THREE-PARAMETER MOCK THETA FUNCTIONS

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Abstract. Mock theta functions were first introduced by Ramanujan. Historically, mock theta functions can be represented as Eulerian forms, Appell-Lerch sums, Hecke-type double sums, and Fourier coefficients of meromorphic Jacobi forms. In this paper, in view of the q-Zeilberger algorithm and the Watson–Whipple transformation formula, we establish five three-parameter mock theta functions in Eulerian forms, and express them by Appell–Lerch sums. Especially, the main results generalize some two-parameter mock theta functions. For example, setting $(m, q, x) \to (1, q^{1/2}, xq^{-1/2})$ in

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2+(2m-1)n}}{(xq^m, x^{-1}q^m; q^2)_{n+1}},$$

we derive the universal mock theta function $g_2(x, q)$.

1. Introduction

In this paper, we use the following standard q-series notation [20]. For positive integers n and m,

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),$$

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$

$$(a_1, a_2, \ldots, a_m; q)_n := (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n,$$

$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1;q)_\infty(a_2;q)_\infty \cdots (a_m;q)_\infty.$$

Define

$$j(x;q) := (x;q)_\infty(q/x;q)_\infty(q;q)_\infty,$$

$$J_{a,m} := j(q^a;q^m), \quad \overline{J}_{a,m} := j(-q^a;q^m), \quad J_m := J_{m,3m} = \prod_{i=1}^{\infty}(1-q^{mi}).$$

The (unilateral) basic hypergeometric series ${}_r\phi_s$ is stated as

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} x^n.$$

Ramanujan [37] first introduced seventeen classical mock theta functions in his last letter to Hardy. These functions are defined by q-series convergent for $|q| < 1$ with a complex

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variable q. Generally, Eulerian forms, Appell-Lerch sums, Hecke-type double sums, and Fourier coefficients of meromorphic Jacobi forms are used to express mock theta functions. The properties of mock theta functions were widely studied in the literature. In 1936, some identities related to third and fifth order mock theta functions were proved by Watson [40,41]. Later, using the Hecke-type double sums for the fifth and seventh order mock theta functions provided by Andrews [2], Hickerson [23,24] proved some identities related to these mock theta functions. For example, Hickerson [23] proved

$$f_0(q) = \frac{J_{5,10}J_{2,5}}{J_1} - 2q^2 g_3(q^2, q^{10}),$$
$$f_1(q) = \frac{J_{5,10}J_{4,5}}{J_1} - 2q^3 g_3(q^4, q^{10}),$$

where the fifth order mock theta functions $f_0(q)$ and $f_1(q)$ are stated as

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} \quad \text{and} \quad f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n},$$

and the function $g_3(x,q)$ is defined as

$$g_3(x,q) := x^{-1}\left(-1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(x;q)_{n+1}(x^{-1}q;q)_n}\right) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(x,x^{-1}q;q)_{n+1}}.$$

In 1991, some identities related to sixth order mock theta functions were proved by Andrews and Hickerson [5]. In 2007, two new sixth order mock theta functions were established by Berndt and Chan [7]. By means of some q-series identities, Lovejoy [29] reproved the identities involving the sixth order mock theta functions given by Ramanujan. Later, Motivated by asymptotics of q-series, eight eighth order mock theta functions were discovered by Gordon and McIntosh [21]. In 2007, McIntosh [31] built some relations between the second and eighth order mock theta functions. Choi [13–16] proved eight Ramanujan's identities for the tenth order mock theta functions. Recently, Chen and Wang [12] provided a new method to establish Appell–Lerch series and Hecke-type series representations of some mock theta functions. For other research on classical mock theta functions, one can refer to [3,4,8,22, 25,34,38].

In 2012, Gordon and McIntosh [22] showed that the odd order mock theta functions can be expressed by the function $g_3(x,q)$ and the even order mock theta functions are related to $g_2(x,q)$ where

$$g_2(x,q) := \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{(n^2+n)/2}}{(x,x^{-1}q;q)_{n+1}}.$$

In addition, they showed that $g_3(x,q)$ can be expressed in terms of $g_2(x,q)$ [22, Eq. (6.1)]. Since each of the classical mock theta functions can be represented by $g_3(x,q)$ or $g_2(x,q)$, these two functions are called universal mock theta functions. Some other two-parameter mock theta functions are stated as follows [19,36].

$$N(x,q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(xq,x^{-1}q;q)_n}, \qquad K(x,q) = \sum_{n=0}^{\infty} \frac{(q;q^2)_n(-1)^n q^{n^2}}{(xq^2,x^{-1}q^2;q^2)_n},$$
$$K_1(x,q) = \sum_{n=0}^{\infty} \frac{(q;q^2)_n(-1)^n q^{(n+1)^2}}{(xq,x^{-1}q;q^2)_{n+1}}, \qquad K_2(x,q) = \sum_{n=0}^{\infty} \frac{(-1;q)_n q^{(n^2+n)/2}}{(xq,x^{-1}q;q)_n},$$

$$S_2(x;q) = (1+x^{-1})\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{n+1}}{(xq,x^{-1}q;q^2)_{n+1}},$$

$$S_4(x;q) = (1+x^{-1})\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n}{(-x;q)_{n+1}(-x^{-1}q;q)_n}.$$

Besides the relations between universal mock theta functions and classical mock theta functions, some two-parameter mock theta functions are related to ranks and cranks [6,18,28,30]. For more properties of universal mock theta functions, one can refer to [9,10,19,26,27,32,33]. In 2011, Choi [17] found some generalizations of classical mock theta functions. For example, Choi introduced

$$f(\alpha,z;q) = \sum_{n=0}^{\infty}\frac{q^{n^2-3n}\alpha^n z^{2n}}{(-z,-\alpha z/q;q)_n}.$$

The purpose of this paper is to find some new mock theta functions by introducing an integer parameter m. We establish the following three-parameter mock theta functions, and then express them in terms of Appell–Lerch sums.

$$A_m(x;q) := \sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2mn}}{(xq^m,x^{-1}q^m;q^2)_{n+1}},$$

$$B_m(x;q) := \sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2+(2m-1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}},$$

$$C_m(x;q) := \sum_{n=0}^{\infty}\frac{(q^2;q^4)_n(-1)^n q^{2mn}}{(xq^m,x^{-1}q^m;q^2)_{n+1}},$$

$$D_m(x;q) := \sum_{n=0}^{\infty}\frac{(-q;q)_{2n} q^{(2m-1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}},$$

$$E_m(x;q) := \sum_{n=0}^{\infty}\frac{(-1;q)_{2n} q^{(2m+1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}.$$

Recall the following definition of Appell–Lerch sums which was introduced by Hickerson and Mortenson [25].

Definition 1.1. *Let $x,\ z \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$ with neither z nor xz an integral power of q. Then*

$$m(x,q,z) = \frac{1}{j(z;q)}\sum_{r=-\infty}^{\infty}\frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz}.$$

Changing r to $r+1$ in the above series becomes another useful form of $m(x,q,z)$:

$$m(x,q,z) = \frac{-z}{j(z;q)}\sum_{r=-\infty}^{\infty}\frac{(-1)^r q^{\binom{r+1}{2}} z^r}{1-q^r xz}. \tag{1.1}$$

They provided some properties of Appell–Lerch sums.

Proposition 1.2. *[25] For generic $x, z \in \mathbb{C}^*$,*

$$m(x,q,z) = m(x,q,zq), \tag{1.2}$$

$$m(x,q,z) = x^{-1}m(x^{-1},q,z^{-1}). \tag{1.3}$$

Following [25], the term "generic" means that the parameters do not cause poles in the Appell–Lerch sums. They [25] also found that

$$m(q,q^2,-1)=\frac{1}{2}. \tag{1.4}$$

In addition, Mortenson and Hickerson [25] established the following result.

$$g_2(x,q)=\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{(n^2+n)/2}}{(x,x^{-1}q;q)_{n+1}}=\frac{1}{J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-xq^n}=-x^{-1}m(x^{-2}q,q^2,x). \tag{1.5}$$

In 2014, Mortenson [35] expressed the following series in terms of Appell–Lerch sums. Some of these identities were first proved in [1].

$$(1+x^{-1})\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n q^{(n+1)^2}}{(-xq,-x^{-1}q;q^2)_{n+1}}=m(x,q,-1)-\frac{J_{1,2}^2}{2j(-x;q)}, \tag{1.6}$$

$$(1+x^{-1})\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{n+1}}{(xq,x^{-1}q;q^2)_{n+1}}=-m(x,q^2,q), \tag{1.7}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n q^{n^2}}{(-x;q^2)_{n+1}(-x^{-1}q^2;q^2)_n}&=m(x,q,-1)+\frac{J_{1,2}^2}{2j(-x;q)}\\&=2m(x,q,-1)-m(x,q,\sqrt{-x^{-1}q})\\&=m(-x^2q,q^4,-q^{-1})-xq^{-1}m(-x^2q^{-1},q^4,-q),\end{aligned} \tag{1.8}$$

$$\sum_{n=0}^{\infty}{}^{*}\frac{(q;q^2)_n(-1)^n}{(-x;q)_{n+1}(-x^{-1}q;q)_n}=m(x,q,-1), \tag{1.9}$$

$$\sum_{n=0}^{\infty}\frac{(q^2;q^4)_n(-1)^n q^{2n^2}}{(-x;q^4)_{n+1}(-x^{-1}q^4;q^4)_n}=m(x,q^2,q)+\frac{\overline{J}_{1,4}^2 j(-xq^2;q^4)}{j(-x;q^4)j(xq;q^2)},$$

where $\sum^*$ denotes convergence problems and indicates that some "summability" convention is required. In [1, Pages 36–37], Andrews considered the q-Whipple's theorem [20, Appendix (III.18)]

$$\begin{aligned}&{}_8\phi_7\left(\begin{matrix}a, & \sqrt{a}q, & -\sqrt{a}q, & b, & c, & d, & e, & q^{-N}\\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{N+1}\end{matrix};q,\frac{a^2q^{N+2}}{bcde}\right)\\&=\frac{(aq,aq/de;q)_N}{(aq/d,aq/e;q)_N}{}_4\phi_3\left(\begin{matrix}aq/bc, & d, & e, & q^{-N}\\ & aq/b, & aq/c, & deq^{-N}/a\end{matrix};q,q\right).\end{aligned} \tag{1.10}$$

Setting $a=\alpha$, $b=-x^{-1}$, $c=-x$, $d=q^{1/2}$, and $e=-q^{1/2}$ in (1.10), and then letting $N\to\infty$ and $\alpha\to 1^-$, he derived

$$\lim_{\alpha\to 1^-}\sum_{n=0}^{\infty}\frac{(\alpha q;q)_n(q;q^2)_n(-\alpha)^n}{(q,-\alpha xq,-\alpha q/x;q)_n}=\frac{1}{\overline{J}_{0,1}}\sum_{n=-\infty}^{\infty}\frac{(1+x)(1+x^{-1})q^{n(n+1)/2}}{(1+xq^n)(1+x^{-1}q^n)}. \tag{1.11}$$

Thus, the right-hand side of (1.11) is a suitable representation of the divergent series

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n}{(-xq;q)_n(-x^{-1}q;q)_n}.$$

In this paper, we obtain the following main results.

Theorem 1.3. *For $m \geq 0$, we have*

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2mn}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}$$
$$= (-1)^{\lceil \frac{m}{2}\rceil} q^{-\frac{m^2+m}{2}} B_m^{(1)}\left(m(x^2q,q^4,q^{2m+3}) + (-1)^{m+1}xq^{-1}m(x^2q^{-1},q^4,q^{2m+1})\right)$$
$$+ (-1)^{\lceil \frac{m}{2}\rceil} q^{-\frac{m^2+m}{2}} \sum_{k=1}^{m} B_{k,m}^{(1)}\Big(m(q^{4k-2m-1},q^4,q^{2m+3})$$
$$+ (-1)^m q^{-m+2k-2} m(q^{4k-2m-3},q^4,q^{2m+1})\Big),$$

where

$$B_m^{(1)} = \frac{(-1)^{m+1}x(xq^{-m+2};q^2)_{m-1}}{(-xq^{-m+1};q^2)_m}, \tag{1.12}$$

$$B_{k,m}^{(1)} = \frac{(-q)^{k-1}(-q;q^2)_{k-1}(-q;q^2)_{m-k}}{(1+xq^{m-2k+1})(q^2;q^2)_{k-1}(q^2;q^2)_{m-k}}. \tag{1.13}$$

Remark: Notice that here and in what follows, we define $\sum_{n=j}^{k} = 0$ if $j > k$.

Theorem 1.4. *For $m \geq 1$, we have*

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2+(2m-1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}$$
$$= (-1)^{\lceil \frac{m}{2}\rceil} x^{\frac{1-(-1)^m}{2}} q^{\lceil \frac{-m^2+(-1)^m-1}{2}\rceil} B_m^{(2)} m(x^2q^{1+(-1)^m},q^4,x^{-1}q^{\frac{(-1)^m-1}{2}})$$
$$+ \sum_{k=1}^{m-1}(-1)^{\lceil \frac{m}{2}\rceil} q^{\lceil \frac{-m^2+(-1)^m-1}{2}\rceil}(-q^{2k-m})^{\frac{1-(-1)^m}{2}}$$
$$\times B_{k,m}^{(2)} m(q^{-2m+4k+(-1)^m+1},q^4,-q^{-2k+m+\frac{(-1)^m-1}{2}}),$$

where

$$B_m^{(2)} = \frac{(-1)^{m+1}(xq^{-m+2};q^2)_{m-1}}{(-xq^{-m+2};q^2)_{m-1}}, \tag{1.14}$$

$$B_{k,m}^{(2)} = \frac{(-1)^{k-1}(-q^2;q^2)_{k-1}(-1;q^2)_{m-k}}{(q^2;q^2)_{k-1}(q^2;q^2)_{m-k-1}(1+xq^{m-2k})}. \tag{1.15}$$

Theorem 1.5. *For $m \geq 0$, we have*

$$\sum_{n=0}^{\infty}\frac{(q^2;q^4)_n(-1)^n q^{2mn}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}$$
$$= 2q^{-m^2-m}(-q^2;q^2)_{m-1}B_m^{(3)} m(-xq^{-m},q^2,-1)$$
$$+ 2q^{-m^2-m}(-q^2;q^2)_{m-1}\sum_{k=0}^{m-1}\left(B_{m-k,m}^{(3)} m(q^{-2k-1},q^2,-1) + \overline{B}_{m-k,m}^{(3)} m(-q^{-2k-1},q^2,-1)\right),$$

where

$$B_m^{(3)} = -\frac{x^{m+1}(xq^{-m+2};q^2)_{m-1}}{(x^2q^{-2m+2};q^4)_m}, \tag{1.16}$$

$$B_{k,m}^{(3)} = \frac{(-1)^{k-1}q^{k^2-1}(-q;q^2)_{k-1}(-q;q^2)_{m-k}}{2(1+xq^{m+1-2k})(q^4;q^4)_{k-1}(q^4;q^4)_{m-k}}, \tag{1.17}$$

$$\overline{B}^{(3)}_{k,m} = \frac{q^{k^2-1}(q;q^2)_{k-1}(q;q^2)_{m-k}}{2(1-xq^{m+1-2k})(q^4;q^4)_{k-1}(q^4;q^4)_{m-k}}. \tag{1.18}$$

Theorem 1.6. *For $m \geq 1$, we have*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{(2m-1)n}}{(xq^m, x^{-1}q^m;q^2)_{n+1}} \\
&= (-1)^m q^{-m^2}(q;q^2)_{m-1}B^{(4)}_m m(xq^{-m+1},q^2,q) \\
&\quad + (-1)^m q^{-m^2}(q;q^2)_{m-1}\sum_{k=1}^{m-1} B^{(4)}_{m-k,m} m(-q^{-2k+1},q^2,q) \\
&\quad + (-1)^m q^{-m^2}(q;q^2)_{m-1}\sum_{k=0}^{m-1} \overline{B}^{(4)}_{m-k,m} m(-q^{-2k},q^2,q),
\end{aligned}$$

where

$$B^{(4)}_m = \frac{(-1)^{m-1}x^m(xq^{-m+2};q^2)_{m-1}}{(-xq^{-m+1};q)_{2m-1}}, \tag{1.19}$$

$$B^{(4)}_{k,m} = -\frac{q^{k^2}(-q^2;q^2)_{k-1}(-1;q^2)_{m-k}}{(q;q)_{2k-1}(q;q)_{2m-2k-1}(1+xq^{m-2k})}, \tag{1.20}$$

$$\overline{B}^{(4)}_{k,m} = \frac{q^{k^2-k}(-q;q^2)_{k-1}(-q;q^2)_{m-k}}{(q;q)_{2k-2}(q;q)_{2m-2k}(1+xq^{m-2k+1})}. \tag{1.21}$$

Theorem 1.7. *For $m \geq 1$, we have*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1;q)_{2n}q^{(2m+1)n}}{(xq^m, x^{-1}q^m;q^2)_{n+1}} \\
&= 2(-1)^{m+1} q^{-(m+1)^2}(q;q^2)_{m}B^{(5)}_m m(xq^{-1-m},q^2,q) \\
&\quad + 2(-1)^{m+1} q^{-(m+1)^2}(q;q^2)_{m}\sum_{k=0}^{m} B^{(5)}_{m-k,m} m(-q^{-2k-1},q^2,q) \\
&\quad + 2(-1)^{m+1} q^{-(m+1)^2}(q;q^2)_{m}\sum_{k=0}^{m-1} \overline{B}^{(5)}_{m-k,m} m(-q^{-2k-2},q^2,q),
\end{aligned}$$

where

$$B^{(5)}_m = \frac{(-1)^{m+1}x^{m+2}(xq^{-m+2};q^2)_{m-1}}{(-xq^{-m};q)_{2m+1}}, \tag{1.22}$$

$$B^{(5)}_{k,m} = \frac{q^{k^2+2k}(-q^2;q^2)_{k-1}(-1;q^2)_{m-k}}{(q;q)_{2k}(q;q)_{2m-2k}(1+xq^{m-2k})}, \tag{1.23}$$

$$\overline{B}^{(5)}_{k,m} = -\frac{q^{k^2+k-1}(-q;q^2)_{k-1}(-q;q^2)_{m-k}}{(q;q)_{2k-1}(q;q)_{2m-2k+1}(1+xq^{m+1-2k})}. \tag{1.24}$$

Letting $(m,q,x) \to (1, q^{1/2}, xq^{-1/2})$ in Theorem 1.4 and using (1.3), we derive (1.5). The identity (1.7) can be established by setting $m=1$ in Theorem 1.6. If we replace m, q, and x by 0, $-q$, and $-x$ in Theorem 1.3, respectively, then after simplification, we deduce (1.8). Setting $m=0$, $q \to q^{1/2}$, and $x \to -x$ in Theorem 1.5 implies (1.9). Notice that replacing m and q by 1 and $-q$ in Theorem 1.3, respectively, and multiplying both sides by $(1+x^{-1})$, we obtain a different representation of (1.6). Furthermore, some new two-parameter mock theta functions can be obtained from the main results. For example, setting $m = 2$ in Theorem 1.4 and using (1.4), we derive

$$(1+x)\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2+3n+2}}{(x;q^2)_{n+2}(x^{-1}q^2;q^2)_{n+1}}+\frac{1}{1-x}=m(x^2q^2,q^4,x^{-1}).$$

This paper is organized as follows. In Section 2, some preliminary results are stated. In Section 3, we prove Theorems 1.3-1.7.

2. Preliminaries

In this paper, in order to prove the main results, the following identities are needed. For $|q| < 1$,

$$\begin{aligned} j(x;q) &= j(q/x;q),\\ j(qx;q) &= -x^{-1}j(x;q),\\ j(q^nx;q) &= (-1)^n q^{-\binom{n}{2}}x^{-n}j(x;q), \quad n\in\mathbb{Z}, && (2.1)\\ (a;q)_{-n} &= \frac{1}{(aq^{-n};q)_n}, \quad n\geq 0, && (2.2)\\ (a;q)_{n+k} &= (a;q)_n(aq^n;q)_k, && (2.3)\\ (aq^{-n};q)_n &= (q/a;q)_n(-a/q)^n q^{-\binom{n}{2}}, && (2.4)\\ (a;q)_{n-k} &= \frac{(a;q)_n}{(q^{1-n}/a;q)_k}\left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk}, && (2.5)\\ \frac{(a;q)_{n-k}}{(b;q)_{n-k}} &= \frac{(a;q)_n(q^{1-n}/b;q)_k}{(b;q)_n(q^{1-n}/a;q)_k}\left(\frac{b}{a}\right)^k. && (2.6) \end{aligned}$$

In order to prove the main theorems, the following results are needed.

Lemma 2.1. *[20, Appendix (III.17)] The Watson–Whipple transformation formula is stated as*

$$\begin{aligned} &{}_8\phi_7\left(\begin{matrix} a, & \sqrt{a}q, & -\sqrt{a}q, & b, & c, & d, & e, & f \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix};q,\frac{a^2q^2}{bcdef}\right)\\ &=\frac{(aq,aq/de,aq/df,aq/ef;q)_\infty}{(aq/d,aq/e,aq/f,aq/def;q)_\infty}{}_4\phi_3\left(\begin{matrix} aq/bc, & d, & e, & f \\ & aq/b, & aq/c, & def/a \end{matrix};q,q\right). \qquad (2.7) \end{aligned}$$

Lemma 2.2. *For $m\geq 1$, we have*

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{(d,e;q^2)_n}{(xq^m,x^{-1}q^m;q^2)_{n+1}}\left(\frac{q^{2m+2}}{de}\right)^n\\ &=\frac{(q^{2m+2}/d,q^{2m+2}/e;q^2)_\infty}{(q^2,q^{2m+2}/de;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(q^{2n+2};q^2)_{m-1}(d,e;q^2)_n(-1)^nq^{n^2+2mn+3n}}{(q^{2m+2}/d,q^{2m+2}/e;q^2)_n(1-xq^{2n+m})}\left(\frac{1}{de}\right)^n. \end{aligned}$$

Proof. For $m\geq 1$, setting $q\to q^2$, $a=q^{2m}$, $b=x^{-1}q^m$, $c=xq^m$, and $f\to\infty$ in (2.7), then dividing both sides by $(1-xq^m)(1-x^{-1}q^m)$, after simplification, we deduce

$$\sum_{n=0}^{\infty}\frac{(d,e;q^2)_n}{(xq^m,x^{-1}q^m;q^2)_{n+1}}\left(\frac{q^{2m+2}}{de}\right)^n$$

$$
\begin{aligned}
&= \frac{(q^{2m+2}/d, q^{2m+2}/e; q^2)_\infty}{(q^2, q^{2m+2}/de; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(1-q^{4n+2m})(q^{2n+2}; q^2)_{m-1} (d, e; q^2)_n (-1)^n q^{n^2+2mn+3n}}{(q^{2m+2}/d, q^{2m+2}/e; q^2)_n (1-xq^{2n+m})(1-x^{-1}q^{2n+m})} \left(\frac{1}{de}\right)^n \\
&= \frac{(q^{2m+2}/d, q^{2m+2}/e; q^2)_\infty}{(q^2, q^{2m+2}/de; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q^{2n+2}; q^2)_{m-1} (d, e; q^2)_n (-1)^n q^{n^2+2mn+3n}}{(q^{2m+2}/d, q^{2m+2}/e; q^2)_n} \left(\frac{1}{de}\right)^n \\
&\quad \times \left(\frac{x^{-1}q^{2n+m}}{1-x^{-1}q^{2n+m}} + \frac{1}{1-xq^{2n+m}} \right) \\
&= \frac{(q^{2m+2}/d, q^{2m+2}/e; q^2)_\infty}{(q^2, q^{2m+2}/de; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q^{2n+2}; q^2)_{m-1} (d, e; q^2)_n (-1)^{n+1} q^{n^2+2mn+3n}}{(q^{2m+2}/d, q^{2m+2}/e; q^2)_n (1-xq^{-2n-m})} \left(\frac{1}{de}\right)^n \\
&\quad + \frac{(q^{2m+2}/d, q^{2m+2}/e; q^2)_\infty}{(q^2, q^{2m+2}/de; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q^{2n+2}; q^2)_{m-1} (d, e; q^2)_n (-1)^n q^{n^2+2mn+3n}}{(q^{2m+2}/d, q^{2m+2}/e; q^2)_n (1-xq^{2n+m})} \left(\frac{1}{de}\right)^n \\
&= \frac{(q^{2m+2}/d, q^{2m+2}/e; q^2)_\infty}{(q^2, q^{2m+2}/de; q^2)_\infty} \left(\sum_{n=-\infty}^{-m} + \sum_{n=0}^{\infty} \right) \frac{(q^{2n+2}; q^2)_{m-1} (d, e; q^2)_n (-1)^n q^{n^2+2mn+3n}}{(q^{2m+2}/d, q^{2m+2}/e; q^2)_n (1-xq^{2n+m})} \left(\frac{1}{de}\right)^n \\
&= \frac{(q^{2m+2}/d, q^{2m+2}/e; q^2)_\infty}{(q^2, q^{2m+2}/de; q^2)_\infty} \\
&\quad \times \left(\sum_{n=-\infty}^{\infty} - \sum_{n=-m+1}^{-1} \right) \frac{(q^{2n+2}; q^2)_{m-1} (d, e; q^2)_n (-1)^n q^{n^2+2mn+3n}}{(q^{2m+2}/d, q^{2m+2}/e; q^2)_n (1-xq^{2n+m})} \left(\frac{1}{de}\right)^n,
\end{aligned}
$$

where the penultimate equality is obtained by setting $n \to -n-m$ in the first sum, and then using (2.2), (2.3), and (2.4). Finally, we find that $(q^{2n+2}; q^2)_{m-1} = 0$ for $-m+1 \le n \le -1$. Therefore, we complete the lemma. □

Lemma 2.3. *[11, Theorem 2.1] We have, provided both sides converge,*

$$
\begin{aligned}
&(q;q)_\infty \frac{\prod_{k=1}^{L} (a_k z; q)_\infty \prod_{k=1}^{M} (b_k/z; q)_\infty}{\prod_{k=1}^{L} (c_k z; q)_\infty \prod_{k=1}^{M} (d_k/z; q)_\infty} \\
&= \frac{(a_1/c_1, \ldots, a_L/c_1; q)_\infty (b_1 c_1, \ldots, b_M c_1; q)_\infty}{(c_2/c_1, \ldots, c_L/c_1; q)_\infty (c_1 d_1, \ldots, c_1 d_M; q)_\infty} \\
&\quad \times \sum_{n=0}^{\infty} \frac{\left(\frac{a_1 \cdots a_L}{c_1 \cdots c_L}\right)^n (c_1 q/a_1, \ldots, c_1 q/a_L; q)_n (c_1 d_1, \ldots, c_1 d_M; q)_n}{(q;q)_n (c_1 q/c_2, \ldots, c_1 q/c_L; q)_n (b_1 c_1, \ldots, b_M c_1; q)_n (1 - c_1 q^n z)} + idem(c_1; c_2, \ldots, c_L) \\
&\quad + \frac{(a_1 d_1, \ldots, a_L d_1; q)_\infty (b_1/d_1, \ldots, b_M/d_1; q)_\infty}{(c_1 d_1, \ldots, c_L d_1; q)_\infty (d_2/d_1, \ldots, d_M/d_1; q)_\infty} \\
&\quad \times \sum_{n=0}^{\infty} \frac{d_1 \left(\frac{b_1 \cdots b_M q}{d_1 \cdots d_M}\right)^n (d_1 q/b_1, \ldots, d_1 q/b_M; q)_n (c_1 d_1, \ldots, c_L d_1; q)_n}{(q;q)_n (a_1 d_1, \ldots, a_L d_1; q)_n (d_1 q/d_2, \ldots, d_1 q/d_M; q)_n (z - d_1 q^n)} + idem(d_1; d_2, \ldots, d_M),
\end{aligned}
$$

where

$$
F(b_1, b_2, \ldots, b_m) + idem(b_1; b_2, \ldots, b_n) := \sum_{i=1}^{n} F(b_i; b_2, \ldots, b_{i-1}, b_1, b_{i+1}, \ldots, b_m).
$$

Lemma 2.4. *For $n \ge 0$ and $m \ge 1$, we have*

$$
\frac{(q^{2n+2}; q^2)_{m-1}}{(-q^{2n+1}; q^2)_m} = \sum_{k=1}^{m} \frac{A_{k,m}^{(1)}}{1 + q^{2n+2k-1}},
$$

where

$$A_{k,m}^{(1)} = \frac{(-q)^{k-1}(-q;q^2)_{k-1}(-q;q^2)_{m-k}}{(q^2;q^2)_{k-1}(q^2;q^2)_{m-k}}.$$

Proof. We prove this lemma by applying the q-Zeilberger algorithm. The Maple program is stated in Appendix I. Instead, we can also prove the lemma by using Bailey pairs and Bailey's lemma.

A pair of sequences $(\alpha_n(a,k), \beta_n(a,k))$ is a WP-Bailey pair if they satisfy

$$\beta_n(a,k) = \sum_{j=0}^{n} \frac{(k/a;q)_{n-j}(k;q)_{n+j}}{(q;q)_{n-j}(aq;q)_{n+j}} \alpha_j(a,k). \tag{2.8}$$

Then recall the following pair which was initially provided by Singh [39]:

$$\alpha_n(a,k) = \frac{(\sqrt{a}q, -\sqrt{a}q, a, \rho_1, \rho_2, a^2q/k\rho_1\rho_2;q)_n}{(\sqrt{a}, -\sqrt{a}, q, aq/\rho_1, aq/\rho_2, k\rho_1\rho_2/a;q)_n} \left(\frac{k}{a}\right)^n, \tag{2.9}$$

$$\beta_n(a,k) = \frac{(k\rho_1/a, k\rho_2/a, k, aq/\rho_1\rho_2;q)_n}{(aq/\rho_1, aq/\rho_2, k\rho_1\rho_2/a, q;q)_n}. \tag{2.10}$$

Replacing k by ak and then letting $a \to 0$ in (2.8)-(2.10), we obtain

$$\frac{(k\rho_1, k\rho_2;q)_n}{(k\rho_1\rho_2, q;q)_n} = \sum_{j=0}^{n} \frac{(k;q)_{n-j}(\rho_1,\rho_2;q)_j}{(q;q)_{n-j}(k\rho_1\rho_2, q;q)_j} k^j. \tag{2.11}$$

Next, letting $q \to q^2$ and $n = m-1$, and then setting $k = -q$, $\rho_1 = -q^{2n+1}$, and $\rho_2 = -q$ in (2.11), we find

$$\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+1};q^2)_m} = \sum_{j=0}^{m-1} \frac{(-q)^j(-q;q^2)_j(-q;q^2)_{m-j-1}}{(1+q^{2n+2j+1})(q^2;q^2)_j(q^2;q^2)_{m-j-1}}.$$

Finally, shifting the summation interval on the right-hand side of the above equation, we complete the proof. □

Using the above lemma, we derive the following corollary.

Corollary 2.5. *For $n \geq 0$ and $m \geq 1$, we have*

$$\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+1};q^2)_m(1-xq^{2n+m})} = \frac{B_m^{(1)}}{1-xq^{2n+m}} + \sum_{k=1}^{m} \frac{B_{k,m}^{(1)}}{1+q^{2n+2k-1}},$$

where $B_m^{(1)}$ and $B_{k,m}^{(1)}$ are defined in (1.12) *and* (1.13)*, respectively.*

Proof. According to Lemma 2.4, we have

$$\begin{aligned}
&\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+1};q^2)_m(1-xq^{2n+m})} \\
&= \sum_{k=1}^{m} \frac{A_{k,m}^{(1)}}{(1+q^{2n+2k-1})(1-xq^{2n+m})} \\
&= \sum_{k=1}^{m} \frac{A_{k,m}^{(1)}}{1+xq^{m-2k+1}} \left(\frac{xq^{m-2k+1}}{1-xq^{2n+m}} + \frac{1}{1+q^{2n+2k-1}}\right)
\end{aligned}$$

$$= \sum_{k=1}^{m} \frac{A_{k,m}^{(1)} x q^{m-2k+1}}{(1+xq^{m-2k+1})(1-xq^{2n+m})} + \sum_{k=1}^{m} \frac{A_{k,m}^{(1)}}{(1+xq^{m-2k+1})(1+q^{2n+2k-1})}$$

$$= \sum_{k=1}^{m} \frac{(-1)^{k-1} x q^{m-k} (-q;q^2)_{k-1}(-q;q^2)_{m-k}}{(1+xq^{m-2k+1})(q^2;q^2)_{k-1}(q^2;q^2)_{m-k}} \cdot \frac{1}{1-xq^{2n+m}} + \sum_{k=1}^{m} \frac{B_{k,m}^{(1)}}{1+q^{2n+2k-1}}. \tag{2.12}$$

Notice that

$$\begin{aligned}\sum_{k=1}^{m} \frac{(-1)^{k-1} x q^{m-k} (-q;q^2)_{k-1}(-q;q^2)_{m-k}}{(1+xq^{m-2k+1})(q^2;q^2)_{k-1}(q^2;q^2)_{m-k}} &= \sum_{k=0}^{m-1} \frac{(-q)^k (-q;q^2)_k (-q;q^2)_{m-k-1}}{(1+x^{-1}q^{2k-m+1})(q^2;q^2)_k (q^2;q^2)_{m-k-1}} \\ &= \frac{(x^{-1}q^{-m+2};q^2)_{m-1}}{(-x^{-1}q^{-m+1};q^2)_m} \\ &= B_m^{(1)}, \end{aligned} \tag{2.13}$$

where we replace q by q^2, and then set $n = m-1$, $k = -q$, $\rho_1 = -q$, and $\rho_2 = -x^{-1}q^{-m+1}$ in (2.11) to obtain the second equality, and the last step follows from (1.12) and (2.4).

Hence, applying (2.12) and (2.13) yields the corollary. □

Lemma 2.6. *For $n \geq 0$ and $m \geq 1$, we have*

$$\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+2};q^2)_{m-1}} = (-1)^{m-1} + \sum_{k=1}^{m-1} \frac{A_{k,m}^{(2)}}{1+q^{2n+2k}}, \tag{2.14}$$

where

$$A_{k,m}^{(2)} = \frac{(-1)^{k-1}(-q^2;q^2)_{k-1}(-1;q^2)_{m-k}}{(q^2;q^2)_{k-1}(q^2;q^2)_{m-k-1}}.$$

Proof. We use the q-Zeilberger algorithm to prove the lemma, and the program is given in Appendix II. It can be shown that three terms of (2.14) satisfy the same recurrence relation. □

Corollary 2.7. *For $n \geq 0$ and $m \geq 1$, we have*

$$\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+2};q^2)_{m-1}(1-xq^{2n+m})} = \frac{B_m^{(2)}}{1-xq^{2n+m}} + \sum_{k=1}^{m-1} \frac{B_{k,m}^{(2)}}{1+q^{2n+2k}},$$

where $B_m^{(2)}$ and $B_{k,m}^{(2)}$ are defined in (1.14) and (1.15), respectively.

Proof. In view of Lemma 2.6, we have

$$\begin{aligned}&\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+2};q^2)_{m-1}(1-xq^{2n+m})} \\ &= \left((-1)^{m-1} + \sum_{k=1}^{m-1} \frac{A_{k,m}^{(2)}}{1+q^{2n+2k}}\right) \cdot \frac{1}{1-xq^{2n+m}} \\ &= \frac{(-1)^{m-1}}{1-xq^{2n+m}} + \sum_{k=1}^{m-1} \frac{A_{k,m}^{(2)}}{1+xq^{m-2k}} \left(\frac{1}{1+q^{2n+2k}} + \frac{xq^{m-2k}}{1-xq^{2n+m}}\right) \\ &= \frac{(-1)^{m-1}}{1-xq^{2n+m}} + \sum_{k=1}^{m-1} \frac{B_{k,m}^{(2)}}{1+q^{2n+2k}} + \sum_{k=1}^{m-1} \frac{(-1)^{k-1} x q^{m-2k} (-q^2;q^2)_{k-1}(-1;q^2)_{m-k}}{(q^2;q^2)_{k-1}(q^2;q^2)_{m-k-1}(1+xq^{m-2k})} \cdot \frac{1}{1-xq^{2n+m}}\end{aligned}$$

$$= \left((-1)^{m-1} + \sum_{k=1}^{m-1} \frac{(-1)^{k-1}(-q^2;q^2)_{k-1}(-1;q^2)_{m-k}}{(q^2;q^2)_{k-1}(q^2;q^2)_{m-k-1}(1+x^{-1}q^{2k-m})} \right) \cdot \frac{1}{1-xq^{2n+m}} + \sum_{k=1}^{m-1} \frac{B_{k,m}^{(2)}}{1+q^{2n+2k}}. \tag{2.15}$$

The following identity is still proved by using the q-Zeilberger algorithm. The Maple program is stated in Appendix III.

$$\sum_{k=1}^{m-1} \frac{(-1)^{k-1}(-q^2;q^2)_{k-1}(-1;q^2)_{m-k}}{(q^2;q^2)_{k-1}(q^2;q^2)_{m-k-1}(1+x^{-1}q^{2k-m})} = \frac{(-1)^{m+1}(xq^{-m+2};q^2)_{m-1}}{(-xq^{-m+2};q^2)_{m-1}} + (-1)^m. \tag{2.16}$$

Then substituting the above identity into (2.15), we obtain

$$\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+2};q^2)_{m-1}(1-xq^{2n+m})} = \frac{(-1)^{m+1}(xq^{-m+2};q^2)_{m-1}}{(-xq^{-m+2};q^2)_{m-1}} \cdot \frac{1}{1-xq^{2n+m}} + \sum_{k=1}^{m-1} \frac{B_{k,m}^{(2)}}{1+q^{2n+2k}}$$

which complete the proof. □

Lemma 2.8. *For $n \geq 0$ and $m \geq 1$, we have*

$$\frac{(q^{2n+2};q^2)_{m-1}}{(q^{4n+2};q^4)_m} = \sum_{k=1}^{m} \frac{A_{k,m}^{(3)}}{1+q^{2n+2k-1}} + \sum_{k=1}^{m} \frac{\overline{A}_{k,m}^{(3)}}{1-q^{2n+2k-1}}, \tag{2.17}$$

where

$$A_{k,m}^{(3)} = \frac{(-1)^{k-1}q^{k^2-1}(-q;q^2)_{k-1}(-q;q^2)_{m-k}}{2(q^4;q^4)_{k-1}(q^4;q^4)_{m-k}}, \tag{2.18}$$

$$\overline{A}_{k,m}^{(3)} = \frac{q^{k^2-1}(q;q^2)_{k-1}(q;q^2)_{m-k}}{2(q^4;q^4)_{k-1}(q^4;q^4)_{m-k}}. \tag{2.19}$$

Proof. We prove this lemma by utilizing the q-Zeilberger algorithm. It can be shown that three terms of (2.17) satisfy the same recurrence relation. The Maple program is stated in Appendix IV. □

Corollary 2.9. *For $n \geq 0$ and $m \geq 1$, we have*

$$\frac{(q^{2n+2};q^2)_{m-1}}{(q^{4n+2};q^4)_m(1-xq^{2n+m})} = \frac{B_m^{(3)}}{1-xq^{2n+m}} + \sum_{k=1}^{m} \frac{B_{k,m}^{(3)}}{1+q^{2n+2k-1}} + \sum_{k=1}^{m} \frac{\overline{B}_{k,m}^{(3)}}{1-q^{2n+2k-1}},$$

where $B_m^{(3)}$, $B_{k,m}^{(3)}$, and $\overline{B}_{k,m}^{(3)}$ are defined in (1.16), (1.17), *and* (1.18), *respectively.*

Proof. First, using Lemma 2.8, we derive

$$\begin{aligned}
&\frac{(q^{2n+2};q^2)_{m-1}}{(q^{4n+2};q^4)_m(1-xq^{2n+m})} \\
&= \left(\sum_{k=1}^{m} \frac{A_{k,m}^{(3)}}{1+q^{2n+2k-1}} + \sum_{k=1}^{m} \frac{\overline{A}_{k,m}^{(3)}}{1-q^{2n+2k-1}} \right) \cdot \frac{1}{1-xq^{2n+m}} \\
&= \left(\sum_{k=1}^{m} \frac{A_{k,m}^{(3)}}{1+x^{-1}q^{-m-1+2k}} - \sum_{k=1}^{m} \frac{\overline{A}_{k,m}^{(3)}xq^{m+1-2k}}{1-xq^{m+1-2k}} \right) \cdot \frac{1}{1-xq^{2n+m}} \\
&\quad + \sum_{k=1}^{m} \frac{A_{k,m}^{(3)}x^{-1}q^{-m-1+2k}}{1+x^{-1}q^{-m-1+2k}} \cdot \frac{1}{1+q^{2n+2k-1}} + \sum_{k=1}^{m} \frac{\overline{A}_{k,m}^{(3)}}{1-xq^{m+1-2k}} \cdot \frac{1}{1-q^{2n+2k-1}}
\end{aligned}$$

$$= \left(\sum_{k=1}^{m} \frac{A_{k,m}^{(3)}}{1+x^{-1}q^{-m-1+2k}} - \sum_{k=1}^{m} \frac{\overline{A}_{k,m}^{(3)} xq^{m+1-2k}}{1-xq^{m+1-2k}} \right) \cdot \frac{1}{1-xq^{2n+m}} + \sum_{k=1}^{m} \frac{B_{k,m}^{(3)}}{1+q^{2n+2k-1}} + \sum_{k=1}^{m} \frac{\overline{B}_{k,m}^{(3)}}{1-q^{2n+2k-1}}. \tag{2.20}$$

Next, based on (2.18) and (2.19), we find

$$\begin{aligned}
&\sum_{k=1}^{m} \frac{A_{k,m}^{(3)}}{1+x^{-1}q^{-m-1+2k}} - \sum_{k=1}^{m} \frac{\overline{A}_{k,m}^{(3)} xq^{m+1-2k}}{1-xq^{m+1-2k}} \\
&= \sum_{k=0}^{m-1} \frac{A_{k+1,m}^{(3)}}{1+x^{-1}q^{-m+1+2k}} - \sum_{k=0}^{m-1} \frac{\overline{A}_{m-k,m}^{(3)} xq^{-m+1+2k}}{1-xq^{-m+1+2k}} \\
&= \frac{(-q;q^2)_{m-1}}{2(q^4;q^4)_{m-1}} \sum_{k=0}^{m-1} \frac{q^{(2m+2)k}(q^{-2m+2}, -q^{-2m+2}, -q;q^2)_k}{(q^2, -q^2, -q^{-2m+3};q^2)_k} \cdot \frac{1}{1+x^{-1}q^{-m+1+2k}} \\
&\quad - \frac{xq^{m^2-m}(q;q^2)_{m-1}}{2(q^4;q^4)_{m-1}} \sum_{k=0}^{m-1} \frac{q^{2k}(q^{-2m+2}, -q^{-2m+2}, q;q^2)_k}{(q^2, -q^2, q^{-2m+3};q^2)_k} \cdot \frac{1}{1-xq^{-m+1+2k}},
\end{aligned} \tag{2.21}$$

where we set $k \to k+1$ in the first term and $k \to m-k$ in the second term to obtain the first equality, and we apply (2.5) to derive the last step.

The following identity is proved by applying the q-Zeilberger algorithm. It can be shown that the three terms of (2.22) satisfy the same recurrence relation. The Maple program is stated in Appendix V.

$$\begin{aligned}
&\frac{(-q;q^2)_{m-1}}{2(q^4;q^4)_{m-1}} \sum_{k=0}^{m-1} \frac{q^{(2m+2)k}(q^{-2m+2}, -q^{-2m+2}, -q;q^2)_k}{(q^2, -q^2, -q^{-2m+3};q^2)_k} \cdot \frac{1}{1+x^{-1}q^{-m+1+2k}} \\
&- \frac{xq^{m^2-m}(q;q^2)_{m-1}}{2(q^4;q^4)_{m-1}} \sum_{k=0}^{m-1} \frac{q^{2k}(q^{-2m+2}, -q^{-2m+2}, q;q^2)_k}{(q^2, -q^2, q^{-2m+3};q^2)_k} \cdot \frac{1}{1-xq^{-m+1+2k}} \\
&= -\frac{x^{m+1}(xq^{-m+2};q^2)_{m-1}}{(x^2q^{-2m+2};q^4)_m}.
\end{aligned} \tag{2.22}$$

Finally, from (2.21) and (2.22), it can be seen that

$$\sum_{k=1}^{m} \frac{A_{k,m}^{(3)}}{1+x^{-1}q^{-m-1+2k}} - \sum_{k=1}^{m} \frac{\overline{A}_{k,m}^{(3)} xq^{m+1-2k}}{1-xq^{m+1-2k}} = -\frac{x^{m+1}(xq^{-m+2};q^2)_{m-1}}{(x^2q^{-2m+2};q^4)_m}. \tag{2.23}$$

Therefore, combining (2.20) and (2.23), we complete the proof. □

Lemma 2.10. *For $n \geq 0$ and $m \geq 1$, we have*

$$\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+1};q)_{2m-1}} = \sum_{k=1}^{m-1} \frac{A_{k,m}^{(4)}}{1+q^{2n+2k}} + \sum_{k=1}^{m} \frac{\overline{A}_{k,m}^{(4)}}{1+q^{2n+2k-1}},$$

where

$$A_{k,m}^{(4)} = -\frac{q^{k^2}(-q^2;q^2)_{k-1}(-1;q^2)_{m-k}}{(q;q)_{2k-1}(q;q)_{2m-2k-1}},$$

$$\overline{A}_{k,m}^{(4)} = \frac{q^{k^2-k}(-q;q^2)_{k-1}(-q;q^2)_{m-k}}{(q;q)_{2k-2}(q;q)_{2m-2k}}.$$

Proof. The proof is similar to that of Lemma 2.8. □

Corollary 2.11. *For $n \geq 0$ and $m \geq 1$, we have*

$$\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+1};q)_{2m-1}(1-xq^{2n+m})} = \frac{B_m^{(4)}}{1-xq^{2n+m}} + \sum_{k=1}^{m-1}\frac{B_{k,m}^{(4)}}{1+q^{2n+2k}} + \sum_{k=1}^{m}\frac{\overline{B}_{k,m}^{(4)}}{1+q^{2n+2k-1}},$$

where $B_m^{(4)}$, $B_{k,m}^{(4)}$, and $\overline{B}_{k,m}^{(4)}$ are defined in (1.19), (1.20), *and* (1.21), *respectively.*

Proof. In view of Lemma 2.10, we derive

$$\begin{aligned}
&\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+1};q)_{2m-1}(1-xq^{2n+m})}\\
&=\sum_{k=1}^{m-1}\frac{A_{k,m}^{(4)}}{1+q^{2n+2k}}\cdot\frac{1}{1-xq^{2n+m}} + \sum_{k=1}^{m}\frac{\overline{A}_{k,m}^{(4)}}{1+q^{2n+2k-1}}\cdot\frac{1}{1-xq^{2n+m}}\\
&=\sum_{k=1}^{m-1}\frac{A_{k,m}^{(4)}}{1+x^{-1}q^{2k-m}}\left(\frac{x^{-1}q^{2k-m}}{1+q^{2n+2k}}+\frac{1}{1-xq^{2n+m}}\right)\\
&\quad+\sum_{k=1}^{m}\frac{\overline{A}_{k,m}^{(4)}}{1+xq^{m-2k+1}}\left(\frac{1}{1+q^{2n+2k-1}}+\frac{xq^{m-2k+1}}{1-xq^{2n+m}}\right)\\
&=\sum_{k=1}^{m-1}\frac{A_{k,m}^{(4)}}{1+x^{-1}q^{2k-m}}\cdot\frac{x^{-1}q^{2k-m}}{1+q^{2n+2k}} + \sum_{k=1}^{m}\frac{\overline{A}_{k,m}^{(4)}}{1+xq^{m-2k+1}}\cdot\frac{1}{1+q^{2n+2k-1}}\\
&\quad+\sum_{k=1}^{m-1}\frac{A_{k,m}^{(4)}}{1+x^{-1}q^{2k-m}}\cdot\frac{1}{1-xq^{2n+m}} + \sum_{k=1}^{m}\frac{\overline{A}_{k,m}^{(4)}}{1+xq^{m-2k+1}}\cdot\frac{xq^{m-2k+1}}{1-xq^{2n+m}}\\
&=\sum_{k=1}^{m-1}\frac{B_{k,m}^{(4)}}{1+q^{2n+2k}} + \sum_{k=1}^{m}\frac{\overline{B}_{k,m}^{(4)}}{1+q^{2n+2k-1}}\\
&\quad+\left(\sum_{k=0}^{m-2}\frac{A_{k+1,m}^{(4)}}{1+x^{-1}q^{2k-m+2}} + \sum_{k=0}^{m-1}\frac{\overline{A}_{m-k,m}^{(4)}xq^{-m+2k+1}}{1+xq^{-m+2k+1}}\right)\cdot\frac{1}{1-xq^{2n+m}},
\end{aligned}\tag{2.24}$$

where we set $k \to k+1$ in the third term and $k \to m-k$ in the fourth term to obtain the last step. Next,

$$\begin{aligned}
&\sum_{k=0}^{m-2}\frac{A_{k+1,m}^{(4)}}{1+x^{-1}q^{2k-m+2}} + \sum_{k=0}^{m-1}\frac{\overline{A}_{m-k,m}^{(4)}xq^{-m+2k+1}}{1+xq^{-m+2k+1}}\\
&=-\sum_{k=0}^{m-2}\frac{q^{k^2+2k+1}(-q^2;q^2)_k(-1;q^2)_{m-k-1}}{(q;q^2)_{k+1}(q^2;q^2)_k(q;q^2)_{m-k-1}(q^2;q^2)_{m-k-2}(1+x^{-1}q^{2k-m+2})}\\
&\quad+x\sum_{k=0}^{m-1}\frac{q^{(m-k)^2-3(m-k)+m+1}(-q;q^2)_{m-k-1}(-q;q^2)_k}{(q;q^2)_{m-k-1}(q^2;q^2)_{m-k-1}(q;q^2)_k(q^2;q^2)_k(1+xq^{2k-m+1})}\\
&=(-1)^{m-1}\frac{x^m(xq^{-m+2};q^2)_{m-1}}{(-xq^{-m+1};q)_{2m-1}},
\end{aligned}\tag{2.25}$$

where we derive the last step by utilizing the q-Zeilberger algorithm, and the proof is similar to that of (2.22).

So, combining (2.24) and (2.25), we complete the proof. □

Lemma 2.12. *For $n \geq 0$ and $m \geq 1$, we have*

$$\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n};q)_{2m+1}} = \sum_{k=0}^{m} \frac{A_{k,m}^{(5)}}{1+q^{2n+2k}} + \sum_{k=1}^{m} \frac{\overline{A}_{k,m}^{(5)}}{1+q^{2n+2k-1}},$$

where

$$A_{k,m}^{(5)} = \frac{q^{k^2+2k}(-q^2;q^2)_{k-1}(-1;q^2)_{m-k}}{(q;q)_{2k}(q;q)_{2m-2k}},$$
$$\overline{A}_{k,m}^{(5)} = -\frac{q^{k^2+k-1}(-q;q^2)_{k-1}(-q;q^2)_{m-k}}{(q;q)_{2k-1}(q;q)_{2m-2k+1}}.$$

Proof. The proof is similar to that of Lemma 2.8. □

Corollary 2.13. *For $n \geq 0$ and $m \geq 1$, we have*

$$\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n};q)_{2m+1}(1-xq^{2n+m})} = \frac{B_m^{(5)}}{1-xq^{2n+m}} + \sum_{k=0}^{m} \frac{B_{k,m}^{(5)}}{1+q^{2n+2k}} + \sum_{k=1}^{m} \frac{\overline{B}_{k,m}^{(5)}}{1+q^{2n+2k-1}},$$

where $B_m^{(5)}$, $B_{k,m}^{(5)}$, and $\overline{B}_{k,m}^{(5)}$ are defined in (1.22), (1.23), and (1.24), respectively.

Proof. Based on Lemma 2.12, we have

$$\begin{aligned}
&\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n};q)_{2m+1}(1-xq^{2n+m})} \\
&= \sum_{k=0}^{m} \frac{A_{k,m}^{(5)}x^{-1}q^{-m+2k}}{1+x^{-1}q^{-m+2k}} \cdot \frac{1}{1+q^{2n+2k}} + \sum_{k=1}^{m} \frac{\overline{A}_{k,m}^{(5)}}{1+xq^{m+1-2k}} \cdot \frac{1}{1+q^{2n+2k-1}} \\
&\quad + \left(\sum_{k=0}^{m} \frac{A_{k,m}^{(5)}}{1+x^{-1}q^{-m+2k}} + \sum_{k=1}^{m} \frac{\overline{A}_{k,m}^{(5)}xq^{m+1-2k}}{1+xq^{m+1-2k}} \right) \cdot \frac{1}{1-xq^{2n+m}} \\
&= \sum_{k=0}^{m} \frac{B_{k,m}^{(5)}}{1+q^{2n+2k}} + \sum_{k=1}^{m} \frac{\overline{B}_{k,m}^{(5)}}{1+q^{2n+2k-1}} \\
&\quad + \left(\sum_{k=0}^{m} \frac{A_{k,m}^{(5)}}{1+x^{-1}q^{-m+2k}} + \sum_{k=0}^{m-1} \frac{\overline{A}_{m-k,m}^{(5)}xq^{-m+1+2k}}{1+xq^{-m+1+2k}} \right) \cdot \frac{1}{1-xq^{2n+m}}.
\end{aligned}$$

Notice that using (2.5) and (2.6), we have

$$\begin{aligned}
&\sum_{k=0}^{m} \frac{A_{k,m}^{(5)}}{1+x^{-1}q^{-m+2k}} + \sum_{k=0}^{m-1} \frac{\overline{A}_{m-k,m}^{(5)}xq^{-m+1+2k}}{1+xq^{-m+1+2k}} \\
&= \frac{(-q^2;q^2)_{m-1}}{(q,q^2;q^2)_m} \sum_{k=0}^{m} \frac{q^{(2m+4)k}(q^{-2m},q^{-2m+1},-1;q^2)_k}{(q,q^2,-q^{-2m+2};q^2)_k} \cdot \frac{1}{1+x^{-1}q^{-m+2k}} \\
&\quad - \frac{xq^{m^2}(-q;q^2)_{m-1}}{(1-q)(q;q^2)_m(q^2;q^2)_{m-1}} \sum_{k=0}^{m-1} \frac{q^{2k}(q^{-2m+2},q^{-2m+1},-q;q^2)_k}{(q^2,q^3,-q^{-2m+3};q^2)_k} \cdot \frac{1}{1+xq^{-m+1+2k}} \\
&= (-1)^{m+1}x^{m+2}\frac{(xq^{-m+2};q^2)_{m-1}}{(-xq^{-m};q)_{2m+1}},
\end{aligned}$$

where we obtain the last equality by using the q-Zeilberger algorithm, and the proof is similar to that of (2.22). Therefore, we complete the proof. □

Remark: We can derive most of the identities which are proved by the q-Zeilberger algorithm in this section by using Lemma 2.3. For example, based on (2.6), we deduce

$$\sum_{k=1}^{m}\frac{(-q)^{k-1}(-q;q^2)_{k-1}(-q;q^2)_{m-k}}{(1+q^{2n+2k-1})(q^2;q^2)_{k-1}(q^2;q^2)_{m-k}}$$
$$=\frac{(-q;q^2)_{m-1}}{(q^2;q^2)_{m-1}}\sum_{k=0}^{m-1}\frac{q^{2k}(-q;q^2)_k(q^{-2m+2};q^2)_k}{(1+q^{2n+2k+1})(q^2;q^2)_k(-q^{-2m+3};q^2)_k}. \tag{2.26}$$

For $n\geq 0$ and $m\geq 1$, replacing q by q^2, and then setting $L=2$, $M=0$, $a_1=-q$, $a_2=q^{2m}$, $c_1=1$, $c_2=-q^{2m-1}$, and $z=-q^{2n+1}$ in Lemma 2.3, we obtain

$$(q^2;q^2)_\infty\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+1};q^2)_m}=(q^{2m};q^2)_\infty(-q;q^2)_{m-1}\sum_{k=0}^{m-1}\frac{q^{2k}(-q;q^2)_k(q^{-2m+2};q^2)_k}{(1+q^{2n+2k+1})(q^2;q^2)_k(-q^{-2m+3};q^2)_k}.$$

Therefore, substituting the above identity into (2.26) yields Lemma 2.4.

3. Proofs of Theorems 1.3-1.7

In this section, we prove Theorems 1.3-1.7.

Proof of Theorem 1.3. First, in view of (1.8), we get the case for $m=0$. Next, we prove the theorem for $m\geq 1$. Setting $d\to\infty$ and $e=-q$ in Lemma 2.2, we deduce

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{n^2+2mn}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}$$
$$=\frac{1}{J_{1,4}}\sum_{n=-\infty}^{\infty}\frac{(q^{2n+2};q^2)_{m-1}(-1)^nq^{2n^2+2mn+n}}{(-q^{2n+1};q^2)_m(1-xq^{2n+m})}$$
$$=\frac{1}{J_{1,4}}\sum_{n=-\infty}^{\infty}\left(\frac{B_m^{(1)}(-1)^nq^{2n^2+2mn+n}(1+xq^{2n+m})}{1-x^2q^{4n+2m}}+\sum_{k=1}^{m}\frac{B_{k,m}^{(1)}(-1)^nq^{2n^2+2mn+n}(1-q^{2n+2k-1})}{1-q^{4n+4k-2}}\right)$$
$$=\frac{B_m^{(1)}j(q^{2m+3};q^4)m(x^2q,q^4,q^{2m+3})-B_m^{(1)}xq^{-m-1}j(q^{2m+1};q^4)m(x^2q^{-1},q^4,q^{2m+1})}{J_{1,4}}$$
$$+\sum_{k=1}^{m}\frac{B_{k,m}^{(1)}j(q^{2m+3};q^4)m(q^{4k-2m-1},q^4,q^{2m+3})}{J_{1,4}}$$
$$+\sum_{k=1}^{m}\frac{B_{k,m}^{(1)}q^{2k-2m-2}j(q^{2m+1};q^4)m(q^{4k-2m-3},q^4,q^{2m+1})}{J_{1,4}}, \tag{3.1}$$

where we obtain the second equality by using Corollary 2.5, and the last step follows from (1.1). If m is even, then we change m to $2m$ in (3.1). So, we obtain

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{n^2+4mn}}{(xq^{2m},x^{-1}q^{2m};q^2)_{n+1}}$$
$$=(-1)^mq^{-2m^2-m}B_{2m}^{(1)}m(x^2q,q^4,q^3)+(-1)^{m+1}xq^{-2m^2-m-1}B_{2m}^{(1)}m(x^2q^{-1},q^4,q)$$

$$+\sum_{k=1}^{2m}(-1)^m q^{-2m^2-m}B_{k,2m}^{(1)}m(q^{4k-4m-1},q^4,q^3)$$
$$+\sum_{k=1}^{2m}(-1)^m q^{-2m^2-3m+2k-2}B_{k,2m}^{(1)}m(q^{4k-4m-3},q^4,q), \tag{3.2}$$

where the last step follows from (1.2) and (2.1). Similarly, for odd m, we replace m by $2m-1$ in (3.1) and use (1.2) to obtain

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+4mn-2n}}{(xq^{2m-1},x^{-1}q^{2m-1};q^2)_{n+1}}$$
$$=(-1)^m q^{-2m^2+m}B_{2m-1}^{(1)}m(x^2q,q^4,q)+(-1)^m xq^{-2m^2+m-1}B_{2m-1}^{(1)}m(x^2q^{-1},q^4,q^3)$$
$$+\sum_{k=1}^{2m-1}(-1)^m q^{-2m^2+m}B_{k,2m-1}^{(1)}m(q^{4k-4m+1},q^4,q)$$
$$+\sum_{k=1}^{2m-1}(-1)^{m+1}q^{-2m^2-m+2k-1}B_{k,2m-1}^{(1)}m(q^{4k-4m-1},q^4,q^3). \tag{3.3}$$

In view of (1.2), we see that (3.2) and (3.3) are the even and odd cases of m in Theorem 1.3, respectively. Therefore, we complete the proof. □

Proof of Theorem 1.4. For $m\geq 1$, setting $d\to\infty$ and $e=-q^2$ in Lemma 2.2, and using Corollary 2.7, we deduce

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2+(2m-1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}$$
$$=\frac{1}{J_{2,4}}\sum_{n=-\infty}^{\infty}\frac{(q^{2n+2};q^2)_{m-1}(-1)^n q^{2n^2+2mn}}{(-q^{2n+2};q^2)_{m-1}(1-xq^{2n+m})}$$
$$=\frac{1}{J_{2,4}}\sum_{n=-\infty}^{\infty}\left(\frac{(-1)^n q^{2n^2+2mn}B_m^{(2)}}{1-xq^{2n+m}}+\sum_{k=1}^{m-1}\frac{(-1)^n q^{2n^2+2mn}B_{k,m}^{(2)}}{1+q^{2n+2k}}\right). \tag{3.4}$$

Next, we split m into two cases. First, if m is even, then changing m by $2m$ in (3.4), we derive

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2+(4m-1)n}}{(xq^{2m},x^{-1}q^{2m};q^2)_{n+1}}$$
$$=\frac{1}{J_{2,4}}\sum_{n=-\infty}^{\infty}\left(\frac{(-1)^n q^{2n^2+4mn}B_{2m}^{(2)}}{1-xq^{2n+2m}}+\sum_{k=1}^{2m-1}\frac{(-1)^n q^{2n^2+4mn}B_{k,2m}^{(2)}}{1+q^{2n+2k}}\right)$$
$$=\frac{1}{J_{2,4}}\sum_{n=-\infty}^{\infty}\left(\frac{(-1)^{n+m} q^{2n^2-2m^2}B_{2m}^{(2)}}{1-xq^{2n}}+\sum_{k=1}^{2m-1}\frac{(-1)^{n+m} q^{2n^2-2m^2}B_{k,2m}^{(2)}}{1+q^{2n+2k-2m}}\right), \tag{3.5}$$

where we set $n\to n-m$ in the first sum to obtain the last step.

By changing $n\to -n$, $x\to x^{-1}$, and $q\to q^2$ in (1.5), we find

$$\frac{1}{J_{2,4}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2}}{1-xq^{2n}}=m(x^2q^2,q^4,x^{-1}). \tag{3.6}$$

Then substituting (3.6) into (3.5), we arrive at

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2+(4m-1)n}}{(xq^{2m},x^{-1}q^{2m};q^2)_{n+1}}$$
$$=(-1)^m q^{-2m^2}B_{2m}^{(2)}m(x^2q^2,q^4,x^{-1})+\sum_{k=1}^{2m-1}(-1)^m q^{-2m^2}B_{k,2m}^{(2)}m(q^{-4m+4k+2},q^4,-q^{2m-2k})$$

which implies the even case of m in Theorem 1.4.

Now we turn to prove the case for odd m in Theorem 1.4. Changing m by $2m-1$ in (3.4), we obtain

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2+(4m-3)n}}{(xq^{2m-1},x^{-1}q^{2m-1};q^2)_{n+1}}$$
$$=\frac{1}{J_{2,4}}\sum_{n=-\infty}^{\infty}\left(\frac{(-1)^n q^{2n^2+4mn-2n}B_{2m-1}^{(2)}}{1-xq^{2n+2m-1}}+\sum_{k=1}^{2m-2}\frac{(-1)^n q^{2n^2+4mn-2n}B_{k,2m-1}^{(2)}}{1+q^{2n+2k}}\right)$$
$$=\frac{1}{J_{2,4}}\sum_{n=-\infty}^{\infty}\left(\frac{(-1)^{n+m+1}q^{2n^2+2n-2m^2+2m}B_{2m-1}^{(2)}}{1-xq^{2n+1}}+\sum_{k=1}^{2m-2}\frac{(-1)^{n+m+1}q^{2n^2+2n-2m^2+2m}B_{k,2m-1}^{(2)}}{1+q^{2n+2k-2m+2}}\right) \tag{3.7}$$
$$=(-1)^m x^{-1}q^{-2m^2+2m-1}B_{2m-1}^{(2)}m(x^{-2},q^4,xq)$$
$$+\sum_{k=1}^{2m-2}(-1)^{m+1}q^{-2m^2+4m-2k-2}B_{k,2m-1}^{(2)}m(q^{4m-4k-2},q^4,-q^{-2m+2k+2})$$
$$=(-1)^m xq^{-2m^2+2m-1}B_{2m-1}^{(2)}m(x^2,q^4,x^{-1}q^{-1})$$
$$+\sum_{k=1}^{2m-2}(-1)^{m+1}q^{-2m^2+2k}B_{k,2m-1}^{(2)}m(q^{-4m+4k+2},q^4,-q^{2m-2k-2}),$$

where we set $n\to n-m+1$ in the first sum to obtain (3.7), the penultimate step follows from (1.5), and we obtain the last step by applying (1.3).

Therefore, we complete the proof. □

Proof of Theorem 1.5. In view of (1.9), we have the case for $m=0$. Next, we prove the theorem for $m\geq 1$. Setting $(d,e)\to(q,-q)$ in Lemma 2.2, and then using Corollary 2.9, we deduce

$$\sum_{n=0}^{\infty}\frac{(q^2;q^4)_n(-1)^n q^{2mn}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}$$
$$=\frac{2(-q^2;q^2)_{m-1}}{\overline{J}_{0,2}}\sum_{n=-\infty}^{\infty}\frac{(q^{2n+2};q^2)_{m-1}q^{n^2+n+2mn}}{(q^{4n+2};q^4)_m(1-xq^{2n+m})}$$
$$=\frac{2(-q^2;q^2)_{m-1}}{\overline{J}_{0,2}}\sum_{n=-\infty}^{\infty}\frac{B_m^{(3)}q^{n^2+n+2mn}}{1-xq^{2n+m}}+\frac{2(-q^2;q^2)_{m-1}}{\overline{J}_{0,2}}\sum_{k=1}^{m}\sum_{n=-\infty}^{\infty}\frac{B_{k,m}^{(3)}q^{n^2+n+2mn}}{1+q^{2n+2k-1}}$$
$$+\frac{2(-q^2;q^2)_{m-1}}{\overline{J}_{0,2}}\sum_{k=1}^{m}\sum_{n=-\infty}^{\infty}\frac{\overline{B}_{k,m}^{(3)}q^{n^2+n+2mn}}{1-q^{2n+2k-1}}$$

$$
\begin{aligned}
&= 2q^{-m^2-m}(-q^2;q^2)_{m-1}B_m^{(3)}m(-xq^{-m},q^2,-1)\\
&\quad + 2q^{-m^2-m}(-q^2;q^2)_{m-1}\sum_{k=1}^{m}B_{k,m}^{(3)}m(q^{2k-2m-1},q^2,-1)\\
&\quad + 2q^{-m^2-m}(-q^2;q^2)_{m-1}\sum_{k=1}^{m}\overline{B}_{k,m}^{(3)}m(-q^{2k-2m-1},q^2,-1),
\end{aligned}
$$

where the last step follows from (1.1) and (1.2). Hence, we complete the proof. □

Proof of Theorem 1.6. Letting $(d,e)\to(-q,-q^2)$ in Lemma 2.2, and then using Corollary 2.11, we derive

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{(2m-1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}\\
&= \frac{(q;q^2)_{m-1}}{J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2mn}B_m^{(4)}}{1-xq^{2n+m}} + \frac{(q;q^2)_{m-1}}{J_{1,2}}\sum_{k=1}^{m-1}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2mn}B_{k,m}^{(4)}}{1+q^{2n+2k}}\\
&\quad + \frac{(q;q^2)_{m-1}}{J_{1,2}}\sum_{k=1}^{m}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2mn}\overline{B}_{k,m}^{(4)}}{1+q^{2n+2k-1}}\\
&= (-1)^mq^{-m^2}(q;q^2)_{m-1}B_m^{(4)}m(xq^{-m+1},q^2,q)\\
&\quad + (-1)^mq^{-m^2}(q;q^2)_{m-1}\sum_{k=1}^{m-1}B_{k,m}^{(4)}m(-q^{2k-2m+1},q^2,q)\\
&\quad + (-1)^mq^{-m^2}(q;q^2)_{m-1}\sum_{k=1}^{m}\overline{B}_{k,m}^{(4)}m(-q^{2k-2m},q^2,q),
\end{aligned}
$$

where we obtain the last step by utilizing (1.1) and (1.2). Therefore, we complete the proof. □

Proof of Theorem 1.7. Setting $(d,e)\to(-1,-q)$ in Lemma 2.2, and then using Corollary 2.13, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-1;q)_{2n}q^{(2m+1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}\\
&= \frac{2(q;q^2)_m}{J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+(2m+2)n}B_m^{(5)}}{1-xq^{2n+m}} + \frac{2(q;q^2)_m}{J_{1,2}}\sum_{k=0}^{m}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+(2m+2)n}B_{k,m}^{(5)}}{1+q^{2n+2k}}\\
&\quad + \frac{2(q;q^2)_m}{J_{1,2}}\sum_{k=1}^{m}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+(2m+2)n}\overline{B}_{k,m}^{(5)}}{1+q^{2n+2k-1}}\\
&= 2(-1)^{m+1}q^{-(m+1)^2}(q;q^2)_mB_m^{(5)}m(xq^{-1-m},q^2,q)\\
&\quad + 2(-1)^{m+1}q^{-(m+1)^2}(q;q^2)_m\sum_{k=0}^{m}B_{k,m}^{(5)}m(-q^{2k-2m-1},q^2,q)\\
&\quad + 2(-1)^{m+1}q^{-(m+1)^2}(q;q^2)_m\sum_{k=1}^{m}\overline{B}_{k,m}^{(5)}m(-q^{2k-2m-2},q^2,q),
\end{aligned}
$$

where the last step follows from (1.1) and (1.2). Therefore, we complete the proof. □

4. Appendixes

In this section, we provide some Maple programs which are used to prove the main results. The Maple package APCI which is written by Hou can be downloaded from the following link.

http://faculty.tju.edu.cn/HouQinghu/en/lwcg/4184/content/23455.htm#lwcg

4.1. **Appendix I.** We prove Lemma 2.4 by Maple with the following program.

$> with(APCI);$

$> t := \frac{(-q)^{k-1}\cdot qpoch(-q,q^2,k-1)\cdot qpoch(-q,q^2,m-k)}{qpoch(q^2,q^2,k-1)\cdot qpoch(q^2,q^2,m-k)\cdot(1+q^{2n+2k-1})};$

$> re := qZeil(t,m,k,q);$

$> f := \frac{qpoch(q^{2n+2},q^2,m-1)}{qpoch(-q^{2n+1},q^2,m)};$

$> qhyper\_simp(add(re[i]\cdot subs(m=m+i-1,f), i=1..nops(re)));$

4.2. **Appendix II.** We use the following program to prove Lemma 2.6.

$> with(APCI);$

$> t := \frac{(-1)^{k-1}\cdot qpoch(-q^2,q^2,k-1)\cdot qpoch(-1,q^2,m-k)}{qpoch(q^2,q^2,k-1)\cdot qpoch(q^2,q^2,m-k-1)\cdot(1+q^{2n+2k})};$

$> re := qZeil(t,m,k,q);$

$> f := \frac{qpoch(q^{2n+2},q^2,m-1)}{qpoch(-q^{2n+2},q^2,m-1)};$

$> qhyper\_simp(add(re[i]\cdot subs(m=m+i-1,f), i=1..nops(re)));$

$> qhyper\_simp(add(re[i]\cdot subs(m=m+i-1,(-1)^{m-1}), i=1..nops(re)));$

4.3. **Appendix III.** With the aid of the following program, we prove (2.16).

$> with(APCI);$

$> t := \frac{(-1)^{k-1}\cdot qpoch(-q^2,q^2,k-1)\cdot qpoch(-1,q^2,m-k)}{qpoch(q^2,q^2,k-1)\cdot qpoch(q^2,q^2,m-k-1)\cdot(1+\frac{q^{2k-m}}{x})};$

$> re := qZeil(t,m,k,q);$

$> nops(re);$

$> f := \frac{(-1)^{m+1}\cdot qpoch(x\cdot q^{-m+2},q^2,m-1)}{qpoch(-x\cdot q^{-m+2},q^2,m-1)};$

$> qhyper\_simp(re[1]\cdot f + re[3]\cdot subs(m=m+2,f));$

$> qhyper\_simp(re[2]\cdot subs(m=m+1,f) + re[4]\cdot subs(m=m+3,f));$

$> qhyper\_simp(add(re[i]\cdot subs(m=m+i-1,(-1)^{m}), i=1..nops(re)));$

4.4. **Appendix IV.** We prove Lemma 2.8 by utilizing the following program.

$> with(APCI);$

$> t1 := \frac{(-1)^{k-1}\cdot q^{k^2-1}\cdot qpoch(-q,q^2,k-1)\cdot qpoch(-q,q^2,m-k)}{2qpoch(q^4,q^4,k-1)\cdot qpoch(q^4,q^4,m-k)\cdot(1+q^{2n+2k-1})};$

$> t2 := \frac{q^{k^2-1}\cdot qpoch(q,q^2,k-1)\cdot qpoch(q,q^2,m-k)}{2qpoch(q^4,q^4,k-1)\cdot qpoch(q^4,q^4,m-k)\cdot(1-q^{2n+2k-1})};$

$> re1 := qZeil(t1,m,k,q);$

$> re2 := qZeil(t2,m,k,q);$

$> re1 - re2;$

$> f := \frac{qpoch(q^{2n+2}, q^2, m-1)}{qpoch(q^{4n+2}, q^4, m)};$

$> qhyper\_simp(add(re1[i] \cdot subs(m = m + i - 1, f), i = 1..nops(re1)));$

4.5. **Appendix V.** We prove (2.22) by applying the following program.

$> with(APCI);$

$> t1 := \frac{qpoch(-q, q^2, m-1) \cdot q^{(2m+2)k} \cdot qpoch([q^{-2m+2}, -q^{-2m+2}, -q], [q^2, -q^2, -q^{-2m+3}], q^2, k)}{2qpoch(q^4, q^4, m-1) \cdot (1 + \frac{q^{-m+1+2k}}{x})};$

$> t2 := \frac{-x \cdot q^{m^2-m} \cdot qpoch(q, q^2, m-1) \cdot q^{2k} \cdot qpoch([q^{-2m+2}, -q^{-2m+2}, q], [q^2, -q^2, q^{-2m+3}], q^2, k)}{2qpoch(q^4, q^4, m-1) \cdot (1 - x \cdot q^{-m+1+2k})};$

$> re1 := qZeil(t1, m, k, q);$

$> nops(re1);$

$> re2 := qZeil(t2, m, k, q);$

$> re1 - re2;$

$> f := \frac{-x^{m+1} \cdot qpoch(x \cdot q^{-m+2}, q^2, m-1)}{qpoch(x^2 \cdot q^{-2m+2}, q^4, m)};$

$> qhyper\_simp(add(re1[2i+1] \cdot subs(m = m + 2i, f), i = 0..2));$

$> qhyper\_simp(add(re1[2i] \cdot subs(m = m + 2i - 1, f), i = 1..2));$

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