}_n are indexed by *bipartitions* of n, that is, by ordered pairs of partitions (λ,μ) with $|\lambda| + |\mu| = n$. To indicate that (λ,μ) is a bipartition of n we write $(\lambda,\mu) \vdash n$. Let $\chi^{(\lambda,\mu)}$ denote the irreducible character of W^{B}_n indexed by $(\lambda,\mu) \vdash n$ following the construction given in [12, §5.5]. Then, as explained before [12, Lem. 5.5.5],

$$\begin{aligned} 1\!\!1 = 1\!\!1_{++} &= \chi^{((n),\emptyset)} \\ 1\!\!1_{-+} &= \chi^{(\emptyset,(n))} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{sgn} = 1\!\!1_{--} &= \chi^{(\emptyset,(1,1,\dots,1))} \\ 1\!\!1_{+-} &= \chi^{((1,1,\dots,1),\emptyset)}. \end{aligned} \tag{4.1}$$

By [12, Thm. 5.5.6] one also has

$$\chi^{(\lambda,\mu)}\,\mathrm{sgn} = \chi^{(\mu^\top,\lambda^\top)}, \quad \chi^{(\lambda,\mu)}\,1\!\!1_{-+} = \chi^{(\mu,\lambda)}, \quad \text{and} \quad \chi^{(\lambda,\mu)}\,1\!\!1_{+-} = \chi^{(\lambda^\top,\mu^\top)}. \tag{4.2}$$

Let $p, q \in \mathbb{N}$. We identify $W^{\mathsf{B}}_p \times W^{\mathsf{B}}_q$ in the usual way (see [12, §5.5]) with the subgroup of permutations in W^{B}_{p+q} fixing $\{\pm1, \pm2, \dots, \pm p\}$ and $\{\pm(p+1), \pm(p+2), \dots, \pm(p+q)\}$. Write $u \times v$ for the image of $(u,v) \in W^{\mathsf{B}}_p \times W^{\mathsf{B}}_q$ in W^{B}_{p+q}. Given $f : W^{\mathsf{B}}_p \to \mathbb{C}$ and $g : W^{\mathsf{B}}_q \to \mathbb{C}$ define $f \boxtimes g : W^{\mathsf{B}}_p \times W^{\mathsf{B}}_q \to \mathbb{C}$ by the formula $u \times v \mapsto f(u)g(v)$. When f and g are class functions, let

$$f \bullet_{\mathsf{B}} g := \mathrm{Ind}_{W^{\mathsf{B}}_p \times W^{\mathsf{B}}_q}^{W^{\mathsf{B}}_{p+q}}(f \boxtimes g). \tag{4.3}$$

This is an associative, commutative, and bilinear operation. If $\Theta = \left[\begin{smallmatrix}\alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1\end{smallmatrix}\right] \in \mathsf{Index}(W^{\mathsf{B}}_n)$ then

$$\chi^\Theta_{\mathsf{B}} = \chi_{\mathsf{B}}^{\left[\begin{smallmatrix}\alpha_0 \\ \beta_0 \\ \gamma_0\end{smallmatrix}\right]} \bullet_{\mathsf{B}} \mathrm{Ind}_{S_{\alpha_1}}^{W^{\mathsf{B}}_{\alpha_1}}\left(\chi_{\mathsf{A}}^{\left[\begin{smallmatrix}\alpha_1 \\ \beta_1 \\ \gamma_1\end{smallmatrix}\right]}\right) \quad \text{where we define } \chi_{\mathsf{B}}^{\left[\begin{smallmatrix}\alpha_0 \\ \beta_0 \\ \gamma_0\end{smallmatrix}\right]} := \chi_{\mathsf{B}}^{\left[\begin{smallmatrix}\alpha_0 & 0 \\ \beta_0 & \mathrm{id} \\ \gamma_0 & 1\!\!1\end{smallmatrix}\right]} \tag{4.4}$$

and where we interpret the characters on the right as the trivial characters of $W^{\mathsf{B}}_0 := \{1\}$ or $S_0 := \{1\}$ when $\alpha_0 = 0$ or $\alpha_1 = 0$.

[4] This is true, even with our restrictions on γ_0 when $\beta_0 = \mathrm{fpf}$, because we do not distinguish between model triples $(J, \mathcal{K}, \sigma_1)$ and $(J, \mathcal{K}, \sigma_2)$ with $\mathrm{Res}^{W_J}_{C_J(z)}(\sigma_1) = \mathrm{Res}^{W_J}_{C_J(z)}(\sigma_2)$.

If (λ_1,λ_2) and (μ_1,μ_2) are bipartitions then by [12, Lem. 6.1.3] we have

$$\chi^{(\lambda_1,\lambda_2)} \bullet_{\mathsf{B}} \chi^{(\mu_1,\mu_2)} = \sum_{\nu_1,\nu_2} c^{\nu_1}_{\lambda_1\mu_1} c^{\nu_2}_{\lambda_2\mu_2} \chi^{(\nu_1,\nu_2)} \tag{4.5}$$

where $c^\nu_{\lambda\mu} \in \mathbb{N}$ are the Littlewood-Richard coefficients from Section 3.3. In addition, one has

$$\mathrm{Ind}_{S_n}^{W_n^{\mathsf{B}}}(\chi^\nu) = \sum_{\lambda,\mu} c^\nu_{\lambda\mu}\chi^{(\lambda,\mu)} \tag{4.6}$$

by [12, Lem. 6.1.4]. Thus, the Pieri rules for S_n imply that

$$\mathrm{Ind}_{S_n}^{W_n^{\mathsf{B}}}(\mathbb{1}) = \sum_{p+q=n} \chi^{((p),(q))} \quad\text{and}\quad \mathrm{Ind}_{S_n}^{W_n^{\mathsf{B}}}(\mathrm{sgn}) = \sum_{p+q=n} \chi^{((1^p),(1^q))}. \tag{4.7}$$

Let $\mathsf{ERows}_B(n)$ be the set of bipartitions $(\lambda,\mu) \vdash n$ where λ and μ both have all even parts. Define $\mathsf{ECols}_B(n) = \{(\lambda^\top,\mu^\top) : (\lambda,\mu) \in \mathsf{ERows}_B(n)\}$. If n is even then

$$\chi_{\mathsf{B}}^{\left[\begin{smallmatrix} n \\ \mathrm{fpf} \\ \mathbb{1}\end{smallmatrix}\right]} = \sum_{(\lambda,\mu)\in\mathsf{ERows}_B(n)} \chi^{(\lambda,\mu)} \quad\text{and}\quad \chi_{\mathsf{B}}^{\left[\begin{smallmatrix} n \\ \mathrm{fpf} \\ \mathrm{sgn}\end{smallmatrix}\right]} = \sum_{(\lambda,\mu)\in\mathsf{ECols}_B(n)} \chi^{(\lambda,\mu)} \tag{4.8}$$

by [4, Prop. 1]. We note one other character formula for use in the next section:

Proposition 4.1. Suppose $p,q>0$ are integers with $p+q=n$. Define $\Lambda = \Lambda(p,q)$ to be the set of partitions of the form $\lambda = (\max\{p,q\}+r, \min\{p,q\}-r)$ for $0 \le r \le \min\{p,q\}$. Then

$$\chi_{\mathsf{B}}^{\left[\begin{smallmatrix} n \\ (p,q) \\ \mathbb{1}\end{smallmatrix}\right]} = \sum_{\lambda\in\Lambda}\chi^{(\lambda,\emptyset)}, \quad \chi_{\mathsf{B}}^{\left[\begin{smallmatrix} n \\ (p,q) \\ \mathbb{1}_{-+}\end{smallmatrix}\right]} = \sum_{\lambda\in\Lambda}\chi^{(\emptyset,\lambda)}, \quad \chi_{\mathsf{B}}^{\left[\begin{smallmatrix} n \\ (p,q) \\ \mathrm{sgn}\end{smallmatrix}\right]} = \sum_{\lambda\in\Lambda}\chi^{(\emptyset,\lambda^\top)}, \quad \chi_{\mathsf{B}}^{\left[\begin{smallmatrix} n \\ (p,q) \\ \mathbb{1}_{-+}\end{smallmatrix}\right]} = \sum_{\lambda\in\Lambda}\chi^{(\lambda^\top,\emptyset)}.$$

Proof. The centralizer of $\bar{1}\bar{2}\cdots\bar{q}(q+1)(q+2)\cdots n \in \mathcal{K}^{W_n^{\mathsf{B}}}_{(p,q)}$ is $W_q^{\mathsf{B}} \times W_p^{\mathsf{B}}$ and the restriction of any linear character σ of W_n^{B} to this subgroup is $\sigma \boxtimes \sigma$. The result then follows from (4.5). $\square$

4.4 Model projections in type B

We define two maps $\pi_{\mathsf{L}}, \pi_{\mathsf{R}} : \mathsf{Index}(W_n^{\mathsf{B}}) \to \mathsf{Index}(S_n) \sqcup \{0\}$. Let $\Theta = \left[\begin{smallmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{smallmatrix}\right] \in \mathsf{Index}(W_n^{\mathsf{B}})$. Let $(\lambda,\mu) \in \{((n),\emptyset), (\emptyset,(n)), ((1^n),\emptyset), (\emptyset,(1^n))\}$ be such that $\gamma_0 = \chi^{(\lambda,\mu)}$. If $\alpha_0 = 0$ then set

$$\pi_{\mathsf{L}}(\Theta) = \pi_{\mathsf{R}}(\Theta) := \left[\begin{smallmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{smallmatrix}\right].$$

Assume $\alpha_0, \alpha_1 > 0$. If $\alpha_0 \in \{2,4,6,\dots\}$ and $\beta_0 = \mathrm{fpf}$, then $\gamma_0 \in \{\mathbb{1}, \mathrm{sgn}\}$ and we set

$$\pi_{\mathsf{L}}(\Theta) = \pi_{\mathsf{R}}(\Theta) := \Theta = \left[\begin{smallmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{smallmatrix}\right] \in \mathsf{Index}(S_n).$$

If $\beta_0 \in \{\mathrm{id}, \mathrm{id}^+\}$ then define

$$\pi_{\mathsf{L}}(\Theta) := \begin{cases} \left[\begin{smallmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \chi^\lambda & \gamma_1 \end{smallmatrix}\right] & \text{if } \mu = \emptyset \\ 0 & \text{if } \mu \neq \emptyset \end{cases} \quad\text{and}\quad \pi_{\mathsf{R}}(\Theta) := \begin{cases} \left[\begin{smallmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \chi^\mu & \gamma_1 \end{smallmatrix}\right] & \text{if } \lambda = \emptyset \\ 0 & \text{if } \lambda \neq \emptyset. \end{cases}$$

If $\beta_0 = (p,q)$ for positive integers p and q with $p+q = \alpha_0$ then define

$$\pi_{\mathsf{L}}(\Theta) := \begin{cases} \left[\begin{smallmatrix} p & q & \alpha_1 \\ \mathrm{id} & \mathrm{id} & \beta_1 \\ \chi^\lambda & \chi^\lambda & \gamma_1 \end{smallmatrix}\right] & \text{if } \mu = \emptyset \\ 0 & \text{if } \mu \neq \emptyset \end{cases} \quad\text{and}\quad \pi_{\mathsf{R}}(\Theta) := \begin{cases} \left[\begin{smallmatrix} p & q & \alpha_1 \\ \mathrm{id} & \mathrm{id} & \beta_1 \\ \chi^\mu & \chi^\mu & \gamma_1 \end{smallmatrix}\right] & \text{if } \lambda = \emptyset \\ 0 & \text{if } \lambda \neq \emptyset. \end{cases}$$

When $\alpha_0 > 0$ but $\alpha_1 = 0$, we form $\pi_{\mathsf{L}}(\Theta)$ and $\pi_{\mathsf{R}}(\Theta)$ by applying the same formulas as above, and then deleting the last column if the result is nonzero.

Let $\mathcal{R}^{\mathsf{A}}_n$ and $\mathcal{R}^{\mathsf{B}}_n$ denote the $\mathbb{C}$-vector spaces of complex-valued class functions on S_n and W^{B}_n, respectively, and set $\mathcal{R}^{\mathsf{A}} := \bigoplus_{n\in\mathbb{N}} \mathcal{R}^{\mathsf{A}}_n$ and $\mathcal{R}^{\mathsf{B}} := \bigoplus_{n\in\mathbb{N}} \mathcal{R}^{\mathsf{B}}_n$. We use the same symbols π_{L} and π_{R} to denote the linear maps $\mathcal{R}^{\mathsf{B}} \to \mathcal{R}^{\mathsf{A}}$ with

$$\pi_{\mathsf{L}}(\chi^{(\lambda,\mu)}) := \begin{cases} \chi^\lambda & \text{if } \mu = \emptyset \\ 0 & \text{if } \mu \neq \emptyset \end{cases} \quad \text{and} \quad \pi_{\mathsf{R}}(\chi^{(\lambda,\mu)}) := \begin{cases} \chi^\mu & \text{if } \lambda = \emptyset \\ 0 & \text{if } \lambda \neq \emptyset \end{cases}$$

for all bipartitions (λ,μ). Finally, set $\chi^0_{\mathsf{A}} := 0 \in \mathcal{R}^{\mathsf{A}}$.

Lemma 4.2. If $\Theta \in \mathsf{Index}(W^{\mathsf{B}}_n)$ then $\pi_{\mathsf{L}}(\chi^\Theta_{\mathsf{B}}) = \chi^{\pi_{\mathsf{L}}(\Theta)}_{\mathsf{A}}$ and $\pi_{\mathsf{R}}(\chi^\Theta_{\mathsf{B}}) = \chi^{\pi_{\mathsf{R}}(\Theta)}_{\mathsf{A}}$.

Proof. One has $c^\emptyset_{\lambda\mu} = 0$ if λ or μ is nonempty and $c^\emptyset_{\emptyset\emptyset} = 1$. Likewise, one has $c^\nu_{\lambda\emptyset} = 0$ if $\lambda \neq \nu$ and $c^\nu_{\nu\emptyset} = 1$. It follows from these observations via (4.5) and (4.6) that $\pi_{\mathsf{L}}(\chi \bullet_{\mathsf{B}} \mathrm{Ind}^{W^{\mathsf{B}}_n}_{S_n}(\psi)) = \pi_{\mathsf{L}}(\chi) \bullet_{\mathsf{A}} \psi$ for all $\chi \in \mathcal{R}^{\mathsf{B}}$ and $\psi \in \mathcal{R}^{\mathsf{A}}_n$. In view of this identity and (3.2) and (4.4), we see that to show $\pi_{\mathsf{L}}(\chi^\Theta_{\mathsf{B}}) = \chi^{\pi_{\mathsf{L}}(\Theta)}_{\mathsf{A}}$ for all $\Theta \in \mathsf{Index}(W^{\mathsf{B}}_n)$ it suffices to prove this identity when $\Theta = \left[\begin{smallmatrix}\alpha_0\\ \beta_0\\ \gamma_0\end{smallmatrix}\right]$ as in (4.8) and Proposition 4.1. But this follows immediately from the definition of π_{L} and Proposition 3.2. The proof that $\pi_{\mathsf{R}}(\chi^\Theta_{\mathsf{B}}) = \chi^{\pi_{\mathsf{R}}(\Theta)}_{\mathsf{A}}$ is similar. □

4.5 Perfect models in type B

Fix a model index $\Theta = \left[\begin{smallmatrix}\alpha_0 & \alpha_1\\ \beta_0 & \beta_1\\ \gamma_0 & \gamma_1\end{smallmatrix}\right] \in \mathsf{Index}(W^{\mathsf{B}}_n)$.

Lemma 4.3. The character χ^Θ_{B} is not multiplicity-free if any of the following conditions hold:

(a) $\alpha_1 \in \{4, 6, 8, \dots\}$ and $\beta_1 \in \{\mathrm{fpf}, \mathrm{fpf}^+\}$.

(b) $\alpha_0 \in \{2, 4, 6, \dots\}$, $\beta_0 = \mathrm{fpf}$, $\alpha_1 \geq 2$, and $\gamma_0 = \gamma_1 \in \{\mathbb{1}, \mathrm{sgn}\}$.

(c) $\beta_0 = (p, q)$ for some $p, q > 0$ with $p + q = \alpha_0$ and $\alpha_1 > 0$.

Proof. Suppose (a) holds and let $\Psi := \left[\begin{smallmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{smallmatrix}\right]$ and $m := \alpha_1$. To prove that χ^Θ_{B} is not multiplicity-free, it suffices by (4.4) to show that $\mathrm{Ind}^{W^{\mathsf{B}}_m}_{S_m}(\chi^\Psi_{\mathsf{A}})$ is not multiplicity-free. As $\chi^{\overline{\Psi}}_{\mathsf{A}} = \chi^\Psi_{\mathsf{A}}\,\mathrm{sgn}$, we may assume $\gamma_1 = \mathbb{1}$. Proposition 3.2 and (4.6) imply that $\mathrm{Ind}^{W^{\mathsf{B}}_m}_{S_m}(\chi^\Psi_{\mathsf{A}}) = \sum_{\nu \in \mathsf{ERows}(m)} \sum_{\lambda,\mu} c^\nu_{\lambda\mu} \chi^{(\lambda,\mu)}$, which is not multiplicity-free as $c^{(m)}_{(m-2)(2)} = c^{(m-2,2)}_{(m-2)(2)} = 1$.

Suppose (b) holds. We may assume $\beta_1 \in \{\mathrm{id}, \mathrm{id}^+\}$ given the previous paragraph. Let $m = \alpha_0$. It is straightforward from Section 4.3 to check that χ^Θ_{B} contains either $\chi^{((n-m),(m))}$ (when $\gamma_0 = \gamma_1 = \mathbb{1}$) or $\chi^{((1^{n-m}),(1^m))}$ (when $\gamma_0 = \gamma_1 = \mathrm{sgn}$) as a constituent with multiplicity greater than one.

Finally, if (c) holds then one of $\pi_{\mathsf{L}}(\chi^\Theta_{\mathsf{B}})$ or $\pi_{\mathsf{R}}(\chi^\Theta_{\mathsf{B}})$ is nonzero. But if $\pi_{\mathsf{L}}(\chi^\Theta_{\mathsf{B}}) = \chi^{\pi_{\mathsf{L}}(\Theta)}_{\mathsf{A}}$ is nonzero then it cannot be multiplicity-free since $\pi_{\mathsf{L}}(\Theta)$ has three columns. The character $\pi_{\mathsf{R}}(\chi^\Theta_{\mathsf{B}})$ likewise cannot be nonzero and multiplicity-free. Hence χ^Θ_{B} must not be multiplicity-free. □

Let $\mathsf{ORows}_B(n, q)$ be the set of bipartitions $(\lambda, \mu) \vdash n$ such that $\lambda \cup \mu$ has exactly q odd parts. Define $\mathsf{OCols}_B(n, q) = \{(\lambda^\top, \mu^\top) : (\lambda, \mu) \in \mathsf{ORows}_B(n, q)\}$.

Proposition 4.4. Suppose $\Theta \in \mathsf{Index}(W^{\mathsf{B}}_n)$ and χ^Θ_{B} is multiplicity-free. Then Θ has one of the following forms:

(a) $\left[\begin{smallmatrix} k & n-k\\ \mathrm{id}/\mathrm{id}^+ & \mathrm{id}/\mathrm{id}^+\\ \gamma_0 & \gamma_1\end{smallmatrix}\right]$ for some $0 \leq k \leq n$, $\gamma_0 \in \{\mathbb{1}, \mathrm{sgn}, \mathbb{1}_{+-}, \mathbb{1}_{-+}\}$, and $\gamma_1 \in \{\mathbb{1}, \mathrm{sgn}\}$.

(b) $\begin{bmatrix} 2k & n-2k \\ \text{fpf} & \text{id}/\text{id}^+ \\ \mathbb{1} & \text{sgn} \end{bmatrix}$ for some $0 \le k \le \lfloor n/2 \rfloor$, in which case $\chi_{\mathsf{B}}^{\Theta} = \sum_{(\lambda,\mu)\in \mathsf{ORows}_B(n,n-2k)} \chi^{(\lambda,\mu)}$.

(c) $\begin{bmatrix} 2k & n-2k \\ \text{fpf} & \text{id}/\text{id}^+ \\ \text{sgn} & \mathbb{1} \end{bmatrix}$ for some $0 \le k \le \lfloor n/2 \rfloor$, in which case $\chi_{\mathsf{B}}^{\Theta} = \sum_{(\lambda,\mu)\in \mathsf{OCols}_B(n,n-2k)} \chi^{(\lambda,\mu)}$.

(d) $\begin{bmatrix} n & 0 \\ (p,q) & \text{id}/\text{id}^+ \\ \gamma_0 & \mathbb{1} \end{bmatrix}$ for some $p, q > 0$ such that $p + q = n$ and $\gamma_0 \in \{\mathbb{1}, \text{sgn}, \mathbb{1}_{+-}, \mathbb{1}_{-+}\}$.

Proof. The given cases account for all model indices in $\mathsf{Index}(W_n^{\mathsf{B}})$ not excluded by Lemma 4.3. The formulas in parts (b) and (c) follow by combining (4.4)–(4.6) with (4.8). □

Theorem 4.5. Assume $n \ge 2$. The following sets are inequivalent perfect models for W_n^{B}:

$$\mathcal{P}_n^{\mathsf{B}} := \left\{ \mathbb{T}^{\Theta} : \Theta = \begin{bmatrix} 2k & n-2k \\ \text{fpf} & \text{id} \\ \mathbb{1} & \text{sgn} \end{bmatrix} \text{ for } 0 \le k \le \lfloor \tfrac{n}{2} \rfloor \right\},$$

$$\hat{\mathcal{P}}_n^{\mathsf{B}} := \left\{ \mathbb{T}^{\Theta} : \Theta = \begin{bmatrix} 2 & n-2 \\ \text{id} & \text{id} \\ \mathbb{1} & \text{sgn} \end{bmatrix},\ \begin{bmatrix} 2 & n-2 \\ \text{id} & \text{id} \\ \mathbb{1}_{-+} & \text{sgn} \end{bmatrix},\ \text{or } \begin{bmatrix} 2k & n-2k \\ \text{fpf} & \text{id} \\ \mathbb{1} & \text{sgn} \end{bmatrix} \text{ for } k = 0, 2, 3, 4, \dots \lfloor \tfrac{n}{2} \rfloor \right\}.$$

If $n \ne 3$ then each perfect model for W_n^{B} is strongly equivalent to one of $\mathcal{P}_n^{\mathsf{B}}$, $\overline{\mathcal{P}_n^{\mathsf{B}}}$, $\hat{\mathcal{P}}_n^{\mathsf{B}}$, or $\overline{\hat{\mathcal{P}}_n^{\mathsf{B}}}$.

Proof. It is clear from part (b) of Proposition 4.4 that $\mathcal{P}_n^{\mathsf{B}}$ is a perfect model for W_n^{B}. It follows that $\hat{\mathcal{P}}_n^{\mathsf{B}}$ is also a perfect model since

$$\chi_{\mathsf{B}}^{\begin{bmatrix} 2 & n-2 \\ \text{id} & \text{id} \\ \mathbb{1} & \text{sgn} \end{bmatrix}} + \chi_{\mathsf{B}}^{\begin{bmatrix} 2 & n-2 \\ \text{id} & \text{id} \\ \mathbb{1}_{-+} & \text{sgn} \end{bmatrix}} = \left(\chi_{\mathsf{B}}^{\begin{bmatrix} 2 \\ \text{id} \\ \mathbb{1} \end{bmatrix}} + \chi_{\mathsf{B}}^{\begin{bmatrix} 2 \\ \text{id} \\ \mathbb{1}_{-+} \end{bmatrix}} \right) \bullet_{\mathsf{B}} \mathrm{Ind}_{S_{n-2}}^{W_{n-2}^{\mathsf{B}}} \left(\chi_{\mathsf{A}}^{\begin{bmatrix} n-2 \\ \text{id} \\ \text{sgn} \end{bmatrix}} \right) = \chi^{\begin{bmatrix} 2 & n-2 \\ \text{fpf} & \text{id} \\ \mathbb{1} & \text{sgn} \end{bmatrix}}.$$

We have checked the desired result using the computer algebra system GAP [10] when $n \le 4$, so assume $n \ge 5$. Suppose $\mathcal{M}$ is a set of model indices $\Theta \in \mathsf{Index}(W_n^{\mathsf{B}})$ such that $\{\mathbb{T}^{\Theta} : \Theta \in \mathcal{M}\}$ is a perfect model for W_n^{B}. Every perfect model for W_n^{B} arises in this way. Define $\mathcal{M}_{\mathsf{L}} := \{\pi_{\mathsf{L}}(\Theta) : \Theta \in \mathcal{M}\} \setminus \{0\}$ and $\mathcal{M}_{\mathsf{R}} := \{\pi_{\mathsf{R}}(\Theta) : \Theta \in \mathcal{M}\} \setminus \{0\}$. Lemma 4.2 implies that $\{\mathbb{T}^{\Theta} : \Theta \in \mathcal{M}_{\mathsf{L}}\}$ and $\{\mathbb{T}^{\Theta} : \Theta \in \mathcal{M}_{\mathsf{R}}\}$ are perfect models for S_n. After possibly replacing $\mathcal{M}$ by $\overline{\mathcal{M}} = \{\overline{\Theta} : \Theta \in \mathcal{M}\}$, we may assume that $\{\mathbb{T}^{\Theta} : \Theta \in \mathcal{M}_{\mathsf{L}}\}$ is a sgn-model in the terminology of Remark 3.5.

Define $\Theta_k := \begin{bmatrix} 2k & n-2k \\ \text{fpf} & \text{id} \\ \mathbb{1} & \text{sgn} \end{bmatrix}$ for $0 \le k \le \lfloor n/2 \rfloor$. Remark 3.5 tells us that $\mathcal{M}_{\mathsf{L}}$ must contain elements of each of the forms (3.6), (3.7), (3.8), and (3.9). There are limited possibilities for model indices $\Theta \in \mathsf{Index}(W_n^{\mathsf{B}})$ with $\chi_{\mathsf{B}}^{\Theta}$ multiplicity-free that can serve as the preimages for these elements under π_{L}. It follows by inspecting Proposition 4.4 that $\mathcal{M}$ must contain a unique model index strongly equivalent to Θ_k for at least each $2 \le k \le \lfloor n/2 \rfloor$. Since $\pi_{\mathsf{R}}(\Theta_k) = \pi_{\mathsf{L}}(\Theta_k) = \Theta_k$, it follows from Remark 3.5 that $\{\mathbb{T}^{\Theta} : \Theta \in \mathcal{M}_{\mathsf{R}}\}$ is also a sgn-model.

By similar reasoning, for $\mathcal{M}_{\mathsf{L}}$ to contain an index of the form (3.4), $\mathcal{M}$ must contain a unique element strongly equivalent to $\Theta_0 = \begin{bmatrix} 0 & n \\ \text{fpf} & \text{id} \\ \mathbb{1} & \text{sgn} \end{bmatrix} \sim \begin{bmatrix} 0 & n \\ \text{id} & \text{id} \\ \mathbb{1} & \text{sgn} \end{bmatrix}$ or $\Theta_0' := \begin{bmatrix} n & 0 \\ \text{id} & \text{id} \\ \mathbb{1}_{+-} & \mathbb{1} \end{bmatrix}$. For $\mathcal{M}_{\mathsf{R}}$ to contain an index of the form (3.4), $\mathcal{M}$ must contain a unique element strongly equivalent to Θ_0 or $\Theta_0'' := \begin{bmatrix} n & 0 \\ \text{id} & \text{id} \\ \text{sgn} & \mathbb{1} \end{bmatrix}$. Likewise, for $\mathcal{M}_{\mathsf{L}}$ to contain an index of the form (3.5), $\mathcal{M}$ must contain a unique element strongly equivalent to Θ_1, $\Theta_1' := \begin{bmatrix} 2 & n-2 \\ \text{id} & \text{id} \\ \mathbb{1} & \text{sgn} \end{bmatrix}$, or $\Psi_1' := \begin{bmatrix} n-2 & 2 \\ \text{id} & \text{id} \\ \mathbb{1}_{+-} & \mathbb{1} \end{bmatrix}$. For $\mathcal{M}_{\mathsf{R}}$ to contain an index of the form (3.5), $\mathcal{M}$ must contain a unique element strongly equivalent to Θ_1, $\Theta_1'' := \begin{bmatrix} 2 & n-2 \\ \text{id} & \text{id} \\ \mathbb{1}_{-+} & \text{sgn} \end{bmatrix}$, or $\Psi_1'' := \begin{bmatrix} n-2 & 2 \\ \text{id} & \text{id} \\ \text{sgn} & \mathbb{1} \end{bmatrix}$.

Thus, $\mathcal{M}$ contains a subset strongly equivalent to $\mathcal{M}^0 \sqcup \mathcal{M}^1 \sqcup \mathcal{M}^2$ where $\mathcal{M}^0$ is either $\{\Theta_0\}$ or $\{\Theta_0', \Theta_0''\}$; $\mathcal{M}^1$ is either $\{\Theta_1\}$, $\{\Theta_1', \Theta_1''\}$, $\{\Theta_1', \Psi_1''\}$, $\{\Psi_1', \Theta_1''\}$, or $\{\Psi_1', \Psi_1''\}$; and $\mathcal{M}^2 := \{\Theta_k : 2 \le k \le \lfloor n/2 \rfloor\}$. In fact, we must have $\mathcal{M} \sim \mathcal{M}^0 \sqcup \mathcal{M}^1 \sqcup \mathcal{M}^2$ since it is impossible to add any further indices without violating our assumption that $\{\mathbb{T}^\Theta : \Theta \in \mathcal{M}_\mathsf{L}\}$ and $\{\mathbb{T}^\Theta : \Theta \in \mathcal{M}_\mathsf{R}\}$ are perfect models for S_n.

If $\mathcal{M}^0 = \{\Theta_0\}$ and $\mathcal{M}^1 \in \{\{\Theta_1\}, \{\Theta_1', \Theta_1''\}\}$ then $\{\mathbb{T}^\Theta : \Theta \in \mathcal{M}\}$ is strongly equivalent to $\mathcal{P}_n^\mathsf{B}$ or $\hat{\mathcal{P}}_n^\mathsf{B}$ as desired. All of the other choices for $\mathcal{M}^0$ and $\mathcal{M}^1$ are all impossible since they would lead to $\sum_{\Theta\in\mathcal{M}} \chi_\mathsf{B}^\Theta(1) = \sum_{\Theta\in\mathcal{M}^0\sqcup\mathcal{M}^1\sqcup\mathcal{M}^2} \chi_\mathsf{B}^\Theta(1) < \sum_{k=0}^{\lfloor n/2\rfloor} \chi_\mathsf{B}^{\Theta_k}(1) = \sum_{\chi\in\mathrm{Irr}(W_n^\mathsf{B})} \chi(1)$, contradicting the fact that $\{\mathbb{T}^\Theta : \Theta \in \mathcal{M}\}$ is a model. We conclude that if $\mathcal{P}$ is a perfect model for W_n^B when $n \neq 3$ then $\mathcal{P}$ or $\overline{\mathcal{P}}$ is strongly equivalent to $\mathcal{P}_n^\mathsf{B}$ or $\hat{\mathcal{P}}_n^\mathsf{B}$. □

Example 4.6. There are 2 more equivalence classes of perfect models for W_3^B, represented by

$$\left\{\mathbb{T}^\Theta : \Theta = \begin{bmatrix} 1 & 2 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathbb{1} \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ \mathrm{id} & \mathrm{id} \\ \mathrm{sgn} & \mathbb{1} \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1}_{+-} & \mathbb{1} \end{bmatrix}, \text{ or } \begin{bmatrix} 3 & 0 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1}_{-+} & \mathbb{1} \end{bmatrix}\right\} \text{ and}$$
$$\left\{\mathbb{T}^\Theta : \Theta = \begin{bmatrix} 1 & 2 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1}_{-+} & \mathbb{1} \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1}_{+-} & \mathbb{1} \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathbb{1} \end{bmatrix}, \text{ or } \begin{bmatrix} 3 & 0 \\ \mathrm{id} & \mathrm{id} \\ \mathrm{sgn} & \mathbb{1} \end{bmatrix}\right\}.$$

5 Model classification in type D

In this section we continue to fix an integer $n \geq 2$ and consider the Coxeter group $W = W_n^\mathsf{D}$ with generating set $S = \{s_{-1}, s_2, s_3, \dots, s_{n-1}\}$. Recall that W_n^D is the subgroup of permutations $w \in W_n^\mathsf{B}$ for which $|\{i \in [n] : w(i) < 0\}|$ is even. Our main result here is Theorem 5.8.

5.1 Perfect conjugacy classes in type D

Let $w \mapsto w^\diamond$ denote the Coxeter automorphism of W_n^D interchanging $s_{-1} \leftrightarrow s_1$ while fixing the other simple generators. This is the restriction of $\mathrm{Ad}(s_0) \in \mathrm{Aut}(W_n^\mathsf{B})$. Write w_0 for the longest element in W_n^D. If n is even then $\diamond$ is an outer automorphism and $w_0 = \bar{1}\bar{2}\bar{3}\cdots\bar{n}$ is central. If n is odd then $w_0 = 1\bar{2}\bar{3}\cdots\bar{n}$ and $\diamond = \mathrm{Ad}(w_0)$. Thus $w_0^+ := (w_0, \mathrm{Ad}(w_0)) \in W^+$ is equal to w_0 if n is even and to $(w_0, \diamond)$ when n is odd.

If $n \neq 4$ then the Coxeter automorphisms of W_n^D are $\{\mathrm{id}, \diamond\}$. The Coxeter diagram of W_4^D is

$$\begin{array}{ccccc} s_1 & \text{——} & s_2 & \text{——} & s_3 \\ & & | & & \\ & & s_{-1} & & \end{array}$$

so there are two Coxeter automorphisms of W_4^D of order three that fix s_2. We denote these by

$$\circlearrowright : s_{-1} \mapsto s_1 \mapsto s_3 \mapsto s_{-1} \quad \text{and} \quad \circlearrowleft : s_{-1} \mapsto s_3 \mapsto s_1 \mapsto s_{-1}. \tag{5.1}$$

There are three nontrivial Coxeter involutions of W_4^D, given by $\diamond$, $\circlearrowright \diamond \circlearrowleft$, and $\circlearrowleft \diamond \circlearrowright$. The latter two interchange $s_{-1} \leftrightarrow s_3$ and $s_1 \leftrightarrow s_3$, respectively, while fixing the other simple generators.[5]

Fix $p, q > 0$ with $p + q = n$. If q is even then let $\mathcal{K}_{(p,q)}^{W_n^\mathsf{D}} := \mathcal{K}_{(p,q)}^{W_n^\mathsf{B}}$ be the set of $w \in W_n^\mathsf{D}$ with $|\{i \in [n] : w(i) = i\}| = p$ and $|\{i \in [n] : w(i) = -i\}| = q$. If q is odd then define

$$\mathcal{K}_{(p,q)}^{W_n^\mathsf{D}} := \left\{(ws_0, \diamond) : w \in \mathcal{K}_{(p,q)}^{W_n^\mathsf{B}}\right\} \subseteq (W_n^\mathsf{D})^+.$$

The unique minimal-length element of $\mathcal{K}_{(p,q)}^{W_n^\mathsf{D}}$ is either

$$\bar{1}\bar{2}\cdots\bar{q}(q+1)(q+2)\cdots n \in W_n^\mathsf{D} \quad \text{or} \quad (1\bar{2}\cdots\bar{q}(q+1)(q+2)\cdots n, \diamond) \in (W_n^\mathsf{D})^+$$

[5]The automorphism $\circlearrowright \diamond \circlearrowleft$ interchanges s_{-1} and s_3 because $s_{-1}^{\circlearrowright\diamond\circlearrowleft} := ((s_{-1}^{\circlearrowright})^{\diamond}))^{\circlearrowleft} = (s_1^{\diamond})^{\circlearrowleft} = s_{-1}^{\circlearrowleft} = s_3$.

according to whether q is even or odd. Let $\mathcal{K}_{\mathrm{id}}^{W_n^{\mathsf{D}}} := \{1\}$ and $\mathcal{K}_{\mathrm{id}^+}^{W_n^{\mathsf{D}}} := \{w_0^+\}$. Recall that if n is even then $\mathcal{K}_{\mathrm{fpf}}^{W_n^{\mathsf{B}}}$ is the set of elements $z = z^{-1} \in W_n$ with $|z(i)| \neq i$ for all $i \in [n]$. If z belongs to this set and $z(i) = -j$ for some $i, j \in [n]$ then $i \neq j$ and $z(j) = -i$, so $z \in W_n^{\mathsf{D}}$. Define

$$\mathcal{K}_{\mathrm{fpf}}^{W_n^{\mathsf{D}}} := \left\{ z \in \mathcal{K}_{\mathrm{fpf}}^{W_n^{\mathsf{B}}} : |\{i \in [n] : w(i) < 0\}| \text{ is divisible by } 4 \right\} \quad \text{and} \quad \mathcal{K}_{\mathrm{fpf}^\diamond}^{W_n^{\mathsf{D}}} = \mathcal{K}_{\mathrm{fpf}}^{W_n^{\mathsf{B}}} - \mathcal{K}_{\mathrm{fpf}}^{W_n^{\mathsf{D}}}.$$

The unique minimal-length elements of $\mathcal{K}_{\mathrm{fpf}}^{W_n^{\mathsf{D}}}$ and $\mathcal{K}_{\mathrm{fpf}^\diamond}^{W_n^{\mathsf{D}}}$ are respectively

$$s_1 s_3 s_5 \cdots s_{n-1} \quad \text{and} \quad s_{-1} s_3 s_5 \cdots s_{n-1}.$$

When $n \in \{2, 3\}$ or $n \geq 5$ the distinct perfect conjugacy classes in W^+ consist of $\mathcal{K}_{\mathrm{id}}^{W_n^{\mathsf{D}}}$, $\mathcal{K}_{\mathrm{id}^+}^{W_n^{\mathsf{D}}}$, and $\mathcal{K}_{(p,q)}^{W_n^{\mathsf{D}}}$ for all $p, q > 0$ with $p + q = n$, along with $\mathcal{K}_{\mathrm{fpf}}^{W_n^{\mathsf{D}}}$ and $\mathcal{K}_{\mathrm{fpf}^\diamond}^{W_n^{\mathsf{D}}}$ if n is even [23, Ex. 9.2]. When $n = 1$ the only perfect conjugacy class is $\mathcal{K}_{\mathrm{id}}^{W_1^{\mathsf{D}}} = \mathcal{K}_{\mathrm{id}^+}^{W_1^{\mathsf{D}}} = \{1\}$ since $W_1^{\mathsf{D}} = \{1\}$ is trivial. The case $n = 4$ is exceptional. For $p, q > 0$ with $p + q = 4$ define

$$\mathcal{K}_{(p,q,\circlearrowright)}^{W_4^{\mathsf{D}}} := \left\{ z^{\circlearrowright} : z \in \mathcal{K}_{(p,q)}^{W_4^{\mathsf{D}}} \right\} \quad \text{and} \quad \mathcal{K}_{(p,q,\circlearrowleft)}^{W_4^{\mathsf{D}}} := \left\{ z^{\circlearrowleft} : z \in \mathcal{K}_{(p,q)}^{W_4^{\mathsf{D}}} \right\}.$$

There are 11 perfect conjugacy classes in $(W_4^{\mathsf{D}})^+$, consisting of $\mathcal{K}_{\mathrm{id}}^{W_4^{\mathsf{D}}}$ and $\mathcal{K}_{\mathrm{id}^+}^{W_4^{\mathsf{D}}}$ together with

$$\begin{aligned}
&\mathcal{K}_{(3,1)}^{W_4^{\mathsf{D}}} \ni (1, \diamond), && \mathcal{K}_{(3,1,\circlearrowright)}^{W_4^{\mathsf{D}}} \ni (1, \circlearrowright \diamond \circlearrowleft), && \mathcal{K}_{(3,1,\circlearrowleft)}^{W_4^{\mathsf{D}}} \ni (1, \circlearrowleft \diamond \circlearrowright), \\
&\mathcal{K}_{(2,2)}^{W_4^{\mathsf{D}}} \ni s_{-1}s_1, && \mathcal{K}_{(2,2,\circlearrowright)}^{W_4^{\mathsf{D}}} = \mathcal{K}_{\mathrm{fpf}}^{W_4^{\mathsf{D}}} \ni s_1 s_3, && \mathcal{K}_{(2,2,\circlearrowleft)}^{W_4^{\mathsf{D}}} = \mathcal{K}_{\mathrm{fpf}^\diamond}^{W_4^{\mathsf{D}}} \ni s_3 s_{-1}, \\
&\mathcal{K}_{(1,3)}^{W_4^{\mathsf{D}}} \ni (1\bar{2}\bar{3}4, \diamond), && \mathcal{K}_{(1,3,\circlearrowright)}^{W_4^{\mathsf{D}}} \ni (4321, \circlearrowright \diamond \circlearrowleft), && \mathcal{K}_{(1,3,\circlearrowleft)}^{W_4^{\mathsf{D}}} \ni (\bar{4}32\bar{1}, \circlearrowleft \diamond \circlearrowright).
\end{aligned}$$

This listed elements of each class are the unique ones of minimal length.

5.2 Model indices in type D

If $n > 2$ then $\mathbb{1}$ and sgn are the only linear characters of W_n^{D}. If $n = 2$ then there are four linear characters given by $\mathbb{1}_{++} := \mathbb{1}$, $\mathbb{1}_{--} := \mathrm{sgn}$, $\mathbb{1}_{+-}$, and $\mathbb{1}_{-+}$, where the last two satisfy $\mathbb{1}_{\pm\mp} : s_{-1} \mapsto \pm 1$ and $\mathbb{1}_{\pm\mp} : s_1 \mapsto \mp 1$. One has $\mathbb{1}^\diamond = \mathbb{1}$, $\mathrm{sgn}^\diamond = \mathrm{sgn}$, and $\mathbb{1}_{\pm\mp}^\diamond = \mathbb{1}_{\mp\pm}$ where for any function $f : W_n^{\mathsf{D}} \to \mathbb{C}$ we write $f^\diamond$ for the map with $w \mapsto f(w^\diamond)$.

Let $\mathsf{Index}(W_n^{\mathsf{D}})$ denote the set of 3×2 arrays, to be called *model indices* for W_n^{D}, of the form

$$\Theta = \left[\begin{smallmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{smallmatrix}\right]$$

where $\alpha_1 \in \{-n, n\} \sqcup \{0, 1, \ldots, n-2\}$ and $\alpha_0 = n - |\alpha_1|$; where β_0 is either

- a symbol in $\{\mathrm{id}, \mathrm{id}^+, \mathrm{fpf}, \mathrm{fpf}^\diamond\}$, with $\beta_0 \in \{\mathrm{fpf}, \mathrm{fpf}^\diamond\}$ allowed only if α_0 is even,
- when $\alpha_0 > 2$, a pair of positive integers (p, q) with $p + q = \alpha_0$, or
- when $\alpha_0 = 4$, one of the triples $(3, 1, \circlearrowright)$, $(3, 1, \circlearrowleft)$, $(1, 3, \circlearrowright)$, or $(1, 3, \circlearrowleft)$;

where β_1 is a symbol in $\{\mathrm{id}, \mathrm{id}^+, \mathrm{fpf}, \mathrm{fpf}^+\}$, with $\beta_1 \in \{\mathrm{fpf}, \mathrm{fpf}^+\}$ only if $|\alpha_1| \in \{4, 6, 8, \ldots\}$; and where γ_0 and γ_1 are linear characters of $W_{\alpha_0}^{\mathsf{D}}$ and $S_{|\alpha_1|}$. Let $\Theta = \left[\begin{smallmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{smallmatrix}\right] \in \mathsf{Index}(W_n^{\mathsf{D}})$. Define

$$J_0 := \begin{cases} \varnothing & \text{if } \alpha_0 = 0 \\ \{s_{-1}, s_1, s_2, \ldots, s_{\alpha_0 - 1}\} & \text{otherwise} \end{cases} \quad \text{and} \quad J_1 := \{s_1, s_2, \ldots, s_{|\alpha_1|-1}\}.$$

Define $\varphi_1 : S^+_{|\alpha_1|} \to \langle J_1\rangle^+$ to be the isomorphism sending $s_j \mapsto s_{\alpha_0+j}$ for $j \in [|\alpha_1|-1]$. When $\alpha_0 \neq 0$ define $\mathcal{K}_0 = \mathcal{K}_{\beta_0}^{W^{\mathsf{B}}_{\alpha_0}}$. When $\alpha_1 \neq 0$ define $\mathcal{K}_1$ to be the image of $\mathcal{K}_{\beta_1}^{S_{|\alpha_1|}}$ under φ_1. Let

$$\mathbb{T}^\Theta := \begin{cases} (J_0, \mathcal{K}_0, \gamma_0) & \text{if } \alpha_0 = n, \\ (J_1, \mathcal{K}_1, \gamma_1) & \text{if } \alpha_1 = n, \\ (J_1, \mathcal{K}_1, \gamma_1)^\diamond & \text{if } \alpha_1 = -n \\ (J_0, \mathcal{K}_0, \gamma_0) \otimes (J_1, \mathcal{K}_1, \gamma_1) & \text{otherwise.} \end{cases}$$

This gives a model triple $\mathbb{T}^\Theta$ for W_n^{D} which is factorizable[6], and by Theorems 1.6 and 2.2 every multiplicity-free model triple for W_n^{D} arises from this construction. As usual we also define

$$\chi^\Theta_{\mathsf{D}} := \chi^{\mathbb{T}^\Theta}.$$

If $\alpha_0 \in \{0,1\}$ then $\mathbb{T}^\Theta$ does not depend on β_0 or γ_0 and if $\alpha_1 = 0$ then $\mathbb{T}^\Theta$ does not depend on β_1 or γ_1. If $\alpha_1 = 1$ then $\mathbb{T}^\Theta$ is unaffected by changing $\beta_1 = \mathrm{id}$ to id^+ (or vice versa) or $\gamma_1 = \mathbb{1}$ to sgn (or vice versa).

Let $\Theta = \left[\begin{smallmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{smallmatrix}\right] \in \mathsf{Index}(W_n^{\mathsf{D}})$. Define $\overline{\Theta} := \left[\begin{smallmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \bar\gamma_0 & \bar\gamma_1 \end{smallmatrix}\right]$ where $\bar\gamma_i := \gamma_i\,\mathrm{sgn}$. Next let

$$\Theta^\vee := \left[\begin{smallmatrix} \alpha_0 & \alpha_1^\diamond \\ \beta_0^\vee & \beta_1^\vee \\ \gamma_0^\diamond & \gamma_1 \end{smallmatrix}\right] \text{ if } n \text{ is odd} \quad\text{and}\quad \Theta^\vee := \left[\begin{smallmatrix} \alpha_0 & \alpha_1 \\ \beta_0^\vee & \beta_1^\vee \\ \gamma_0 & \gamma_1 \end{smallmatrix}\right] \text{ if } n \text{ is even,} \tag{5.2}$$

where we set $\alpha_1^\diamond := -\alpha_1$ when $|\alpha_1| = n$ and $\alpha_1^\diamond := \alpha_1$ otherwise, and define

$$\beta_0^\vee := \begin{cases} (q,p) & \text{if } \beta_0 = (p,q) \\ (q,p,\circlearrowright) & \text{if } \beta_0 = (p,q,\circlearrowright) \text{ and } n \text{ is even} \\ (q,p,\circlearrowright) & \text{if } \beta_0 = (p,q,\circlearrowleft) \text{ and } n \text{ is odd} \\ (q,p,\circlearrowleft) & \text{if } \beta_0 = (p,q,\circlearrowleft) \text{ and } n \text{ is even} \\ (q,p,\circlearrowleft) & \text{if } \beta_0 = (p,q,\circlearrowright) \text{ and } n \text{ is odd} \\ \mathrm{fpf} & \text{if } \beta_0 = \mathrm{fpf} \text{ and } \frac{\alpha_0}{2}+n \text{ is even} \\ \mathrm{fpf} & \text{if } \beta_0 = \mathrm{fpf}^\diamond \text{ and } \frac{\alpha_0}{2}+n \text{ is odd} \\ \mathrm{fpf}^\diamond & \text{if } \beta_0 = \mathrm{fpf}^\diamond \text{ and } \frac{\alpha_0}{2}+n \text{ is even} \\ \mathrm{fpf}^\diamond & \text{if } \beta_0 = \mathrm{fpf} \text{ and } \frac{\alpha_0}{2}+n \text{ is odd} \\ \mathrm{id} & \text{if } \beta_0 = \mathrm{id}^+ \\ \mathrm{id}^+ & \text{if } \beta_0 = \mathrm{id} \end{cases} \quad\text{and}\quad \beta_1^\vee := \begin{cases} \mathrm{fpf} & \text{if } \beta_1 = \mathrm{fpf}^+ \\ \mathrm{fpf}^+ & \text{if } \beta_1 = \mathrm{fpf} \\ \mathrm{id} & \text{if } \beta_1 = \mathrm{id}^+ \\ \mathrm{id}^+ & \text{if } \beta_1 = \mathrm{id}. \end{cases}$$

Next define $\Theta^\diamond := \left[\begin{smallmatrix} \alpha_0 & \alpha_1^\diamond \\ \beta_0^\diamond & \beta_1 \\ \gamma_0^\diamond & \gamma_1 \end{smallmatrix}\right]$ where $\alpha_1^\diamond$ is as above and

$$\beta_0^\diamond := \begin{cases} (p,q,\circlearrowleft) & \text{if } \beta_0 = (p,q,\circlearrowright) \\ (p,q,\circlearrowright) & \text{if } \beta_0 = (p,q,\circlearrowleft) \\ \mathrm{fpf}^\diamond & \text{if } \beta_0 = \mathrm{fpf} \\ \mathrm{fpf} & \text{if } \beta_0 = \mathrm{fpf}^\diamond \\ \beta_0 & \text{otherwise.} \end{cases}$$

We always have $\gamma_0^\diamond = \gamma_0$ unless $\alpha_0 = 2$ and $\gamma_0 = \mathbb{1}_{\pm\mp}$.

[6]Note that $\mathbb{T}^\Theta$ would not be factorizable if we allowed $\beta_0 = (p,q) = (1,1)$ when $\alpha_0 = 2$.

We adapt the relations $\equiv$, $\sim$, and $\approx$ to (sets of) model indices in $\mathsf{Index}(W^{\mathsf{D}}_n)$ in the same way that we did for elements of $\mathsf{Index}(S_n)$ and $\mathsf{Index}(W^{\mathsf{B}}_n)$. It is easy to check that $\mathbb{T}^{\overline{\Theta}} = \overline{\mathbb{T}^{\Theta}}$ and $\mathbb{T}^{\Theta^\diamond} = (\mathbb{T}^{\Theta})^\diamond$, and that if Θ' is formed from Θ by changing any entries equal to id^+ to id, then $\mathbb{T}^{\Theta} \equiv \mathbb{T}^{\Theta'}$. It is somewhat more involved, but still straightforward, to verify that $\mathbb{T}^{\Theta^\vee} = (\mathbb{T}^{\Theta})^\vee$. We have $\Theta \sim \Theta^\vee \sim \Theta^\diamond$ when n is odd and $\Theta \sim \Theta^\vee \approx \Theta^\diamond$ when n is even.

5.3 Littlewood-Richardson coefficients in type D

If $(\lambda,\mu) \vdash n$ with $\lambda \neq \mu$ then the restricted character

$$\chi^{\{\lambda,\mu\}} := \mathrm{Res}^{W^{\mathsf{B}}_n}_{W^{\mathsf{D}}_n}(\chi^{(\lambda,\mu)}) = \mathrm{Res}^{W^{\mathsf{B}}_n}_{W^{\mathsf{D}}_n}(\chi^{(\mu,\lambda)}) \in \mathrm{Irr}(W^{\mathsf{D}}_n)$$

is irreducible. In this case we refer to $\{\lambda,\mu\}$ as an unordered bipartition of n and write $\{\lambda,\mu\} \vdash n$. If n is even and $\nu \vdash n/2$ then

$$\mathrm{Res}^{W^{\mathsf{B}}_n}_{W^{\mathsf{D}}_n}(\chi^{(\nu,\nu)}) = \chi^{[\nu,+]} + \chi^{[\nu,-]}$$

for two different irreducible characters $\chi^{[\nu,\pm]}$. The distinct elements of $\mathrm{Irr}(W^{\mathsf{D}}_n)$ consist of $\chi^{\{\lambda,\mu\}}$ for all $\{\lambda,\mu\} \vdash n$ together with the *degenerate* characters $\chi^{[\nu,\pm]}$ for all $\nu \vdash n/2$ when n is even. We distinguish the degenerate irreducible characters by requiring that $\chi^{[\nu,+]}(w_{\mathrm{fpf}}) - \chi^{[\nu,-]}(w_{\mathrm{fpf}})$ be positive for the element $w_{\mathrm{fpf}} := s_1 s_3 s_5 \cdots s_{n-1}$[7]; by [11, Lem. 3.5] it then holds that

$$\chi^{[\nu,+]}(w_{\mathrm{fpf}}) - \chi^{[\nu,-]}(w_{\mathrm{fpf}}) = 2^{n/2}\chi^{\nu}(1). \tag{5.3}$$

This follows the convention of the data returned by the `CharacterTable("WeylD", n)` command in GAP [10].

In this notation, the linear characters of W^{D}_n are $\mathbb{1} = \chi^{\{(n),\emptyset\}}$ and $\mathrm{sgn} = \chi^{\{(1,1,\dots,1),\emptyset\}}$, together with $\mathbb{1}_{-+} = \chi^{[(1),+]}$ and $\mathbb{1}_{+-} = \chi^{[(1),-]}$ when $n = 2$. As explained in [11, Lem. 3.5] (correcting an error in [12, Rem. 5.6.5]), one has

$$\chi^{\{\lambda,\mu\}}\,\mathrm{sgn} = \chi^{\{\lambda^\top,\mu^\top\}} \quad \text{and} \quad \chi^{[\nu,\pm]}\,\mathrm{sgn} = \begin{cases} \chi^{[\nu^\top,\pm]} & \text{if } n/2 \text{ is even} \\ \chi^{[\nu^\top,\mp]} & \text{if } n/2 \text{ is odd.} \end{cases} \tag{5.4}$$

If $\lambda \neq \mu$ then $(\chi^{\{\lambda,\mu\}})^\diamond = \chi^{\{\lambda,\mu\}}$ and $(\chi^{[\nu,\pm]})^\diamond = \chi^{[\nu,\mp]}$. Finally, note that

$$\mathrm{Ind}^{W^{\mathsf{B}}_n}_{W^{\mathsf{D}}_n}(\chi^{\{\lambda,\mu\}}) = \chi^{(\lambda,\mu)} + \chi^{(\mu,\lambda)} \quad \text{and} \quad \mathrm{Ind}^{W^{\mathsf{B}}_n}_{W^{\mathsf{D}}_n}(\chi^{[\nu,\pm]}) = \chi^{(\nu,\nu)} \tag{5.5}$$

for all $\{\lambda,\mu\} \vdash n$ and $\nu \vdash n/2$ by Frobenius reciprocity.

Let $p, q \in \mathbb{N}$ and recall that $s_{-p} := (p,-p-1)(p+1,-p)$ if $p > 0$. Define

$$J = \begin{cases} \varnothing & \text{if } p \leq 1 \\ \{s_{-1}, s_1, \dots, s_{p-1}\} & \text{if } p \geq 2 \end{cases} \quad \text{and} \quad K = \begin{cases} \varnothing & \text{if } q \leq 1 \\ \{s_{-p-1}, s_{p+1}, s_{p+2}, \dots, s_{p+q-1}\} & \text{if } q \geq 2. \end{cases}$$

We identify $W^{\mathsf{D}}_p \times W^{\mathsf{D}}_q$ with the subgroup $\langle J \sqcup K \rangle \subseteq W^{\mathsf{D}}_{p+q}$. Write $u \times v$ for the image of $(u,v) \in W^{\mathsf{D}}_p \times W^{\mathsf{D}}_q$ in W^{D}_{p+q}. Given maps $f : W^{\mathsf{D}}_p \to \mathbb{C}$ and $g : W^{\mathsf{D}}_q \to \mathbb{C}$ define $f \boxtimes g : W^{\mathsf{D}}_p \times W^{\mathsf{D}}_q \to \mathbb{C}$ by the formula $u \times v \mapsto f(u)g(v)$. When f and g are class functions, let

$$f \bullet_{\mathsf{D}} g := \mathrm{Ind}^{W^{\mathsf{D}}_{p+q}}_{W^{\mathsf{D}}_p \times W^{\mathsf{D}}_q}(f \boxtimes g). \tag{5.6}$$

[7] The convention in [11, §3] for distinguishing $\chi^{[\nu,\pm]}$ is to require that $\chi^{[\nu,+]}(w_{\mathrm{fpf}}) - \chi^{[\nu,-]}(w_{\mathrm{fpf}})) = (-2)^{n/2}\chi^{\nu}(1)$. This gives the same labels as what we use if and only if n is a multiple of 4.

This is another associative and bilinear operation. If $\Theta = \left[\begin{smallmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{smallmatrix}\right] \in \mathsf{Index}(W_n^{\mathsf{D}})$ then

$$\chi^{\Theta}_{\mathsf{D}} = \chi_{\mathsf{D}}^{\left[\begin{smallmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{smallmatrix}\right]} \bullet_{\mathsf{D}} \mathrm{Ind}_{S_{\alpha_1}}^{W^{\mathsf{D}}_{\alpha_1}}\left(\chi_{\mathsf{A}}^{\left[\begin{smallmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{smallmatrix}\right]}\right) \quad \text{where we define } \chi_{\mathsf{D}}^{\left[\begin{smallmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{smallmatrix}\right]} := \chi_{\mathsf{D}}^{\left[\begin{smallmatrix} \alpha_0 & 0 \\ \beta_0 & \mathrm{id} \\ \gamma_0 & \mathbb{1} \end{smallmatrix}\right]} \tag{5.7}$$

and where we interpret the characters on the right as the trivial characters of $W_0^{\mathsf{D}} := \{1\}$ or $S_0 := \{1\}$ when $\alpha_0 = 0$ or $\alpha_1 = 0$. We need to explain how to evaluate this formula when $\alpha_0 = 0$ and $\alpha_1 = -n < 0$. For that case, we define $S_{-n} := S_n^{\diamond} = \langle s_{-1}, s_2, s_3, \dots, s_{n-1}\rangle \subseteq W_n^{\mathsf{D}}$, viewing $S_{-1} = \{1\}$, and extend any $f : S_n \to \mathbb{C}$ to a map $S_{-n} \cup S_n \to \mathbb{C}$ by setting $f(w) := f(w^{\diamond})$ for $w \in S_{-n}$. This is well-defined as $\diamond$ fixes every element of $S_{-n} \cap S_n \cong S_{n-1}$.

There are coefficients $d^{\Upsilon}_{\Lambda\Gamma} \in \mathbb{N}$ whenever Λ, Γ, and Υ are unordered bipartitions or symbols of the form $[\nu, \pm]$ such that

$$\chi^{\Lambda} \bullet_{\mathsf{D}} \chi^{\Gamma} = \sum_{\Upsilon} d^{\Upsilon}_{\Lambda\Gamma}\chi^{\Upsilon}. \tag{5.8}$$

Taylor [24, Prop. 2.7] has shown how to express these numbers in terms of the Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$ from Section 3.3. Namely, if $\lambda_1 \neq \lambda_2$, λ, $\mu_1 \neq \mu_2$, μ, $\nu_1 \neq \nu_2$, ν are partitions and $\epsilon_\lambda, \epsilon_\mu, \epsilon_\nu \in \{\pm\}$ are signs then we have:

(a) $d^{\{\nu_1,\nu_2\}}_{\{\lambda_1,\lambda_2\}\{\mu_1,\mu_2\}} = c^{\nu_1}_{\lambda_1\mu_1}c^{\nu_2}_{\lambda_2\mu_2} + c^{\nu_1}_{\lambda_1\mu_2}c^{\nu_2}_{\lambda_2\mu_1} + c^{\nu_1}_{\lambda_2\mu_1}c^{\nu_2}_{\lambda_1\mu_2} + c^{\nu_1}_{\lambda_2\mu_2}c^{\nu_2}_{\lambda_2\mu_2}$,

(b) $d^{\{\nu_1,\nu_2\}}_{[\lambda,\epsilon_\lambda]\{\mu_1,\mu_2\}} = d^{\{\nu_1,\nu_2\}}_{\{\mu_1,\mu_2\}[\lambda,\epsilon_\lambda]} = c^{\nu_1}_{\lambda\mu_1}c^{\nu_2}_{\lambda\mu_2} + c^{\nu_1}_{\lambda\mu_2}c^{\nu_2}_{\lambda\mu_1}$,

(c) $d^{\{\nu_1,\nu_2\}}_{[\lambda,\epsilon_\lambda][\mu,\epsilon_\mu]} = c^{\nu_1}_{\lambda\mu}c^{\nu_2}_{\lambda\mu}$,

(d) $d^{[\nu,\epsilon_\nu]}_{\{\lambda_1,\lambda_2\}\{\mu_1,\mu_2\}} = c^{\nu}_{\lambda_1\mu_1}c^{\nu}_{\lambda_2\mu_2} + c^{\nu}_{\lambda_1\mu_2}c^{\nu}_{\lambda_2\mu_1}$,

(e) $d^{[\nu,\epsilon_\nu]}_{[\lambda,\epsilon_\lambda]\{\mu_1,\mu_2\}} = d^{[\nu,\epsilon_\nu]}_{\{\mu_1,\mu_2\}[\lambda,\epsilon_\lambda]} = c^{\nu}_{\lambda\mu_1}c^{\nu}_{\lambda\mu_2}$, and

(f) $d^{[\nu,\epsilon_\nu]}_{[\lambda,\epsilon_\lambda][\mu,\epsilon_\mu]} = \frac{1}{2}c^{\nu}_{\lambda\mu}(c^{\nu}_{\lambda\mu} + \epsilon_1\epsilon_2\epsilon_3)$ where to evaluate $\epsilon_1\epsilon_2\epsilon_3 \in \{\pm 1\}$ we replace $\pm$ by ± 1.[8]

Let $\nu \vdash n$. In view of (4.6) and (5.5), if n is odd then

$$\mathrm{Ind}_{S_n}^{W_n^{\mathsf{D}}}(\chi^\nu) = \mathrm{Ind}_{S_{-n}}^{W_n^{\mathsf{D}}}(\chi^\nu) = \sum_{\{\lambda,\mu\}\vdash n} c^{\nu}_{\lambda\mu}\chi^{\{\lambda,\mu\}} \tag{5.9}$$

while if n is even then there are numbers $c^{\nu}_{[\lambda,\pm]} \in \mathbb{N}$ for $\lambda \vdash \frac{n}{2}$ with $c^{\nu}_{[\lambda,+]} + c^{\nu}_{[\lambda,-]} = c^{\nu}_{\lambda\lambda}$ and

$$\mathrm{Ind}_{S_{\pm n}}^{W_n^{\mathsf{D}}}(\chi^\nu) = \sum_{\{\lambda,\mu\}\vdash n} c^{\nu}_{\lambda\mu}\chi^{\{\lambda,\mu\}} + \sum_{\lambda\vdash\frac{n}{2}} \left(c^{\nu}_{[\lambda,+]}\chi^{[\lambda,\pm]} + c^{\nu}_{[\lambda,-]}\chi^{[\lambda,\mp]}\right). \tag{5.10}$$

As noted in [24, §1], it appears to be an open problem to give a general formula for $c^{\nu}_{[\lambda,\pm]}$. One can compute these numbers without much difficulty when $c^{\nu}_{\lambda\lambda} = 1$. For example:

Proposition 5.1. If n is even then $\mathrm{Ind}_{S_{\pm n}}^{W_n^{\mathsf{D}}}(\mathbb{1}) = \sum_{p=0}^{\frac{n}{2}-1} \chi^{\{(p),(n-p)\}} + \chi^{[(\frac{n}{2}),\pm]}$ and

$$\mathrm{Ind}_{S_{\pm n}}^{W_n^{\mathsf{D}}}(\mathrm{sgn}) = \sum_{p=0}^{\frac{n}{2}-1} \chi^{\{(1^p),(1^{n-p})\}} + \begin{cases} \chi^{[(1^{n/2}),\pm]} & \text{if } n/2 \text{ is even} \\ \chi^{[(1^{n/2}),\mp]} & \text{if } n/2 \text{ is odd.}\end{cases}$$

Proof. The second formula follows from the first by (5.4). In view of (4.7) we just need to show that $\chi^{[(\frac{n}{2}),+]}$ is a constituent of $\mathrm{Ind}_{S_n}^{W_n^{\mathsf{D}}}(\mathbb{1})$ and $\chi^{[(\frac{n}{2}),-]}$ is a constituent of $\mathrm{Ind}_{S_{-n}}^{W_n^{\mathsf{D}}}(\mathbb{1})$. For this, it is enough by (5.3) to check that $\mathrm{Ind}_{S_n}^{W_n^{\mathsf{D}}}(\mathbb{1})(w_{\mathrm{fpf}}) - \mathrm{Ind}_{S_{-n}}^{W_n^{\mathsf{D}}}(\mathbb{1})(w_{\mathrm{fpf}}) > 0$ for $w_{\mathrm{fpf}} := s_1 s_3 \cdots s_{n-1}$. This follows from (2.1) since $g \cdot w_{\mathrm{fpf}} \cdot g^{-1} \notin S_{-n}$ for all $g \in W_n^{\mathsf{D}}$. ☐

[8] This formula still holds if one defines $\chi^{[\nu,\pm]}$ such that $\chi^{[\nu,+]}(w_{\mathrm{fpf}}) - \chi^{[\nu,-]}(w_{\mathrm{fpf}}) = (-2)^{n/2}\chi^{\nu}(1)$ as in [11, §3], since switching to this convention would reverse either zero or two of the signs ϵ_1, ϵ_2, ϵ_3.

Let $\mathsf{ERows}_D(n)$ be the set of unordered bipartitions $\{\lambda,\mu\} \vdash n$ where $\lambda \neq \mu$ have all even parts. Let $\mathsf{ECols}_D(n) = \{\{\lambda^\top,\mu^\top\} : \{\lambda,\mu\} \in \mathsf{ERows}_D(n)\}$.

Proposition 5.2. Suppose n is even, $\beta \in \{\mathrm{fpf}, \mathrm{fpf}^\diamond\}$, and $\Theta = \left[\begin{smallmatrix} n \\ \beta \\ \gamma \end{smallmatrix}\right]$. If $\gamma = \mathbb{1}$ then

$$\chi^\Theta_{\mathsf{D}} = \sum_{\{\lambda,\mu\}\in\mathsf{ERows}_D(n)} \chi^{\{\lambda,\mu\}} + \begin{cases} \sum_{\nu\in\mathsf{ERows}(n/2)} \chi^{[\nu,+]} & \text{if } \beta = \mathrm{fpf} \\ \sum_{\nu\in\mathsf{ERows}(n/2)} \chi^{[\nu,-]} & \text{if } \beta = \mathrm{fpf}^\diamond \end{cases}$$

and if $\gamma = \mathrm{sgn}$ then

$$\chi^\Theta_{\mathsf{D}} = \sum_{\{\lambda,\mu\}\in\mathsf{ECols}_D(n)} \chi^{\{\lambda,\mu\}} + \begin{cases} \sum_{\nu\in\mathsf{ECols}(n/2)} \chi^{[\nu,+]} & \text{if } \beta = \mathrm{fpf} \\ \sum_{\nu\in\mathsf{ECols}(n/2)} \chi^{[\nu,-]} & \text{if } \beta = \mathrm{fpf}^\diamond \end{cases}$$

where the sums over $\nu \in \mathsf{ERows}(n/2)$ and $\nu \in \mathsf{ECols}(n/2)$ are zero when $n \not\equiv 0 \pmod 4$.

Proof. Since $\chi^{\Theta^\diamond}_{\mathsf{D}} = (\chi^\Theta_{\mathsf{D}})^\diamond$ and $\chi^{\overline{\Theta}}_{\mathsf{D}} = \mathrm{sgn}\,\chi^\Theta_{\mathsf{D}}$, we may assume that $\beta = \mathrm{fpf}$ and $\gamma = \mathbb{1}$. The centralizers of $s_1s_3\cdots s_{n-1}$ in W^{D}_n and W^{B}_n coincide, so $\mathrm{Ind}_{W^{\mathsf{D}}_n}^{W^{\mathsf{B}}_n}(\chi^\Theta_{\mathsf{D}}) = \chi^\Theta_{\mathsf{B}}$. By (4.8) and (5.5) we therefore have $\chi^\Theta_{\mathsf{D}} = \sum_{\{\lambda,\mu\}\in\mathsf{ERows}_D(n)} \chi^{\{\lambda,\mu\}} + \sum_{\nu\in\mathsf{ERows}(n/2)} \chi^{[\nu,\epsilon_\nu]}$ for some choice of $\epsilon_\nu \in \{\pm\}$. To show that ϵ_ν is always $+$ it suffices by (5.3) to check that if n is divisible by 4 then

$$\mathrm{Ind}_{C_{\mathrm{fpf}}}^{W^{\mathsf{D}}_n}(\mathbb{1})(w_{\mathrm{fpf}}) - \mathrm{Ind}_{C^\diamond_{\mathrm{fpf}}}^{W^{\mathsf{D}}_n}(\mathbb{1})(w_{\mathrm{fpf}}) = 2^{n/2} \sum_{\nu\in\mathsf{ERows}(n/2)} \chi^\nu(1)$$

where C_{fpf} is the W^{D}_n-centralizer of $w_{\mathrm{fpf}} = s_1s_3\cdots s_{n-1}$ and $C^\diamond_{\mathrm{fpf}} = (C_{\mathrm{fpf}})^\diamond$ is the W^{D}_n-centralizer of $w^\diamond_{\mathrm{fpf}} := s_{-1}s_3\cdots s_{n-1}$. Because $|C_{\mathrm{fpf}}| = |C^\diamond_{\mathrm{fpf}}|$, the induced character formula (2.1) gives

$$\begin{aligned} \mathrm{Ind}_{C_{\mathrm{fpf}}}^{W^{\mathsf{D}}_n}(\mathbb{1})(w_{\mathrm{fpf}}) - \mathrm{Ind}_{C^\diamond_{\mathrm{fpf}}}^{W^{\mathsf{D}}_n}(\mathbb{1})(w_{\mathrm{fpf}}) &= |\mathcal{K}^{W^{\mathsf{D}}_n}_{\mathrm{fpf}} \cap C_{\mathrm{fpf}}| - |\mathcal{K}^{W^{\mathsf{D}}_n}_{\mathrm{fpf}} \cap C^\diamond_{\mathrm{fpf}}| \\ &= |\mathcal{K}^{W^{\mathsf{D}}_n}_{\mathrm{fpf}} \cap C_{\mathrm{fpf}}| - |\mathcal{K}^{W^{\mathsf{D}}_n}_{\mathrm{fpf}^\diamond} \cap C_{\mathrm{fpf}}|. \end{aligned}$$

Let $A^+_i := \{2i-1, 2i\}$, $A^-_i := \{1-2i, -2i\}$, and $A_i := A^+_i \sqcup A^-_i$ for $i \in [n/2]$. Each $w \in C_{\mathrm{fpf}}$ determines a permutation of $A_1, A_2, \ldots, A_{n/2}$. If $w \in \mathcal{K}^{W^{\mathsf{D}}_n}_{\mathrm{fpf}} \cap C_{\mathrm{fpf}}$ (respectively, $w \in \mathcal{K}^{W^{\mathsf{D}}_n}_{\mathrm{fpf}^\diamond} \cap C_{\mathrm{fpf}}$) then this permutation is an involution with $w(A^+_i) = A^\pm_j$ whenever $w(A_i) = A_j$, such that the number of $i \in [n]$ with $w(A^+_i) = A^-_i$ is even (respectively, odd). These properties characterize the elements of $\mathcal{K}^{W^{\mathsf{D}}_n}_{\mathrm{fpf}} \cap C_{\mathrm{fpf}}$ and $\mathcal{K}^{W^{\mathsf{D}}_n}_{\mathrm{fpf}^\diamond} \cap C_{\mathrm{fpf}}$, and so there is a bijection from $\mathcal{K}^{W^{\mathsf{D}}_n}_{\mathrm{fpf}^\diamond} \cap C_{\mathrm{fpf}}$ to the subset of elements $w \in \mathcal{K}^{W^{\mathsf{D}}_n}_{\mathrm{fpf}} \cap C_{\mathrm{fpf}}$ with $w(A_i) = A_i$ for at least one $i \in [n/2]$. Thus $|\mathcal{K}^{W^{\mathsf{D}}_n}_{\mathrm{fpf}} \cap C_{\mathrm{fpf}}| - |\mathcal{K}^{W^{\mathsf{D}}_n}_{\mathrm{fpf}^\diamond} \cap C_{\mathrm{fpf}}|$ is the number of $w \in W^{\mathsf{D}}_n$ with $A_i = w^2(A_i) \neq w(A_i) \in \{A_1, A_2, \ldots, A_{n/2}\}$ for each $i \in [n/2]$, such that if $w(A_i) = A_j$ then $w : A_i \to A_j$ is any of the four bijections with $w(A^+_i) = A^\pm_j$. The number of such permutations is $4^{n/4}$ times the number of fixed-point-free involutions of $[n/2]$. This product is $2^{n/2} \sum_{\nu\in\mathsf{ERows}(n/2)} \chi^\nu(1)$ by (3.3) as needed. □

Proposition 5.3. Suppose p and q are positive integers with $p + q = n$. Then

$$\chi_{\mathsf{D}}^{\left[\begin{smallmatrix} n \\ (p,q) \\ \mathbb{1} \end{smallmatrix}\right]} = \sum_{j=0}^{\min(p,q)} \chi^{\{(n-j,j),\emptyset\}} \quad\text{and}\quad \chi_{\mathsf{D}}^{\left[\begin{smallmatrix} n \\ (p,q) \\ \mathrm{sgn} \end{smallmatrix}\right]} = \sum_{j=0}^{\min(p,q)} \chi^{\{(2^j,1^{n-j}),\emptyset\}}.$$

Proof. Let $\Theta := \begin{bmatrix} n \\ (p,q) \\ \mathbb{1} \end{bmatrix}$. For either parity of q, the centralizer of the unique minimal-length element of $\mathcal{K}^{W_n^{\mathsf{D}}}_{(p,q)}$ in W_n^{D} is the intersection $H := (W_q^{\mathsf{B}} \times W_p^{\mathsf{B}}) \cap W_n^{\mathsf{D}}$. Let $K := W_q^{\mathsf{B}} \times W_p^{\mathsf{B}}$. By Frobenius reciprocity we have $\mathrm{Ind}_H^K(\mathbb{1}) = \chi^{((q),\emptyset)} \boxtimes \chi^{((p),\emptyset)} + \chi^{(\emptyset,(q))} \boxtimes \chi^{(\emptyset,(p))}$ so

$$\mathrm{Ind}_{W_n^{\mathsf{D}}}^{W_n^{\mathsf{B}}}(\chi_{\mathsf{D}}^{\Theta}) = \mathrm{Ind}_{W_n^{\mathsf{D}}}^{W_n^{\mathsf{B}}}\mathrm{Ind}_H^{W_n^{\mathsf{D}}}(\mathbb{1}) = \mathrm{Ind}_K^{W_n^{\mathsf{B}}}\mathrm{Ind}_H^K(\mathbb{1}) = \sum_{j=0}^{\min(p,q)} (\chi^{((n-j,j),\emptyset)} + \chi^{((n-j,j),\emptyset)})$$

by (4.5). The formula for $\chi_{\mathsf{D}}^{\Theta}$ follows by (5.5). The other formula holds since $\chi_{\mathsf{D}}^{\overline{\Theta}} = \chi_{\mathsf{D}}^{\Theta}\,\mathrm{sgn}$. $\square$

Our last proposition records a calculation we performed in the algebra system GAP [10]:

Proposition 5.4. Suppose $\beta \in \{(1,3,\theta),(3,1,\theta)\}$ where $\theta \in \{\circlearrowright,\circlearrowleft\}$. Then

$$\chi_{\mathsf{D}}^{\begin{bmatrix} 4 \\ \beta \\ \mathbb{1} \end{bmatrix}} = \begin{cases} \chi^{\{(4),\emptyset\}} + \chi^{[(2),+]} & \text{if } \theta = \circlearrowright \\ \chi^{\{(4),\emptyset\}} + \chi^{[(2),-]} & \text{if } \theta = \circlearrowleft \end{cases} \quad \text{and} \quad \chi_{\mathsf{D}}^{\begin{bmatrix} 4 \\ \beta \\ \mathrm{sgn} \end{bmatrix}} = \begin{cases} \chi^{\{(1,1,1,1),\emptyset\}} + \chi^{[(1,1),+]} & \text{if } \theta = \circlearrowright \\ \chi^{\{(1,1,1,1),\emptyset\}} + \chi^{[(1,1),-]} & \text{if } \theta = \circlearrowleft. \end{cases}$$

5.4 Model projections in type D

We define a map $\pi_{\mathsf{D}} : \mathsf{Index}(W_n^{\mathsf{D}}) \to \mathsf{Index}(S_n) \sqcup \{0\}$. Let $\Theta = \begin{bmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{bmatrix} \in \mathsf{Index}(W_n^{\mathsf{D}})$. First set

$$\pi_{\mathsf{D}}(\Theta) := \begin{bmatrix} |\alpha_1| \\ \beta_1 \\ \gamma_1 \end{bmatrix} \text{ if } \alpha_0 = 0 \quad \text{and} \quad \pi_{\mathsf{D}}(\Theta) := 0 \text{ if } \gamma_0 \notin \{\mathbb{1}, \mathrm{sgn}\} \text{ (only possible if } \alpha_0 = 2).$$

Assume $\alpha_0 \neq 0$ and $\gamma_0 \in \{\mathbb{1}, \mathrm{sgn}\}$. If $\alpha_1 \neq 0$ then we define

$$\pi_{\mathsf{D}}(\Theta) := \begin{cases} \begin{bmatrix} \alpha_0 & \alpha_1 \\ \mathrm{fpf} & \beta_1 \\ \gamma_0 & \gamma_1 \end{bmatrix} & \text{if } \beta_0 \in \{\mathrm{fpf}, \mathrm{fpf}^\diamond\} \\ \begin{bmatrix} \alpha_0 & \alpha_1 \\ \mathrm{id} & \beta_1 \\ \gamma_0 & \gamma_1 \end{bmatrix} & \text{if } \beta_0 \in \{\mathrm{id}, \mathrm{id}^+, (1,3,\circlearrowright), (1,3,\circlearrowleft), (3,1,\circlearrowright), (3,1,\circlearrowleft)\} \\ \begin{bmatrix} p & q & \alpha_1 \\ \mathrm{id} & \mathrm{id} & \beta_1 \\ \gamma_0 & \gamma_0 & \gamma_1 \end{bmatrix} & \text{if } \beta_0 = (p,q) \text{ for } p,q > 0 \text{ with } p+q = \alpha_0. \end{cases}$$

When $\alpha_1 = 0$, we form $\pi_{\mathsf{D}}(\Theta)$ by applying the same formula, and then deleting the last column if the result is nonzero.

Define $\mathcal{R}^{\mathsf{D}} := \bigoplus_{n\in\mathbb{N}} \mathcal{R}_n^{\mathsf{D}}$ where $\mathcal{R}_n^{\mathsf{D}}$ is the $\mathbb{C}$-vector space of class functions $W_n^{\mathsf{D}} \to \mathbb{C}$. We use the same symbol π_{D} to denote the linear map $\mathcal{R}^{\mathsf{D}} \to \mathcal{R}^{\mathsf{A}}$ with

$$\pi_{\mathsf{D}}(\chi^{[\lambda,\pm]}) := 0 \quad \text{and} \quad \pi_{\mathsf{D}}(\chi^{\{\lambda,\mu\}}) := \begin{cases} \chi^{\lambda} & \text{if } \mu = \emptyset \\ \chi^{\mu} & \text{if } \lambda = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for all partitions $\lambda \neq \mu$. Finally, set $\chi_{\mathsf{A}}^0 := 0 \in \mathcal{R}^{\mathsf{A}}$.

Lemma 5.5. If $\Theta \in \mathsf{Index}(W_n^{\mathsf{D}})$ then $\pi_{\mathsf{D}}(\chi_{\mathsf{D}}^{\Theta}) = \chi_{\mathsf{A}}^{\pi_{\mathsf{D}}(\Theta)}$.

Proof. The identities (5.8)–(5.10) imply that $\pi_{\mathsf{D}}(\chi \bullet_{\mathsf{D}} \mathrm{Ind}_{S_n}^{W_n^{\mathsf{B}}}(\psi)) = \pi_{\mathsf{D}}(\chi) \bullet_{\mathsf{A}} \psi$ for $\chi \in \mathcal{R}^{\mathsf{D}}$ and $\psi \in \mathcal{R}_n^{\mathsf{A}}$. Given this together with (3.2) and (5.7), to show that $\pi_{\mathsf{D}}(\chi_{\mathsf{D}}^{\Theta}) = \chi_{\mathsf{A}}^{\pi_{\mathsf{D}}(\Theta)}$ for all $\Theta \in \mathsf{Index}(W_n^{\mathsf{B}})$ it suffices to prove this identity for model indices of the form $\Theta = \begin{bmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{bmatrix}$. This follows from the definition of π_{D} on comparing Proposition 3.2 with Propositions 5.2, 5.3, and 5.4. $\square$

5.5 Perfect models in type D

Fix a model index $\Theta = \begin{bmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{bmatrix} \in \mathsf{Index}(W_n^{\mathsf{D}})$.

Lemma 5.6. The character $\chi_{\mathsf{D}}^{\Theta}$ is not multiplicity-free if any of the following conditions hold:

(a) $\alpha_1 \in \{4, 6, 8, \dots\}$ and $\beta_1 \in \{\mathrm{fpf}, \mathrm{fpf}^+\}$.

(b) $\alpha_0 \in \{4, 6, \dots\}$, $\beta_0 \in \{\mathrm{fpf}, \mathrm{fpf}^\diamond\}$, $\alpha_1 \geq 2$, and $\gamma_0 = \gamma_1 \in \{\mathbb{1}, \mathrm{sgn}\}$.

(c) $\beta_0 = (p, q)$ for some $p, q > 0$ with $p + q = \alpha_0 \geq 3$ and $\alpha_1 \geq 1$.

Proof. Suppose (a) holds and let $\Psi := \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix}$ and $m := \alpha_1$. As in the proof of Lemma 4.3, it is enough by (5.7) to show that the character $\mathrm{Ind}_{S_m}^{W_m^{\mathsf{D}}}(\chi_{\mathsf{A}}^{\Psi})$ is not multiplicity-free when $\gamma_1 = \mathbb{1}$. When $m = 4$ this can be checked by hand or using a computer algebra system. The proof of Lemma 4.3 shows that $\mathrm{Ind}_{W_m^{\mathsf{D}}}^{W_m^{\mathsf{B}}}(\mathrm{Ind}_{S_m}^{W_m^{\mathsf{D}}}(\chi_{\mathsf{A}}^{\Psi}))$ contains $\chi^{((m-2),(2))}$ with multiplicity at least two, and when $m > 4$ this can only occur in view of (5.5) if $\chi^{\{(m-2),(2)\}}$ appears with multiplicity at least two in $\mathrm{Ind}_{S_m}^{W_m^{\mathsf{D}}}(\chi_{\mathsf{A}}^{\Psi})$.

If (b) or (c) holds then $\pi_{\mathsf{D}}(\chi_{\mathsf{D}}^{\Theta}) = \chi_{\mathsf{A}}^{\pi_{\mathsf{D}}(\Theta)} \neq 0$ is not multiplicity-free by Lemma 3.1, so $\chi_{\mathsf{D}}^{\Theta}$ must also not be multiplicity-free. $\square$

Let $\mathsf{ORows}_D(n, q)$ be the set of unordered bipartitions $\{\lambda, \mu\} \vdash n$ such that $\lambda \cup \mu$ has exactly q odd parts. Define $\mathsf{OCols}_D(n, q) = \{\{\lambda^\top, \mu^\top\} : \{\lambda, \mu\} \in \mathsf{ORows}_D(n, q)\}$.

Proposition 5.7. Suppose $\Theta \in \mathsf{Index}(W_n^{\mathsf{D}})$ and $\chi_{\mathsf{D}}^{\Theta}$ is multiplicity-free. Then Θ has one of the following forms:

(a) $\begin{bmatrix} k & n-k \\ \mathrm{id}/\mathrm{id}^+ & \mathrm{id}/\mathrm{id}^+ \\ \gamma_0 & \gamma_0 \end{bmatrix}$ for some $k \in \{3, 4, \dots, n\}$ and $\gamma_0, \gamma_1 \in \{\mathbb{1}, \mathrm{sgn}\}$.

(b) $\begin{bmatrix} 2 & n-2 \\ \mathrm{id}/\mathrm{id}^+/\mathrm{fpf}/\mathrm{fpf}^\diamond & \mathrm{id}/\mathrm{id}^+ \\ \gamma_0 & \gamma_1 \end{bmatrix}$ for some $\gamma_0 \in \{\mathbb{1}, \mathrm{sgn}, \mathbb{1}_{+-}, \mathbb{1}_{-+}\}$, and $\gamma_1 \in \{\mathbb{1}, \mathrm{sgn}\}$.

(c) $\begin{bmatrix} 0 & n \\ \mathrm{id} & \mathrm{id}/\mathrm{id}^+ \\ \mathbb{1} & \gamma_1 \end{bmatrix}$ or $\begin{bmatrix} 0 & -n \\ \mathrm{id} & \mathrm{id}/\mathrm{id}^+ \\ \mathbb{1} & \gamma_1 \end{bmatrix}$ for some $\gamma_1 \in \{\mathbb{1}, \mathrm{sgn}\}$.

(d) $\begin{bmatrix} 2k & n-2k \\ \mathrm{fpf}/\mathrm{fpf}^\diamond & \mathrm{id}/\mathrm{id}^+ \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix}$ for some $0 \leq k \leq \lfloor n/2 \rfloor$, in which case

$$\chi_{\mathsf{D}}^{\Theta} = \sum_{\{\lambda,\mu\} \in \mathsf{ORows}_D(n, n-2k)} \chi^{\{\lambda,\mu\}} + \sum_{\nu \in \mathsf{ORows}(\frac{n}{2}, \frac{n}{2}-k)} \chi^{[\nu,\epsilon_\nu]}$$

for some choice of signs $\epsilon_\nu \in \{\pm\}$, where the second sum is zero if n is odd.

(e) $\begin{bmatrix} 2k & n-2k \\ \mathrm{fpf}/\mathrm{fpf}^\diamond & \mathrm{id}/\mathrm{id}^+ \\ \mathrm{sgn} & \mathbb{1} \end{bmatrix}$ for some $0 \leq k \leq \lfloor n/2 \rfloor$, in which case

$$\chi_{\mathsf{D}}^{\Theta} = \sum_{\{\lambda,\mu\} \in \mathsf{OCols}_D(n, n-2k)} \chi^{\{\lambda,\mu\}} + \sum_{\nu \in \mathsf{OCols}(\frac{n}{2}, \frac{n}{2}-k)} \chi^{[\nu,\epsilon_\nu]}$$

for some choice of signs $\epsilon_\nu \in \{\pm\}$, where the second sum is zero if n is odd.

(f) $\begin{bmatrix} n & 0 \\ (p,q) & \mathrm{id} \\ \gamma_0 & \mathbb{1} \end{bmatrix}$ for some $p, q > 0$ such that $2 < p + q = n$ and $\gamma_0 \in \{\mathbb{1}, \mathrm{sgn}\}$.

(g) $\begin{bmatrix} 4 & n-4 \\ \beta & \mathrm{id}/\mathrm{id}^+ \\ \gamma_0 & \gamma_1 \end{bmatrix}$ for some $\beta \in \{(1, 3, \circlearrowright), (1, 3, \circlearrowleft), (3, 1, \circlearrowright), (3, 1, \circlearrowleft)\}$ and $\gamma_0, \gamma_1 \in \{\mathbb{1}, \mathrm{sgn}\}$.

The signs in parts (d) and (e) can be determined using (5.8) and Propositions 5.1 and 5.2, but we will not need this for our applications.

Proof. The given cases account for all model indices in $\mathsf{Index}(W_n^{\mathsf{D}})$ not excluded by Lemma 5.6. The formulas in parts (d) and (e) follow by combining (5.7) with Propositions 5.1 and 5.2. $\square$

Theorem 5.8. Assume $n \geq 4$. If n is even then W_n^{D} has no perfect models. If n is odd then

$$\mathcal{P}_n^{\mathsf{D}} := \left\{ \mathbb{T}^\Theta : \Theta = \begin{bmatrix} 2k & n-2k \\ \mathrm{fpf} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix} \text{ for } 0 \leq k \leq \lfloor \tfrac{n}{2} \rfloor \right\}$$

is a perfect model for W_n^{D}, and each perfect model for W_n^{D} is strongly equivalent to $\mathcal{P}_n^{\mathsf{D}}$ or $\overline{\mathcal{P}_n^{\mathsf{D}}}$.

The equivalence class of $\mathcal{P}_3^{\mathsf{D}}$ gives the extra models for S_4 described in Example 3.4.

Proof. Our argument is similar to the proof of Theorem 4.5. When n is odd it is clear from Proposition 5.7 that $\mathcal{P}_n^{\mathsf{D}}$ is a perfect model for W_n^{D}.

Suppose $\mathcal{M}$ is a set of model indices $\Theta \in \mathsf{Index}(W_n^{\mathsf{D}})$ such that $\{\mathbb{T}^\Theta : \Theta \in \mathcal{M}\}$ is a perfect model for W_n^{D}. Every perfect model for W_n^{D} arises in this way. Define $\mathcal{M}_{\mathsf{A}} := \{\pi_{\mathsf{D}}(\Theta) : \Theta \in \mathcal{M}\} \setminus \{0\}$. Then $\{\mathbb{T}^\Theta : \Theta \in \mathcal{M}_{\mathsf{A}}\}$ is a perfect model for S_n by Lemma 5.5. After possibly replacing $\mathcal{M}$ by $\overline{\mathcal{M}} := \{\overline{\Theta} : \Theta \in \mathcal{M}\}$, we may assume that $\{\mathbb{T}^\Theta : \Theta \in \mathcal{M}_{\mathsf{A}}\}$ is a sgn-model.

Again let $\Theta_k := \begin{bmatrix} 2k & n-2k \\ \mathrm{fpf} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix}$ for $0 \leq k \leq \lfloor n/2 \rfloor$. If n is odd then we have

$$\Theta_k \sim \begin{bmatrix} 2k & n-2k \\ \mathrm{fpf} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix}^\diamond = \begin{bmatrix} 2k & n-2k \\ \mathrm{fpf}^\diamond & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix} \sim \begin{bmatrix} 2k & n-2k \\ \mathrm{fpf} & \mathrm{id}^+ \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix} \sim \begin{bmatrix} 2k & n-2k \\ \mathrm{fpf}^\diamond & \mathrm{id}^+ \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix}.$$

Since W_2^{D} is abelian we have

$$\Theta_2 \equiv \begin{bmatrix} 2 & n-2 \\ \beta_0 & \beta_1 \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix} \quad \text{for all } \beta_0 \in \{\mathrm{id}, \mathrm{id}^+, \mathrm{fpf}, \mathrm{fpf}^\diamond\} \text{ and } \beta_1 \in \{\mathrm{id}, \mathrm{id}^+\}.$$

Now assume $n \geq 5$ is odd. Remark 3.5 tells us that $\mathcal{M}_{\mathsf{A}}$ must contain elements of each of the forms (3.6), (3.7), (3.8), and (3.9). By considering the limited possibilities for model indices $\Theta \in \mathsf{Index}(W_n^{\mathsf{D}})$ with χ_{D}^Θ multiplicity-free that can serve as the preimages for these elements under π_{D}, we deduce from Proposition 5.7 that $\mathcal{M}$ must contain a unique model index strongly equivalent to Θ_k for at least each $2 \leq k \leq \lfloor n/2 \rfloor$.

By similar reasoning, for $\mathcal{M}_{\mathsf{A}}$ to contain an index of the form (3.4), $\mathcal{M}$ must contain a unique element strongly equivalent to $\Theta_0 = \begin{bmatrix} 0 & n \\ \mathrm{fpf} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix} \sim \begin{bmatrix} 0 & n \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix} \sim \begin{bmatrix} 0 & -n \\ \mathrm{id} & \mathrm{id} \\ \mathbb{1} & \mathrm{sgn} \end{bmatrix}$ or $\Psi_0 := \begin{bmatrix} n & 0 \\ \mathrm{id} & \mathrm{id} \\ \mathrm{sgn} & \mathbb{1} \end{bmatrix}$, and for $\mathcal{M}_{\mathsf{A}}$ to contain an index of the form (3.5), $\mathcal{M}$ must contain a unique element strongly equivalent to Θ_1 or $\Psi_1 := \begin{bmatrix} n-2 & 2 \\ \mathrm{id} & \mathrm{id} \\ \mathrm{sgn} & \mathbb{1} \end{bmatrix}$.

Thus, $\mathcal{M}$ contains a subset of model indices strongly equivalent to $\mathcal{M}^0 \sqcup \mathcal{M}^1 \sqcup \mathcal{M}^2$ where $\mathcal{M}^0$ is either $\{\Theta_0\}$ or $\{\Psi_0\}$, $\mathcal{M}^1$ is either $\{\Theta_1\}$ or $\{\Psi_1\}$, and $\mathcal{M}^2 := \{\Theta_k : 2 \leq k \leq \lfloor n/2 \rfloor\}$. Moreover, all elements $\Phi \in \mathcal{M}$ outside this set must have $\pi_{\mathsf{D}}(\Phi) = 0$, so are of the form

$$\Phi = \begin{bmatrix} 2 & n-2 \\ \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{bmatrix} \quad \text{for some } \gamma_0 \in \{\mathbb{1}_{-+}, \mathbb{1}_{+-}\} \text{ and } \gamma_1 \in \{\mathbb{1}, \mathrm{sgn}\}, \tag{5.11}$$

where $\beta_0 \in \{\mathrm{id}, \mathrm{id}^+, \mathrm{fpf}, \mathrm{fpf}^\diamond\}$ and $\beta_1 \in \{\mathrm{id}, \mathrm{id}^+\}$ are arbitrary. If $\mathcal{M}^0 = \{\Theta_0\}$ and $\mathcal{M}^1 = \{\Theta_1\}$ then $\mathcal{M}^0 \sqcup \mathcal{M}^1 \sqcup \mathcal{M}^2 = \mathcal{P}_n^{\mathsf{D}}$ so $\mathcal{M} \sim \mathcal{P}_n^{\mathsf{D}}$. If this does not occur then the character $\sum_{\Theta \in \mathcal{M}^0 \sqcup \mathcal{M}^1 \sqcup \mathcal{M}^2} \chi_{\mathsf{D}}^\Theta$ is missing several irreducible constituents. Specifically, we know that

$$\chi_{\mathsf{D}}^{\Theta_0} = \sum_{p+q=n} \chi^{\{(1^p),(1^q)\}} \quad \text{and} \quad \chi_{\mathsf{D}}^{\Theta_1} = \sum_{p+q=n-2} \chi^{\{(2,1^p),(1^q)\}} + \sum_{p+q=n-3} \chi^{\{(3,1^p),(1^q)\}}$$

by Proposition 5.7, but one can compute using (5.8) that $\chi_{\mathsf{D}}^{\Psi_0} = \chi^{\{(1^n),\emptyset\}}$ and

$$\chi_{\mathsf{D}}^{\Psi_1} = \chi^{\{(1^{n-2}),(2)\}} + \chi^{\{(2,1^{n-3}),(1)\}} + \chi^{\{(1^{n-1}),(1)\}} + \chi^{\{(3,1^{n-3}),\emptyset\}} + \chi^{\{(2,1^{n-2}),\emptyset\}}.$$

All irreducible constituents of $\chi_{\mathsf{D}}^{\Theta_0} - \chi_{\mathsf{D}}^{\Psi_0}$ and $\chi_{\mathsf{D}}^{\Theta_1} - \chi_{\mathsf{D}}^{\Psi_1}$ must be accounted for in $\sum_{\Theta\in\mathcal{M}} \chi_{\mathsf{D}}^{\Theta}$. However, it is impossible for these constituents to come from model indices of the form (5.11). Indeed, it follows from (5.8) that the character of any such Φ is either

$$\chi_{\mathsf{D}}^{\Phi} = \chi^{[(1),\pm]} \bullet_{\mathsf{D}} \sum_{p+q=n-2} \chi^{\{(1^p),(1^q)\}} = \sum_{\substack{\{\lambda,\mu\}\vdash n \\ \lambda,\mu\in\mathcal{H}}} \chi^{\{\lambda,\mu\}} \tag{5.12}$$

where $\mathcal{H}$ is the set of partitions of the form (1^{k+1}) or $(2,1^k)$ for $k\in\mathbb{N}$, or

$$\chi_{\mathsf{D}}^{\Phi} = \chi^{[(1),\pm]} \bullet_{\mathsf{D}} \sum_{p+q=n-2} \chi^{\{(p),(q)\}} = \sum_{\substack{\{\lambda,\mu\}\vdash n \\ \lambda,\mu\in\mathcal{H}}} \chi^{\{\lambda^\top,\mu^\top\}}. \tag{5.13}$$

In the first case χ_{D}^{Φ} shares the irreducible constituent $\chi^{\{((2,1^{n-3}),(1)\}}$ with $\chi_{\mathsf{D}}^{\Theta_1}$ and in the second case χ_{D}^{Φ} shares the irreducible constituent $\chi^{\{(n-1),(1)\}}$ with $\chi_{\mathsf{D}}^{\Theta_{\lfloor n/2\rfloor}}$. Thus if any Φ of the form (5.11) belongs to $\mathcal{M}$ then (5.12) must hold and $\mathcal{M}^1 = \{\Theta_1'\}$. But then the missing irreducible constituent $\chi^{\{(3),(1^{n-3})\}}$ of $\chi_{\mathsf{D}}^{\Theta_1} - \chi_{\mathsf{D}}^{\Theta_1'}$ does not occur in χ_{D}^{Φ}.

We conclude that it is necessary to have $\mathcal{M}^0 = \{\Theta_0\}$ or $\mathcal{M}^1 = \{\Theta_1\}$, so any perfect model $\mathcal{P}$ for W_n^{D} when $n \geq 5$ is odd has $\mathcal{P} \sim \mathcal{P}_n^{\mathsf{D}}$ or $\overline{\mathcal{P}} \sim \mathcal{P}_n^{\mathsf{D}}$.

Now we turn to the even case. One can check directly that W_4^{D} has no perfect models; we have verified this using the computer algebra system GAP [10]. Assume that $n \geq 6$ is even. The formulas for $\chi_{\mathsf{D}}^{\Psi_0}$ and $\chi_{\mathsf{D}}^{\Psi_1}$ given above are still valid, but now one has $\Theta_k \approx \Theta_k^\diamond$ rather than $\Theta_k \sim \Theta_k^\diamond$. By repeating the argument above, however, we can still deduce that $\mathcal{M}$ contains a unique element strongly equivalent to Θ_k or $\Theta_k^\diamond$ for each $2 \leq k \leq \lfloor n/2 \rfloor$; a unique element strongly equivalent to Θ_k or $\Theta_k^\diamond$ or Ψ_k for each $k \in \{0,1\}$; and all other elements of the form (5.11).

In view of Proposition 5.7, this means that the model indices in $\mathcal{M}$ not of the form (5.11) contribute to the sum $\sum_{\Theta\in\mathcal{M}} \chi_{\mathsf{D}}^{\Theta}$ at most one element from each pair of degenerate irreducible characters $\chi^{[\nu,\pm]} \in \mathrm{Irr}(W_n^{\mathsf{D}})$. But the only degenerate irreducible characters appearing in χ_{D}^{Φ} when Φ has the form (5.11) are $\chi^{[(n/2),\pm]}$ or $\chi^{[(1^{n/2}),\pm]}$. Thus it is impossible to have $\sum_{\Theta\in\mathcal{M}} \chi_{\mathsf{D}}^{\Theta} = \sum_{\chi\in\mathrm{Irr}(W_n^{\mathsf{D}})} \chi$, and we conclude that no perfect models for W_n^{D} exist when $n \geq 6$ is even. $\square$

6 Model classification for exceptional groups

Here we discuss which of the remaining irreducible finite Coxeter groups have perfect models.

6.1 Perfect models for dihedral groups

Fix a positive integer m. The finite dihedral group $\mathsf{I}_2(m)$ is the Coxeter group generated by two elements s and t subject only to the relations $s^2 = t^2 = (st)^m = 1$. We have already encountered the groups $I_2(1) = \{1\}$, $I_2(2) \cong S_2 \times S_2$, $I_2(3) \cong S_3$, and $I_2(4) \cong W_2^{\mathsf{B}}$, so assume $m \geq 5$.

The group $\mathsf{I}_2(m)$ has $2m$ elements and a unique nontrivial Coxeter automorphism interchanging $s \leftrightarrow t$, which we denote by $*$. The longest element is $w_0 = ststs\cdots = tstst\cdots$ (m factors). If m is even then $*$ is an outer automorphism, w_0 is central, and $w_0^+ := (w_0, \mathrm{Ad}(w_0)) = w_0 \in \mathsf{I}_2(m)$. If m is odd then $* = \mathrm{Ad}(w_0)$ and $w_0^+ = (w_0, *) \in \mathsf{I}_2(m)^+$.

For either parity of m, the only perfect conjugacy classes in $\mathsf{I}_2(m)^+$ are $\{1\}$ and $\{w_0^+\}$, which are both central. The only model triples $(J, \mathcal{K}, \sigma)$ for $\mathsf{I}_2(m)$ are therefore

$$\begin{aligned}
\mathbb{T}_\sigma^{\{s,t\}} &:= (\{s,t\}, \{1\}, \sigma) \equiv (\{s,t\}, \{w_0^+\}, \sigma) && \text{for any linear character } \sigma \text{ of } \mathsf{I}_2(m), \\
\mathbb{T}_\sigma^{\{s\}} &:= (\{s\}, \{1\}, \sigma) \equiv (\{s\}, \{s\}, \sigma) && \text{for } \sigma \in \{\mathbb{1}, \mathrm{sgn}\}, \\
\mathbb{T}_\sigma^{\{t\}} &:= (\{t\}, \{1\}, \sigma) \equiv (\{t\}, \{t\}, \sigma) && \text{for } \sigma \in \{\mathbb{1}, \mathrm{sgn}\}, \\
\mathbb{T}^{\varnothing} &:= (\varnothing, \{1\}, \mathbb{1}).
\end{aligned}$$

We can ignore $\mathbb{T}^{\varnothing}$ since its character is not multiplicity-free. One has $(\mathbb{T}_\sigma^{\{s\}})^* = \mathbb{T}_\sigma^{\{t\}}$ so if m is odd then $\mathbb{T}_\sigma^{\{s\}}$ and $\mathbb{T}_\sigma^{\{t\}}$ are strongly equivalent.

Let $\zeta := e^{2\pi\sqrt{-1}/m} \in \mathbb{C}$ and define $\rho_h : \mathsf{I}_2(m) \to \mathbb{C}$ for $h \in \mathbb{N}$ to be the map that sends all elements of odd length to zero and has $\rho_h((st)^k) = \rho_h((ts)^k) = \zeta^{hk} + \zeta^{-hk}$ for $k \in \mathbb{N}$. It is well-known [21, §5.3] that if m is odd then the distinct irreducible characters of $\mathsf{I}_2(m)$ consist of ρ_h for $h \in [\frac{m-1}{2}]$ plus the linear characters $\mathbb{1} : s,t \mapsto 1$ and $\mathrm{sgn} : s,t \mapsto -1$; while if m is even then the distinct irreducible characters of $\mathsf{I}_2(m)$ consist of ρ_h for $h \in [\frac{m}{2} - 1]$ plus the linear characters $\mathbb{1}$, sgn, $\mathbb{1}_{+-}$ and $\mathbb{1}_{-+}$, where $\mathbb{1}_{\pm\mp}$ is the class function sending $s \mapsto \pm 1$ and $t \mapsto \mp 1$.

Evidently $\chi^{\mathbb{T}_\sigma^{\{s,t\}}} = \sigma$. The following identities are straightforward exercises from Frobenius reciprocity. If m is odd then $\chi^{\mathbb{T}_\sigma^{\{s\}}} = \chi^{\mathbb{T}_\sigma^{\{t\}}} = \sigma + \sum_h \rho_h$ for $\sigma \in \{\mathbb{1}, \mathrm{sgn}\}$. If m is even then

$$\begin{aligned}
\chi^{\mathbb{T}_{\mathbb{1}}^{\{s\}}} &= \mathrm{Ind}_{\langle s\rangle}^{\langle s,t\rangle}(\mathbb{1}) = \mathbb{1} + \mathbb{1}_{+-} + \sum_h \rho_h, \quad & \chi^{\mathbb{T}_{\mathrm{sgn}}^{\{s\}}} &= \mathrm{Ind}_{\langle s\rangle}^{\langle s,t\rangle}(\mathrm{sgn}) = \mathrm{sgn} + \mathbb{1}_{-+} + \sum_h \rho_h, \\
\chi^{\mathbb{T}_{\mathbb{1}}^{\{t\}}} &= \mathrm{Ind}_{\langle t\rangle}^{\langle s,t\rangle}(\mathbb{1}) = \mathbb{1} + \mathbb{1}_{-+} + \sum_h \rho_h, \quad & \chi^{\mathbb{T}_{\mathrm{sgn}}^{\{t\}}} &= \mathrm{Ind}_{\langle t\rangle}^{\langle s,t\rangle}(\mathrm{sgn}) = \mathrm{sgn} + \mathbb{1}_{+-} + \sum_h \rho_h.
\end{aligned} \tag{6.1}$$

Recall our notions of model equivalence from Section 2.5. The following now is evident:

Proposition 6.1. Assume $m \geq 5$. If m is odd then $\left\{\mathbb{T}_{\mathbb{1}}^{\{s,t\}}, \mathbb{T}_{\mathrm{sgn}}^{\{s\}}\right\} \approx \left\{\mathbb{T}_{\mathrm{sgn}}^{\{s,t\}}, \mathbb{T}_{\mathbb{1}}^{\{s\}}\right\}$ are perfect models for $\mathsf{I}_2(m)$ and every perfect model is strongly equivalent to one of these. If m is even then $\left\{\mathbb{T}_{\mathbb{1}}^{\{s,t\}}, \mathbb{T}_{\mathbb{1}_{+-}}^{\{s,t\}}, \mathbb{T}_{\mathrm{sgn}}^{\{s\}}\right\} \approx \left\{\mathbb{T}_{\mathbb{1}}^{\{s,t\}}, \mathbb{T}_{\mathbb{1}_{-+}}^{\{s,t\}}, \mathbb{T}_{\mathrm{sgn}}^{\{t\}}\right\} \approx \left\{\mathbb{T}_{\mathrm{sgn}}^{\{s,t\}}, \mathbb{T}_{\mathbb{1}_{-+}}^{\{s,t\}}, \mathbb{T}_{\mathbb{1}}^{\{s\}}\right\} \approx \left\{\mathbb{T}_{\mathrm{sgn}}^{\{s,t\}}, \mathbb{T}_{\mathbb{1}_{+-}}^{\{s,t\}}, \mathbb{T}_{\mathbb{1}}^{\{t\}}\right\}$ are perfect models for $\mathsf{I}_2(m)$ and every perfect model is strongly equivalent to one of these.

6.2 Perfect models for exceptional groups

The finite Coxeter system $(W, S) = (W_3^{\mathsf{H}}, \{h_1, h_2, h_3\})$ of type H_3 has Coxeter diagram

$$h_1 \overset{5}{\text{---}} h_2 \overset{3}{\text{---}} h_3$$

and may be embedded in W_6^{D} by setting $h_1 := s_1 s_3$, $h_2 := s_2 s_4$, and $h_3 := s_{-1} s_5$. There are no nontrivial Coxeter automorphisms of W_3^{H}, the only linear characters are $\mathbb{1}$ and sgn, and the only perfect involutions are 1 and the longest element w_0, which is central.

For each subset $J \subseteq \{h_1, h_2, h_3\}$ and linear character $\sigma : \langle J \rangle \to \mathbb{Q}$ let $\mathbb{T}_\sigma^J$ denote the model triple $(J, \{1\}, \sigma)$. Let $\mathbb{1}_{+-}$ and $\mathbb{1}_{-+}$ be the two linear characters of $\langle h_1, h_3 \rangle \cong S_2 \times S_2$ not given by $\mathbb{1}$ or sgn. We have checked the following propositions using GAP [10]:

Proposition 6.2. The sets $\left\{\mathbb{T}_{\mathbb{1}}^{\{h_1,h_2,h_3\}}, \mathbb{T}_{\mathrm{sgn}}^{\{h_1,h_2,h_3\}}, \mathbb{T}_{\mathbb{1}_{+-}}^{\{h_1,h_3\}}\right\} \approx \left\{\mathbb{T}_{\mathbb{1}}^{\{h_1,h_2,h_3\}}, \mathbb{T}_{\mathrm{sgn}}^{\{h_1,h_2,h_3\}}, \mathbb{T}_{\mathbb{1}_{-+}}^{\{h_1,h_3\}}\right\}$ and $\left\{\mathbb{T}_{\mathbb{1}}^{\{h_1,h_2\}}, \mathbb{T}_{\mathrm{sgn}}^{\{h_2,h_3\}}\right\} \approx \left\{\mathbb{T}_{\mathrm{sgn}}^{\{h_1,h_2\}}, \mathbb{T}_{\mathbb{1}}^{\{h_2,h_3\}}\right\}$ are perfect models for W_3^{H}, and every perfect model is strongly equivalent to one of these four.

Proposition 6.3. The Coxeter groups of types E_6, E_7, E_8, F_4, and H_4 have no perfect models.

A Proofs of Theorems 1.6 and 2.2

This final section contains the proofs of two technical results from Sections 1 and 2. If $\lambda = (\lambda_1, \dots, \lambda_j)$ and $\mu = (\mu_1, \dots, \mu_k)$ are partitions with $j \leq k$, then we define $\lambda + \mu = \mu + \lambda := (\lambda_1 + \mu_1, \dots, \lambda_j + \mu_j, \mu_{j+1}, \dots, \mu_k)$ and $\lambda \cup \mu$ to be the partition sorting $(\lambda_1, \dots, \lambda_j, \mu_1, \dots, \mu_k)$. The arguments below frequently makes use of the following identity [22, Lem. 3.2]:

$$c^{\nu+(1^r)}_{\lambda(\mu+(1^r))} \geq c^\nu_{\lambda\mu} \quad \text{and} \quad c^{\nu\cup(r)}_{\lambda(\mu\cup(r))} \geq c^\nu_{\lambda\mu} \tag{A.1}$$

for all partitions λ, μ, ν and integers $r \in \mathbb{N}$.

Lemma A.1. If λ and μ are any partitions then $c^{\lambda+\mu}_{\lambda\mu} \geq 1$ and $c^{\lambda\cup\mu}_{\lambda\mu} \geq 1$.

Proof. Write $\ell(\mu)$ for the number of nonzero parts of μ. If $\ell(\mu) = 0$ then $\mu = \emptyset$ and the result holds as $c^\lambda_{\lambda\emptyset} = 1$. If $\ell(\mu) = r > 0$ then (A.1) implies that $c^{\lambda+\mu}_{\lambda\mu} \geq c^{\lambda+\tilde\mu}_{\lambda\tilde\mu}$ where $\tilde\mu := (\mu_1 - 1, \mu_2 - 1, \dots, \mu_r - 1)$, and in this case we may assume by induction on $|\lambda| + |\mu|$ that $c^{\lambda+\tilde\mu}_{\lambda\tilde\mu} \geq 1$. The other identity holds since $c^{\lambda\cup\mu}_{\lambda\mu} = c^{(\lambda\cup\mu)^\top}_{\lambda^\top\mu^\top} = c^{\lambda^\top+\mu^\top}_{\lambda^\top\mu^\top} \geq 1$. □

Let (W, S) be an irreducible finite Coxeter system and let $J \subseteq S$ be a subset with $|S \setminus J| \geq 2$. Theorem 1.6 asserts that $\mathrm{Ind}^W_{W_J}(\chi)$ is not multiplicity-free for all $\chi \in \mathrm{Irr}(W)$.

Proof of Theorem 1.6. By the transitivity of induction we may assume that $|S \setminus J| = 2$. We have checked the desired property using GAP [10] for each finite exceptional Coxeter group. It remains to prove the result when $W \in \{S_n, W^{\mathsf{B}}_n, W^{\mathsf{D}}_n\}$ for all $n \geq 3$.

First suppose $W = S_n$ so that $S = \{s_1, s_2, \dots, s_{n-1}\}$ and $S \setminus J = \{s_i, s_j\}$ for some $1 \leq i < j < n$. Then we have $W_J = S_i \times S_{j-i} \times S_{n-j}$ and it suffices to show that $\chi^\lambda \bullet_{\mathsf{A}} \chi^\mu \bullet_A \chi^\nu$ is never multiplicity-free if λ, μ, and ν are nonempty partitions. This is easy to derive from the Pieri rules when two of these partitions are equal to (1).

A partition is a *rectangle* if it has the form $(a^i) = (a, a, \dots, a)$. Stembridge [22, Thm. 3.1] gives necessary and sufficient criteria for $\chi^\lambda \bullet_{\mathsf{A}} \chi^\mu$ to be multiplicity-free. This result implies that $\chi^\mu \bullet_A \chi^\nu$ can only be multiplicity-free if at least one of μ or ν is a rectangle, so $\chi^\lambda \bullet_{\mathsf{A}} \chi^\mu \bullet_A \chi^\nu$ can only be multiplicity-free if at least two of λ, μ, ν are rectangles, say $\lambda = (a^i)$ and $\mu = (b^j)$. Suppose ν is not a rectangle and without loss of generality assume $a \geq b$.

We claim that there is a non-rectangle ρ such that $c^\rho_{(a^i)(b^j)} \geq 1$. If $a > b$ and $i \neq j$, then one can take $\rho = (a^i, b^j)$ since Lemma A.1 implies that $c^{(a^ib^j)}_{(a^i)(b^j)} \geq 1$. If $a = b$ and $\max\{i, j\} > 1$, then one can take $\rho = (2a, a^{i+j-2})$ since using the second identity in (A.1) one can check that $c^{(2a,a^{i+j-2})}_{(a^i)(a^j)} \geq c^{(2a)}_{(a)(a)} = 1$. If $a = b > 1$ and $i = j = 1$, then one can set $\rho = (a+1, a-1)$ since the Pieri rules give $c^{(a+1,a-1)}_{(a)(a)} \geq c^{(a+1)}_{(a)(1)} = 1$. The only remaining case is when $a = b = 1$ and $i = j = 1$ so that $\lambda = \mu = (1)$, which we already considered. We conclude that if λ, μ, and ν are not all rectangles then $\chi^\lambda \bullet_{\mathsf{A}} \chi^\mu \bullet_A \chi^\nu$ is not multiplicity-free.

Now assume further that $\nu = (c^k)$ and $a \geq b \geq c$. A *k-line rectangle* is a rectangular partition with either k rows or k columns. Assume at least two of λ, μ, ν are not 1-line rectangles, say μ and ν. If we define $\rho = (a+1, a^{i-1}, b^{j-1}, b-1)$, then one can check using (A.1) that

$$c^\rho_{(a^i)(b^j)} = c^{(a+1,a^{i-1},b^{j-1},b-1)}_{(a^i)(b^j)} \geq c^{(a+1,a^{i-1},b-1)}_{(a^i)(b)} \geq c^{(a+1,b-1)}_{(a)(b)} = 1$$

but then [22, Thm. 3.1] implies that $\chi^\rho \bullet_A \chi^\nu$ is not multiplicity-free since ρ is not a "fat hook," so $\chi^\lambda \bullet_{\mathsf{A}} \chi^\mu \bullet_A \chi^\nu$ is also not multiplicity-free.

Thus we reduce to the case when at least two of λ, μ, and ν are 1-line rectangles, say λ and μ. If $\lambda = (a)$, $\mu = (b)$ and $\nu = (c^k)$ with $a \geq b$, then the Pieri rules give $c^{(a+b-1,1)}_{(a)(b)} = c^{(a+b)}_{(a)(b)} = 1$

while (A.1) implies that

$$c^{(a+b+c-1,c^{k-1},1)}_{(a+b-1,1)(c^k)} \geq c^{(a+b+c-1,1)}_{(a+b-1,1)(c)} \geq c^{(a+b+c-1)}_{(a+b-1)(c)} = 1$$

and

$$c^{(a+b+c-1,c^{k-1},1)}_{(a+b)(c^k)} \geq c^{(a+b+c-1,1)}_{(a+b)(c)} = 1,$$

so $\chi^\lambda \bullet_{\mathsf{A}} \chi^\mu \bullet_A \chi^\nu$ is not multiplicity-free. Next, if $\lambda = (a)$, $\mu = (1^b)$ and $\nu = (c^k)$ with $a > b > 1, c > 1$, then the Pieri rules give $c^{(a+1,1^{b-1})}_{(a)(1^b)} = c^{(a,1^b)}_{(a)(1^b)} = 1$ while (A.1) implies that

$$c^{(a+c,c^{k-1},1^b)}_{(a+1,1^{b-1})(c^k)} \geq c^{(a+c,1^b)}_{(a+1,1^{b-1})(c)} \geq c^{(a+c,1)}_{(a+1)(c)} \geq c^{(c,1)}_{(1)(c)} = 1$$

and

$$c^{(a+c,c^{k-1},1^b)}_{(a,1^b)(c^k)} \geq c^{(a+c,1^b)}_{(a,1^b)(c)} \geq c^{(a+c)}_{(a)(c)} = 1$$

so $\chi^\lambda \bullet_{\mathsf{A}} \chi^\mu \bullet_A \chi^\nu$ is again not multiplicity-free. The same result follows in the remaining cases since $\chi^\lambda \bullet_{\mathsf{A}} \chi^\mu \bullet_A \chi^\nu$ is multiplicity-free if and only if $\chi^{\lambda^\top} \bullet_{\mathsf{A}} \chi^{\mu^\top} \bullet_A \chi^{\nu^\top}$ is multiplicity-free.

Next let $W = W^{\mathsf{B}}_n$ so that $S = \{s_0, s_1, s_2, \dots, s_{n-1}\}$. There are two cases for J, depending on whether $s_0 \in S \setminus J$. First assume $S \setminus J = \{s_0, s_i\}$ for some $i \in [n-1]$. Then $W_J = S_i \times S_{n-i}$ and it suffices to show that $\mathrm{Ind}^{W^{\mathsf{B}}_n}_{S_i \times S_{n-i}}(\chi^\lambda \boxtimes \chi^\mu)$ is not multiplicity-free for all $\lambda \vdash i$ and $\mu \vdash n-i$. Since $\mathrm{Ind}^{W^{\mathsf{B}}_n}_{S_i \times S_{n-i}} \left(\chi^\lambda \boxtimes \chi^\mu\right) = \mathrm{Ind}^{W^{\mathsf{B}}_n}_{S_n} \mathrm{Ind}^{S_n}_{S_i \times S_{n-i}} \left(\chi^\lambda \boxtimes \chi^\mu\right) = \mathrm{Ind}^{W^{\mathsf{B}}_n}_{S_n} \left(\sum_{\nu \vdash n} c^\nu_{\lambda\mu} \chi^\nu\right)$, Lemma A.1 tells us that our induced character has $\mathrm{Ind}^{W^{\mathsf{B}}_n}_{S_n}(\chi^{\lambda+\mu} + \chi^{\lambda\cup\mu})$ as a constituent. But this is not multiplicity-free by (4.5) since $c^{\lambda+\mu}_{\lambda\mu} \geq 1$ and $c^{\lambda\cup\mu}_{\lambda\mu} \geq 1$.

Alternatively suppose $S \setminus J = \{s_i, s_{i+j}\}$ where $0 < i < i+j < n$. Then $W_J = W^{\mathsf{B}}_i \times S_j \times S_{n-i-j}$ and it suffices to show that

$$\mathrm{Ind}^{W^{\mathsf{B}}_n}_{W^{\mathsf{B}}_i \times S_j \times S_{n-i-j}} \left(\chi^{(\lambda_1,\lambda_2)} \boxtimes \chi^\mu \boxtimes \chi^\nu\right) = \mathrm{Ind}^{W^{\mathsf{B}}_n}_{W^{\mathsf{B}}_i \times W^{\mathsf{B}}_{n-i}} \left(\chi^{(\lambda_1,\lambda_2)} \boxtimes \mathrm{Ind}^{W^{\mathsf{B}}_{n-i}}_{S_j \times S_{n-i-j}} \left(\chi^\mu \boxtimes \chi^\nu\right)\right)$$

is not multiplicity-free for all $(\lambda_1, \lambda_2) \vdash i$, $\mu \vdash j$ and $\nu \vdash n-i-j$. But this is immediate since we have already seen that $\mathrm{Ind}^{W^{\mathsf{B}}_{n-i}}_{S_j \times S_{n-i-j}} \left(\chi^\mu \boxtimes \chi^\nu\right)$ is not multiplicity-free.

Finally let $W = W^{\mathsf{D}}_n$ for $n \geq 4$ so that $S = \{s_{-1}, s_1, s_2, \dots, s_{n-1}\}$. There are again two cases for J, depending on whether $s_{-1} \in S \setminus J$. First suppose $S \setminus J = \{s_{-1}, s_i\}$ for some $i \in [n-1]$. Then $W_J = S_i \times S_{n-i}$ and it suffices to show that $\mathrm{Ind}^{W^{\mathsf{D}}_n}_{S_i \times S_{n-i}} \left(\chi^\lambda \boxtimes \chi^\mu\right)$ is not multiplicity-free for all $\lambda \vdash i$ and $\mu \vdash n-i$. If $\lambda \neq \mu$ then this holds by (5.5) since we have already seen that $\mathrm{Ind}^{W^{\mathsf{B}}_n}_{S_i \times S_{n-i}} \left(\chi^\lambda \boxtimes \chi^\mu\right)$ is not multiplicity-free.

It remains to show that $\mathrm{Ind}^{W^{\mathsf{D}}_n}_{S_k \times S_k} \left(\chi^\lambda \boxtimes \chi^\lambda\right)$ is not multiplicity-free when $n = 2k$ is even and $\lambda \vdash k$. Again using (5.5), it suffices to check that $\chi^{(\lambda,\lambda)}$ appears in $\mathrm{Ind}^{W^{\mathsf{B}}_n}_{S_k \times S_k} \left(\chi^\lambda \boxtimes \chi^\lambda\right)$ with multiplicity at least 3. As we already know that $c^{\lambda+\lambda}_{\lambda\lambda} \geq 1$ and $c^{\lambda\cup\lambda}_{\lambda\lambda} \geq 1$, the desired property follows via (4.5) once we use (A.1) to check that $c^\nu_{\lambda\lambda} \geq 1$ for

$$\nu := \begin{cases} (2\lambda_1, \lambda_2, \lambda_2, \cdots, \lambda_r, \lambda_r) & \text{if } r = \ell(\lambda) > 1 \\ (\lambda_1 + 1, \lambda_1 - 1) & \text{if } \ell(\lambda) = 1. \end{cases}$$

In the second case when $S \setminus J = \{s_1, s_i\}$ for some $i \in \{-1\} \sqcup \{2, 3, \dots, n-1\}$, the desired property follows by a symmetric argument, since this case differs from the one just considered by applying the automorphism $\diamond$. If $S \setminus J = \{i, j\}$ for some $1 \leq i < i+j < n$ then $W_J = W^{\mathsf{D}}_i \times S_j \times S_{n-i-j}$ and the fact that $\mathrm{Ind}^{W^{\mathsf{D}}_n}_{W^{\mathsf{D}}_i \times S_j \times S_{n-i-j}}(\chi)$ is never multiplicity-free is immediate from the cases already examined, as in the argument for type B. $\square$

Suppose (W,S) is an irreducible finite Coxeter system and $\mathbb{T}=(J,\mathcal{K},\sigma)$ is a model triple for W that is not factorizable. Theorem 2.2 is equivalent to the claim that the character $\chi^{\mathbb{T}}$ given by (1.1) is not multiplicity-free. We prove this below.

Proof of Theorem 2.2. Since W is irreducible we must have $J\neq S$. Let z be the unique minimal-length element in $\mathcal{K}$ and let $\theta := \mathsf{aut}(z)$. If $|S\setminus J|\geq 2$ then $\chi^{\mathbb{T}}$ is not multiplicity-free by Theorem 1.6.

Assume $|S\setminus J|=1$. Since θ interchanges two irreducible components of (W_J,J), the irreducible Coxeter system (W,S) must be of type A_{2n-1} for $n\geq 2$ (with $S\setminus J=\{s_n\}$), B_3 (with $S\setminus J=\{s_1\}$), D_4 (with $S\setminus J=\{s_2\}$, D_7 (with $S\setminus J=\{s_3\}$), E_6, or H_3. In all but the first case, one can verify that $\chi^{\mathbb{T}}$ is never multiplicity-free by a finite calculation; we have done this using the computer algebra system GAP [10].

For the remaining case, suppose $W=S_{2n}$ and $J=\{s_i : n\neq i\in[2n-1]\}$ so that $W_J=S_n\times S_n$. Then the automorphism θ must either be the map with $s_i\leftrightarrow s_{n+i}$ for all $i\in[n-1]$ or the map with $s_i\leftrightarrow s_{2n-i}$ for all $i\in[n-1]$. In both cases the only possibility for $\mathcal{K}$ is the set $\{(w\cdot\theta(w)^{-1},\theta) : w\in S_n\}$, whose unique minimal-length element is $z=(1,\theta)$.

The $S_n\times S_n$-centralizer of this element is isomorphic to the diagonal subgroup $\Delta(S_n)=\{(x,y)\in S_n\times S_n : x=y\}\cong S_n$, on which the linear characters of $S_n\times S_n$ each restrict to $\mathbb{1}$ or sgn. To show that $\chi^{\mathbb{T}}$ is not multiplicity-free it suffices to check that $\mathrm{Ind}_{\Delta(S_n)}^{S_{2n}}(\sigma)=\mathrm{Ind}_{S_n\times S_n}^{S_{2n}}\mathrm{Ind}_{\Delta(S_n)}^{S_n\times S_n}(\sigma)$ is not multiplicity-free for each linear character σ of $\Delta(S_n)$.

It is a standard exercise using Frobenius reciprocity and the orthogonality relations for irreducible characters that $\mathrm{Ind}_{\Delta(S_n)}^{S_n\times S_n}(\sigma)$ is either $\sum_{\lambda\vdash n}\chi^\lambda\boxtimes\chi^\lambda$ or $\sum_{\lambda\vdash n}\chi^\lambda\boxtimes\chi^{\lambda^\top}$, so $\mathrm{Ind}_{\Delta(S_n)}^{S_{2n}}(\sigma)$ is either $\sum_{\nu\vdash 2n}\left(\sum_{\lambda\vdash n} c^\nu_{\lambda\lambda}\right)\chi^\nu$ or $\sum_{\nu\vdash 2n}\left(\sum_{\lambda\vdash n} c^\nu_{\lambda\lambda^\top}\right)\chi^\nu$. The second character is not multiplicity-free since $n\geq 2$ and $c^\nu_{\lambda\lambda^\top}=c^\nu_{\lambda^\top\lambda}$. The first character is also not multiplicity-free: when $n=2$ one has $c^{(2,2)}_{(2)(2)}=c^{(2,2)}_{(1,1)(1,1)}=1$, and for $n>2$ one can check that $c^{(n,n-1,1)}_{(n-1,1)(n-1,1)}=2$ using the Pieri rules and the identity $\chi^{(n-1,1)}=\chi^{(n-1)}\bullet_{\mathsf{A}}\chi^{(1)}-\chi^{(n)}$. ☐

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