On Superconnectivity of $(4, g)$-Cages

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Abstract

A (k, g)-*cage* is a graph that has the least number of vertices among all k-regular graphs with girth g. It was conjectured by Fu, Huang and Rodger [3] that all (k, g)-cages are k-connected for every $k \geq 3$. A k-connected graph G is called superconnected if every k-cutset S is trivial. Moreover, if $G - S$ has precisely two components, then G is called tightly superconnected. In [9, 13], the authors showed that every $(4, g)$-cage is 4-connected. In this paper, we proved that every $(4, g)$-cage is tightly superconnected when g is odd.

Key words: cage, superconnected, tightly superconnected

1 Introduction

Throughout this paper, only undirected simple graphs are considered. Unless otherwise defined, we follow [1] for terminology and definitions.

A k-regular graph with girth g is called a (k, g)-*graph*. A (k, g)-*cage* is a (k, g)-graph with the least number of vertices for given k and g. We use $f(k, g)$ to denote the number of vertices in (k, g)-cages. A cutset X of G is called a *non-trivial cutset* if X does not contain the neighborhood $N(u)$ of any vertex $u \notin X$. A k-connected (or k-vertex-connected) graph G is called *superconnected* if for every vertex cutset $S \subseteq V(G)$ with $|S| = k$, S is a trivial cutset. Moreover, if $G - S$ has precisely two components, then G is called *tightly superconnected*. Provided existence of non-trivial cutset, the *superconnectivity* of G is denoted by $\kappa_1 = \kappa_1(G) = \min\{|X| : \text{ X is a non-trivial cutset}\}$. The edge-superconnectivity λ_1 is defined similarly.

Cages were introduced by Tutte in 1947, and have been extensively studied. Most of the work carried out so far focused on the existence problem, whereas very little is known about the structural properties of (k, g)-cages. For more information see survey [12]. Recently, several researchers have studied the connectivity of cages. Fu, Huang and Rodger [3] proved that all cages are 2-connected, and then subsequently showed that all cubic cages are 3-connected. They then conjectured that (k, g)-cages are k-connected. Daven and Rodger [2], and independently Jiang and Mubayi [4], proved that all (k, g)-cages are 3-connected for $k \geq 3$. In [9, 13], some authors also showed that every $(4, g)$-cage is 4-connected. Tang *et al.* [10] conjectured that every $(4, g)$-cage with odd girth is tightly superconnected. In this paper, we show that this conjecture is true.

For the edge-connectivity of (k, g)-cage, Wang, Xu and Wang [11] showed that (k, g)-cages are k-edge-connected when g is odd, and subsequently, Lin, Miller and Rodger [7] proved that (k, g)-cages are k-edge-connected when g is even. Recently, Lin *et al.* [5, 8] proved that (k, g)-cages are edge-superconnected.

2 Main Results

First, we list several known results which will be used in proving our main theorem.

Theorem 1. ([3]) *Let G be a (k, g)-cage with diameter D, where $k \geq 2$ and $g \geq 3$, then $f(k, g) < f(k, g+1)$ and $D \leq g$.*

Theorem 2. ([8]) *Every (k, g)-cage with odd girth $g \geq 5$ is edge-superconnected.*

Regarding to the edge-connectivity of $(k, g) - cages$, Tang *et al.* [10] conjectured the following:

Conjecture 1. ([10]) *Every (k, g)-cage of odd girth $g \geq 5$ has $\lambda_1 = 2k - 2$.*

First, we verify that the conjecture is true for $k = 4$.

Lemma 1. *Every $(4, g)$-cage of girth $g \geq 5$ has $\lambda_1 = 6$*

Proof. Let M be a non-trivial minimal edge cutset of G, from Theorem 2, we know that every $(4, g)$-cage is edge-superconnected, which implies $|M| \geq 5$. Suppose C is a component of $G - M$, then $4|V(C)| - |M| = \sum_{v \in V(C)} d_C(v) \equiv 0 \pmod 2$. Thus $|M|$ must be even, therefore $|M| = 6$. □

The following lemma has been proved in [10].

Lemma 2. ([10]) *Let G be a $(4, g)$-cage with odd girth $g \geq 5$. Assume that there exists a non-trivial cutset X, and C is a component of $G - X$. Then there exists a vertex $u \in V(C)$ such that $d(u, X) \geq (g - 1)/2$.*

We can provide a stronger version of this lemma for $(4, g)$-cages.

Lemma 3. *Let G be a $(4, g)$-cage with odd girth $g \geq 5$. Assume that there exists a non-trivial cutset X, and C is a component of $G - X$. Then* $\max\{d(u, X) : u \in V(C)\} = (g-1)/2$.

Proof. $G - X$ contains exactly two components C and C' since $\lambda_1 = 6$. By Lemma 2, there exists a vertex $v \in V(C')$ such that $d_{C'}(v, X) \geq (g-1)/2$. Since the diameter of G is at most g, then $(g+1)/2 \geq \max\{d(u, X) : u \in V(C)\} \geq (g-1)/2$ by Lemma 2. Suppose there is a vertex $u \in C$ such that $d_C(u, X) = (g+1)/2$, then $d_{C'}(v, X) = (g-1)/2$. Denote $N_C(u) = \{u_1, u_2, u_3, u_4\}$, $N_{C'}(v) = \{v_1, v_2, v_3, v_4\}$ and $X = \{x_1, x_2, x_3, x_4\}$, then $d(u_i, v_j) \geq g - 2$, where $i, j = 1, 2, 3, 4$.

Claim 1. There are at least two pairs of vertices (u_i, v_j) such that $d(u_i, v_j) \geq g - 1$.

Otherwise there exists a vertex $s \in N(u) \cup N(v)$, and $d(u_i, v_j) = g - 2$ if $u_i, v_j \neq s$. Then each vertex in $N(u) - s$ is at distance $(g-1)/2$ to each vertex in X, and there are at least twelve shortest paths at distance $(g-1)/2$ from $N(u) - s$ to X which can not have a common vertex of $N(X) - X$, otherwise a cycle of length shorter than g appears in G. So there are at least twelve edges from X to C, then at most four edges left from X to C', a contradiction to Lemma 1.

Without loss of generality, assume $d(u_1, v_1) = d(u_2, v_2) = g-1$, then we can reconstruct a $(4, g')$-graph as follows: In $G' = G - u - v$, add a vertex y and six edges u_1v_1, u_2v_2, yu_3, yu_4, yv_3 and yv_4. so $|V(G')| < |V(G)|$, and it is clear that $g' \geq g$, a contradiction to Theorem 1. □

To prove that all $(4, g)$-cage with odd girth $g \geq 11$ are tightly superconnected, we assume that there exists a non-trivial cutset S of G. Let G_1 be the smaller component of $G - S$, we know that $|V(G_1)| \leq |V(G)/2| - 2$. We shall use two copies of G_1 to construct a $(4, g')$-graph of order less than $|V(G)|$, where $g' \geq g$, which then yields a contradiction to Theorem 1. Let $S = \{s_1, s_2, s_3, s_4\}$, based on the degree distribution of vertices in S, we will prove in the following two lemmas (the proofs are omitted due to the limits on the number of pages).

Lemma 4. *If $d_{G_1}(s_i) = 2$ and $d_{G_2}(s_i) = 2$ where $s_i \in S$ for $i = 1, 2, 3, 4$, then G is not a $(4, g)$-cage.*

Lemma 5. *If $d_{G_1}(s_1) = d_{G_1}(s_2) = 3$, $d_{G_2}(s_1) = d_{G_2}(s_2) = 1$ and $d_{G_1}(s_3) = d_{G_1}(s_4) = d_{G_2}(s_3) = d_{G_2}(s_4) = 2$, where $s_i \in S$, then G is not a $(4, g)$-cage.*

With this two lemmas, we are be able to prove the main theorem of our results.

Theorem 3. *Every $(4, g)$-cage with odd girth $g \geq 11$ is superconnected.*

Proof. Suppose G is not superconnected, then we choose a non-trivial cutset S of G such that S minimizes the order of the smaller component of $G - S$ among all non-trivial cutsets. Since $4|V(G_1)| - E(S, G_1) = \sum_{v \in V(G_1)} d_{G_1}(v) \equiv 0 \pmod 2$. Thus $E(S, G_1) \equiv 0 \pmod 2$.

Similarly, we have $E(S, G_2) \equiv 0 \pmod 2$. Since every $(4, g)$-cage is edge-superconnected, so we need only to discuss three cases for cutset S shown in Figure 3. (a) and (b) are impossible by Lemmas 4 and 5. For (c), we can simply delete edge s_1s_2 from $G[S]$ and use Lemma 4 to get a contradiction. □

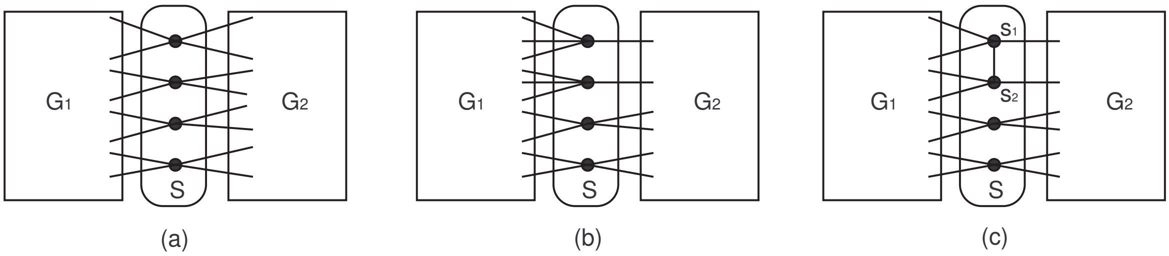

Figure 1: Three possible extremal cutsets of a $(4, g)$-cage G

Corollary 1. *Every* $(4, g)$*-cage with odd girth* $g \geq 11$ *is tightly superconnected.*

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