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Note

NP-completeness of 4-incidence colorability of semi-cubic graphs☆

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Abstract

The incidence coloring conjecture, proposed by Brualdi and Massey in 1993, states that the incidence coloring number of every graph is at most $\Delta+2$, where Δ is the maximum degree of a graph. The conjecture was shown to be false in general by Guiduli in 1997, following the work of Algor and Alon. However, in 2005 Maydanskiy proved that the conjecture holds for any graph with $\Delta \leqslant 3$. It is easily deduced that the incidence coloring number of a semi-cubic graph is 4 or 5. In this paper, we show that it is already NP-complete to determine if a semi-cubic graph is 4-incidence colorable, and therefore it is NP-complete to determine if a general graph is k-incidence colorable.

MSC: 05C15; 68Q17

Keywords: Incidence coloring number; k-incidence colorable; Strong-vertex coloring; Semi-cubic graph; NP-complete

1. Introduction

In this paper we consider undirected, finite and simple graphs only, and use standard notations in graph theory (see [2]). Let $G=(V,E)$ be a graph, and let

$$I(G)=\{(v,e)\colon v\in V, e\in E, \text{ and } v \text{ is incident with } e\}$$

be the set of all *incidences* of G. We say that two incidences (v,e) and (w,f) are *adjacent* if one of the following holds: (1) $v=w$, (2) *the edge* vw *is equal to* e *or* f.

Following Shiu et al. [7] we view G as a digraph by splitting each edge uv into two opposite arcs (u,v) and (v,u). For $e=uv$, we identify the incidence (u,e) with the arc (u,v). By a slight abuse of notation we will refer to the incidence (u,v) whenever it is convenient to do so. Two distinct incidences (u,v) and (x,y) are *adjacent* if one of the following holds: (i) $u=x$, (ii) $v=x$ or $u=y$. The configurations associated with (i) and (ii) are shown in Fig. 1.

We define an *incidence coloring* of G to be a coloring of its incidences in which adjacent incidences are assigned different colors. If $c\colon I(G)\to S$ is an incidence coloring of G and $|S|=k$, then G is called *k-incidence colorable*

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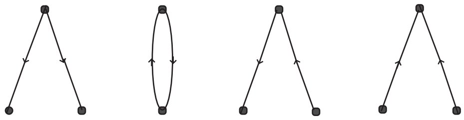

Fig. 1. Examples of adjacent and nonadjacent incidences.

and c is a *k-incidence coloring*, where S is a color-set. The *incidence coloring number* of G, denoted by $\chi_i(G)$, is the smallest number of colors in an incidence coloring.

This concept was first developed by Brualdi and Massey [3] in 1993. They proposed the *incidence coloring conjecture (ICC)*, which states that for every graph G, $\chi_i(G) \leqslant \Delta + 2$, where $\Delta = \Delta(G)$ is the maximum degree of G. In 1997 Guiduli [5] pointed out that the ICC is false, based on an example in [1]. He considered the Paley graphs of order p with $p \equiv 1 \pmod 4$. Following the analysis in [1], he showed that $\chi_i(G) \geqslant \Delta + \Omega(\log \Delta)$, where $\Omega = \frac{1}{8} - o(1)$. By using a tight upper bound for directed star arboricity, he obtained the upper bound $\chi_i(G) \leqslant \Delta + O(\log \Delta)$.

Brualdi and Massey [3] determined the incidence coloring numbers of trees, complete graphs and complete bipartite graphs. They also gave a simple bound for the incidence coloring number as follows.

Theorem 1.1 (*Brualdi and Massey [3]*). *For every graph G, $\Delta(G) + 1 \leqslant \chi_i(G) \leqslant 2\Delta(G)$.*

In [7] Shiu et al. proved that $\chi_i(G) \leqslant 5$ for several classes of cubic (3-regular) 2-connected graphs G, including Hamiltonian cubic graphs. In 2005 Maydanskiy [6] proved that the conjecture (ICC) holds for all graphs with $\Delta \leqslant 3$.

Theorem 1.2 (*Maydanskiy [6]*). *For every graph G with $\Delta(G) \leqslant 3$, $\chi_i(G) \leqslant 5$.*

Definition 1. For a graph G with $\Delta = 3$, if the degree of any vertex of G is 1 or 3, then the graph G is called a semi-cubic graph.

Using Theorems 1.1 and 1.2, we have the following corollary.

Corollary 1.3. *The incidence coloring number of a semi-cubic graph is* 4 *or* 5.

In this paper, we show the following result.

Theorem 1.4. *It is* NP-*complete to determine if a semi-cubic graph is* 4-*incidence colorable. Therefore, it is* NP-*complete to determine if a general graph is k-incidence colorable. So, it is* NP-*hard to determine the incidence coloring number for a general graph.*

2. Incidence coloring of semi-cubic graphs

In a given graph G, $N_G(v)$ denotes the set of vertices of G adjacent to v, and $d_G(v) = |N_G(v)|$ is the degree of a vertex v in G. A vertex of degree k is called a k-vertex. We denote the set of all the incidences of the form (v, u) and (u, v) by O_v and I_v, respectively, where u is adjacent with v. Let CO_v and CI_v denote the set of colors assigned to O_v and I_v in an incidence coloring of G, respectively.

We define a *strong vertex coloring* of G to be a proper vertex coloring such that for any $u, w \in N_G(v)$, u and w are assigned distinct colors. If $c: V(G) \to S$ is a strong vertex coloring of G and $|S| = k$, then G is called *k-strong-vertex colorable* and c is a *k-strong-vertex coloring* of G, where S is a color-set. In this case, we say that G is k-strong-vertex colored.

Lemma 2.1. *Let k be a positive integer. A graph G whose vertices have degree equal to k or* 1 *is* $(k+1)$-*incidence colorable if and only if G is* $(k+1)$-*strong-vertex colorable.*

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Proof. Since any two incidences in O_v are adjacent, $|CO_v|$ is equal to the degree of vertex v in an incidence coloring of G. If G is $(k+1)$-incidence colorable and c is a $(k+1)$-incidence coloring, then $|CI_v|$ is 1 for every vertex v of G. We can color vertex v using CI_v, and obtain a vertex coloring. Since incidences (u, v) and (v, u) are adjacent, $CI_v \neq CI_u$ whence the vertex coloring is proper. If $u, w \in N_G(v)$, then incidences (v, u) and (v, w) are adjacent. Thus u and w are assigned distinct colors in the vertex coloring. So the vertex coloring is a $(k+1)$-strong-vertex coloring and G is $(k+1)$-strong-vertex colorable.

Suppose there exists a $(k+1)$-strong-vertex coloring c of G, we color the incidence (u, v) of G by the color $c(v)$, where $c(v)$ denotes the color assigned to the vertex v in c. If two incidences (u, v) and (x, y) are adjacent, then one of the following holds: (1) $u = x$, $v \in N_G(u)$ and $y \in N_G(u)$; (2) $v = x$ or $u = y$, $vy \in E(G)$. From the definition of strong vertex coloring, incidences (u, v) and (x, y) are assigned different colors. So G is $(k+1)$-incidence colorable. □

3. The gadgets used in the construction

For semi-cubic graphs the 4-incidence colorability problem and the 4-strong-vertex colorability problem are defined formally as follows.

The 4-Incidence Colorability Problem (4ICP for short):

Instance: A semi-cubic graph G.

Question: Is there a 4-incidence coloring of G?

The 4-Strong-Vertex Colorability Problem (4SVCP for short):

Instance: A semi-cubic graph G.

Question: Is there a 4-strong-vertex coloring of G?

For terminology and known results of NP-completeness we refer to Garey and Johnson [4]. The 3SAT problem is stated as follows, which will be used in the sequel.

3SAT;

Instance: Set $U = \{u_1, u_2, \ldots, u_n\}$ of variables, collection $\mathscr{C}$ of clauses over U such that each clause $C_i \in \mathscr{C}$ has three literals $x_{i,1}, x_{i,2}, x_{i,3}$, where a literal $x_{i,j}$ is either a variable u_k or its negation $\overline{u_k}$.

Question: Is there a truth assignment to variables which simultaneously satisfies all the clauses in $\mathscr{C}$, where a clause is satisfied if one or more of its literals has value “true”?

It is clear that both the 4ICP and the 4SVCP are in the class NP. Given an instance $\mathscr{C}$ of the known NP-complete problem 3SAT, we will show how to construct a semi-cubic graph G of polynomial size in terms of the size of the instance $\mathscr{C}$ which is 4-incidence colorable and 4-strong-vertex colorable if and only if $\mathscr{C}$ is satisfiable. The graph G will be constructed from some pieces or “gadgets” which carry out specific tasks. Information will be carried between gadgets by pairs of vertices. In a 4-strong-vertex coloring of the graph G, such a pair of vertices is said to represent the value T (“true”) if the vertices have the same color, and to represent F (“false”) if the vertices have distinct colors. In the following we always use $S = \{1, 2, 3, 4\}$ to denote the set of colors.

3.1. The switch gadgets

First, we give a special semi-cubic graph H as shown in Fig. 2. We call H a Kite graph. It is easy to check that if the Kite graph H is 4-strong-vertex colored, all 1-vertices e, f, g have the same color.

The switch gadget is shown with its symbol in Fig. 3. It may be checked that this gadget is 4-strong-vertex colorable. If this gadget is 4-strong-vertex colored, without loss of generality, we suppose that $c(u)=1, c(w)=2, c(z)=3, c(v)=4$. Then $c(a') \in \{3, 4\}$, $c(c') \in \{2, 4\}$. If $c(a') = 3$ and $c(c') = 2$, then $c(b') = c(d') = 4$. Thus $(c(a'), c(b'), c(c'), c(d'))$

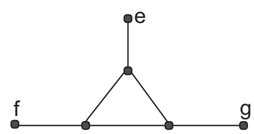

Fig. 2. The Kite graph H.

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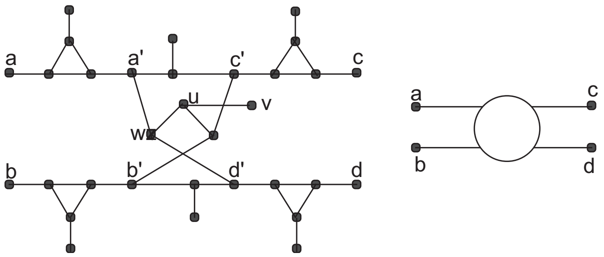

Fig. 3. The switch gadget and its symbolic representation.

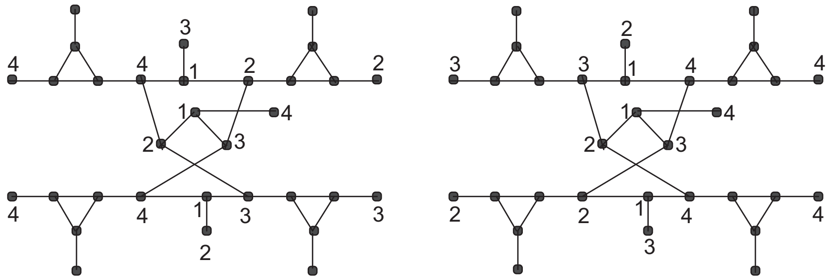

Fig. 4. Two non-equivalent ways to color the two pairs of vertices a, b and c, d.

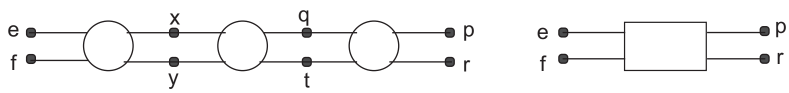

Fig. 5. The big switch gadget and its symbolic representation.

must be (4, 4, 2, 3) or (3, 2, 4, 4). One of the pairs of vertices marked a, b or c, d must then have the same color and the remaining pair of vertices must have distinct colors. Regarding the pair of vertices a, b as the input and the pair c, d as the output, there are only two non-equivalent ways to color the two pairs of vertices a, b and c, d, which is shown in Fig. 4. The gadget changes a representation of T to a representation of F, and vice versa.

3.2. *The variable gadgets*

The "big switch gadget" in Fig. 5 is constructed by using three switch gadgets, identifying the output of the first switch gadget with the input of the second switch gadget, then identifying the output of the second switch gadget with the input of the third switch gadget. We regard the pair of vertices e, f as the input and the pair p, r as the output of the big switch gadget. From the construction of the big switch gadget, we immediately obtain the following lemma. The details of proof are left to the readers.

Lemma 3.1. *Any big switch gadget G for which the input and output are e, f and p, r, respectively, is* 4*-strong-vertex colorable. If G is* 4*-strong-vertex colored, one of the following holds*:

(1) *if $c(e) = c(f)$, then $c(p) \neq c(r)$ and $c(p), c(r)$ may be any two different colors in $S = \{1, 2, 3, 4\}$*;
(2) *if $c(e) \neq c(f)$, then $c(p) = c(r)$ and $c(p)$ may be any color in $S = \{1, 2, 3, 4\}$, where $c(v)$ denotes the color assigned to the vertex v of G.*

From Lemma 3.1, the big switch gadget changes a representation of T to a representation of F, and vice versa.

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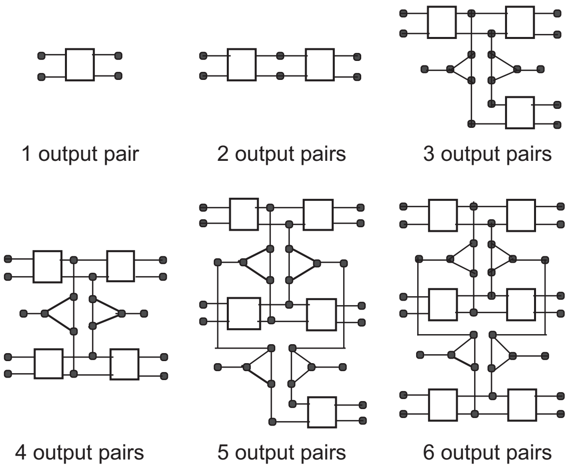

Fig. 6. The variable gadgets having $1, 2, \ldots, 6$ output pairs of vertices, respectively. More generally it is made from k big switch gadgets and $2 \cdot \lfloor (k-1)/2 \rfloor$ Kite graphs H and has k output pairs.

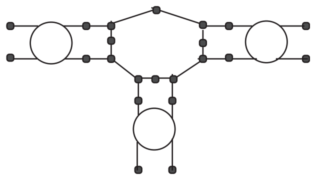

Fig. 7. The clause gadget made from three switch gadgets and a cycle of length 10, having three input pairs of vertices.

The truth or falsity of each variable u_i will be represented by a variable gadget as shown in Fig. 6, in which the gadgets have, respectively, $1, 2, \ldots, 6$ pairs of output vertices. In general, the number of output pairs in the gadget representing u_i should be equal to the total number of occurrences of u_i or $\overline{u_i}$ among the clauses of $\mathscr{C}$. If k pairs of output vertices are needed, we construct a variable gadget which is made from k big switch gadgets and $2 \cdot \lfloor (k-1)/2 \rfloor$ Kite graphs H.

From Lemma 3.1 and the construction of the big switch gadget, we immediately obtain the following lemma. The details of proof are left to the readers.

Lemma 3.2. *Any variable gadget G is 4-strong-vertex colorable, and in any 4-strong-vertex coloring of G, all the output pairs must represent the same value. If the output pairs represent T (“true”), then the color-set of any output pairs may be any color in $S = \{1, 2, 3, 4\}$. If, on the other hand, the output pairs represent F (“false”), then the color-set of any output pairs may be any two different colors in $S = \{1, 2, 3, 4\}$.*

3.3. The clause gadgets

The truth of each clause C_j will be tested by a clause gadget as shown in Fig. 7. The gadget is constructed from three switch gadgets and a cycle of length 10.

The following lemma is crucial for proving our main theorem.

Lemma 3.3. *The clause gadget is 4-strong-vertex colorable if and only if the three input pairs of vertices do not all represent F.*

Proof. Given a clause gadget G as shown in Fig. 8, we suppose that the three output pairs of vertices are $\{r, r'\}$, $\{w, w'\}$ and $\{z, z'\}$, respectively.

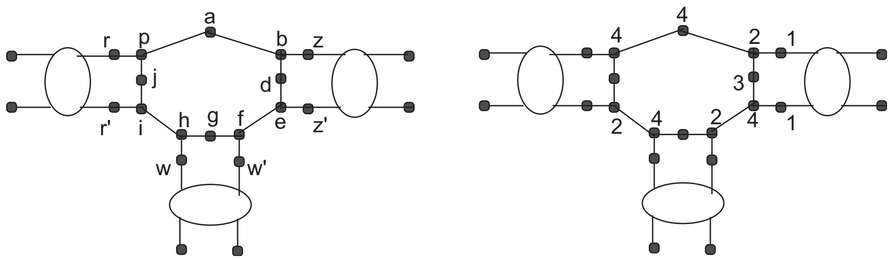

Fig. 8. The three input pairs of vertices all represent *F*.

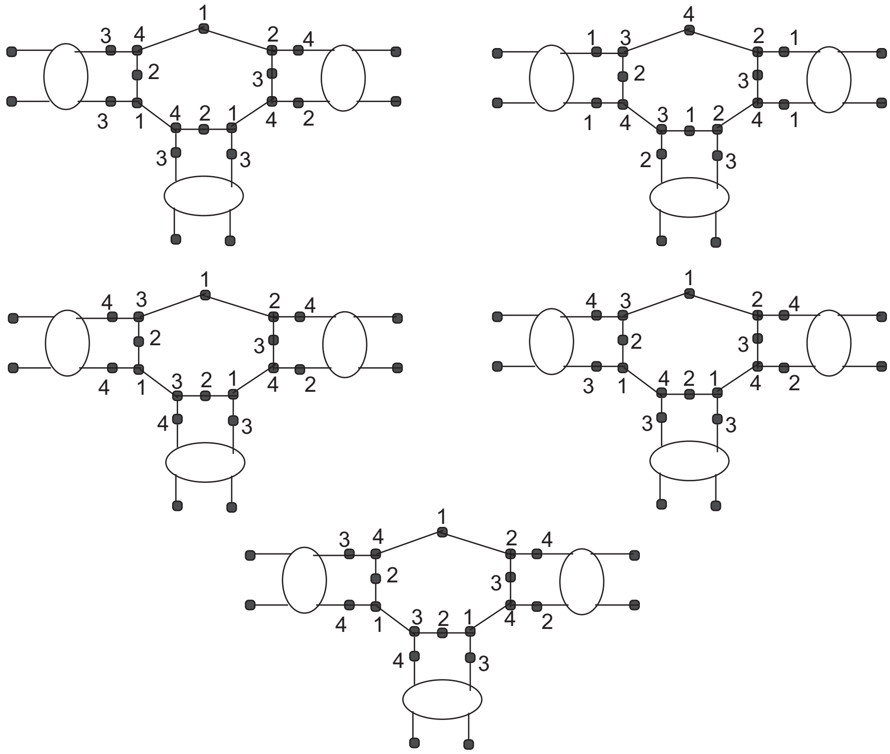

Fig. 9. The three input pairs of vertices that do not all represent *F*.

We suppose that the three input pairs of vertices all represent F. If G is 4-strong-vertex colorable and c is a 4-strong-vertex coloring, then $c(r)=c(r')$, $c(w)=c(w')$ and $c(z)=c(z')$. Remind that $c(v)$ denotes the color assigned to the vertex v in c. Without loss of generality, we suppose that $c(z)$, $c(b)$ and $c(d)$ are 1, 2 and 3, respectively. Then $c(a)=c(e)=4$, $c(f)=2$. Since $c(w)$ is equal to $c(w')$, $\{c(w), c(g)\}=\{c(w'), c(g)\}=\{1,3\}$. Thus $c(h)=4$, $c(i)=2$. By the same token, $c(p)$ must be 4. So $c(a)=c(p)=4$ and $pa \in E(G)$, a contradiction.

If the three input pairs of vertices do not all represent F, we can give a 4-strong-vertex coloring of G, which is shown in Fig. 9. Thus G is 4-strong-vertex colorable. □

4. Proof of the main result

In this section, we prove the main result that it is NP-complete to determine whether the incidence coloring number of a semi-cubic graph is 4 or 5. Thus this problem has no polynomial time algorithm unless $P=\mathrm{NP}$.

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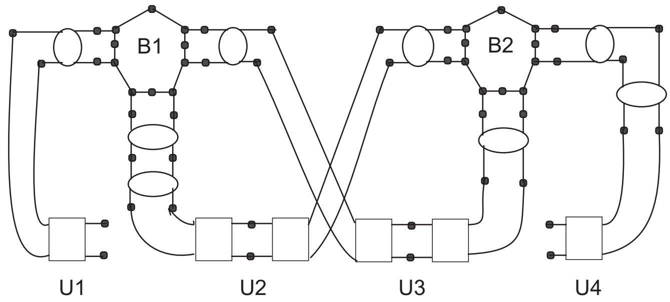

Fig. 10. The graph G'. We can obtain a semi-cubic graph G by adding a pendant edge to every 2-vertex of G'.

Proof of Theorem 1.4. The problem is clearly in the class NP. We exhibit a polynomial reduction from the problem 3SAT. Consider an instance $\mathscr{C}$ of 3SAT and construct from it a semi-cubic graph G as follows.

Each variable u_i corresponds to a variable gadget U_i with one output pair of vertices associated with each occurrence of u_i or $\overline{u_i}$ among the clauses of $\mathscr{C}$. Each clause C_j corresponds to a clause gadget B_j. Suppose literal $x_{j,k}$ in clause C_j is the variable u_i. Then identify the kth input pair of B_j with the associate output pair of U_i. If, on the other hand, $x_{j,k}$ is $\overline{u_i}$, then insert an switch gadget between the kth input pair of B_j and the associated output pair of U_i. The resulting graph G' has some 2-vertices. For every 2-vertex u, we add a pendant edge (u, u'). Now the resulting graph G is a semi-cubic graph.

From the above construction it easily follows that the incidence coloring number is 4 if and only if there is a truth assignment for $\mathscr{C}$. Furthermore, the reduction is clearly polynomial. □

Example 1. Let $\mathscr{C} = \{C_1, C_2\}$ be an instance of the problem 3SAT and

$$C_1 = u_1 \vee \overline{u_2} \vee u_3,$$

$$C_2 = u_2 \vee u_3 \vee \overline{u_4}.$$

By the proof of Theorem 1.4, we can construct the graph G' as shown in Fig. 10 and obtain the semi-cubic graph G by adding a pendant edge to every 2-vertex of G'.

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