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Integral trees of diameter 6☆

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Abstract

A graph G is called integral if all eigenvalues of its adjacency matrix $A(G)$ are integers. In this paper, the trees $T(p,q) \bullet T(r,m,t)$ and $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6 are defined. We determine their characteristic polynomials. We also obtain for the first time sufficient and conditions for them to be integral. To do so, we use number theory and apply a computer search. New families of integral trees of diameter 6 are presented. Some of these classes are infinite. They are different from those in the existing literature. We also prove that the problem of finding integral trees of diameter 6 is equivalent to the problem of solving some Diophantine equations. We give a positive answer to a question of Wang et al. [Families of integral trees with diameters 4, 6 and 8, Discrete Appl. Math. 136 (2004) 349–362].

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1. Introduction

We use G to denote a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix $A = A(G) = [a_{ij}]$ of G is an $n \times n$ symmetric matrix of 0's and 1's with $a_{ij} = 1$ if and only if v_i and v_j are joined by an edge. Let $P(G) = P(G, x) = |xI_n - A|$ denote the characteristic polynomial of G, where here and in the sequel I_n always denotes the $n \times n$ identity matrix. The spectrum of $A(G)$ is also called the *spectrum* of G.

A graph G is called *integral* if all eigenvalues of its characteristic polynomial $P(G, x)$ are integers. The quest for integral graphs was initiated by Harary and Schwenk in 1974 [10]. So far, there are many results on some particular classes of integral graphs, for instance, trees [1,5,6,10–17,22,23,25–28], cubic graphs [4,21], complete r-partite graphs [20,24],

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graphs with maximum degree 4 [2,3], 4-regular integral graphs [8], integral graphs which belong to the class $\overline{\alpha K_{a,b}}$ or $\overline{\alpha K_a \cup \beta K_{b,b}}$ [18,19], etc. Some graph operations, which when applied on integral graphs produce again integral graphs, are described in [10] or [7]. Other results on integral graphs can be found in [1,7,14]. For all other facts or terminology on graph spectra, see [7].

Let the tree $T(m,t)$ of diameter 4 be formed by joining the centers of m copies of $K_{1,t}$ to a new vertex v. Let the tree $T(r,m,t)$ of diameter 6 be obtained by joining the centers of $r(>1)$ copies of $T(m,t)$ to a new vertex w. If $r=1$, then let the tree $T(r,m,t)=T(1,m,t)$ of diameter 4 be formed by joining the center of $T(m,t)$ to a new vertex w. A graph G in which one vertex u is distinguished from the rest is called a *rooted graph*. The distinguished vertex u is called the *root-vertex*, or simply the root. If G is a rooted graph with root v, then G_v denotes the graph formed by deleting v from G. Given two rooted graphs G and H with roots v and w, respectively, we can define the following composite graphs:

(i) $G \bullet H$ is formed from G and H by identifying v and w.
(ii) $G \ominus H$ is formed from disjoint copies of G and H by one edge joining v and w.

Throughout the paper we assume that the centers of $K_{1,s}$, $T(m,t)$, $T(r,m,t)$, $K_{1,s} \bullet T(p,q)$ are the roots of these graphs, respectively. If $r=1$, then the trees $T(p,q) \bullet T(r,m,t) = T(p,q) \ominus T(m,t)$ and $K_{1,s} \bullet T(p,q) \bullet T(r,m,t) = [K_{1,s} \bullet T(p,q)] \ominus T(m,t)$ are trees with diameter 5. If we say the trees $T(p,q) \bullet T(r,m,t)$ and $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ are trees with diameter 6, then it hints $r>1$. The trees $T(p,q) \ominus T(m,t)$ and $[K_{1,s} \bullet T(p,q)] \ominus T(m,t)$ with diameter 5 were studied in [5,13,23].

In this paper, we determine the characteristic polynomials of the trees $T(p,q) \bullet T(r,m,t)$ and $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6. We also obtain for the first time sufficient and necessary conditions for the two classes of trees to be integral. To do so, we use number theory and apply a computer search. New families of integral trees of diameter 6 are presented. Some of these classes are infinite. They are different from those in the existing literature. We also prove that the problem of finding integral trees of diameter 6 is equivalent to the problem of solving some Diophantine equations. The discovery of these integral graphs is a new contribution to the search for such trees. We give a positive answer to a question of Wang et al. [27].

Lemma 1.1 (*Godsil and Mckay [9]*). *$P(G \bullet H, x) = P(G,x)P(H_w,x) + P(G_v,x)P(H,x) - xP(G_v,x)P(H_w,x)$, where v and w are the roots of G and H, respectively.*

Lemma 1.2. (1) (*Li and Lin* [13]) $P(K_{1,t},x) = x^{t-1}(x^2-t)$.
(2) (*Li and Lin* [13]) $P[T(m,t),x] = x^{m(t-1)+1}(x^2-t)^{m-1}[x^2-(m+t)]$.
(3) (*Li and Lin* [13]) $P[T(r,m,t),x] = x^{rm(t-1)+r-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}[x^4-(m+t+r)x^2+rt]$.
(4) (*Wang et al.* [26]) $P(K_{1,s} \bullet G, x) = x^{s-1}[xP(G,x) - sP(G-v,x)]$, *where v is the root of G.*

2. The characteristic polynomials of two classes of trees

In this section, we shall determine the characteristic polynomials of the two classes of trees $T(p,q) \bullet T(r,m,t)$ and $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$.

Theorem 2.1. *$P[T(p,q) \bullet T(r,m,t),x] = x^{rm(t-1)+p(q-1)+r-1}(x^2-q)^{p-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}\{x^6 - (p+q+m+t+r)x^4 + [(p+q)(m+t)+r(q+t)]x^2 - qrt\}$.*

Proof. Assume that the centers v and w of $T(p,q)$ and $T(r,m,t)$ are the roots of these trees, respectively, and let $G=T(p,q)$ and $H=T(r,m,t)$. By Lemma 1.1, we get that

$$P[T(p,q) \bullet T(r,m,t),x] = P[T(p,q),x]P^r[T(m,t),x] + P^p(K_{1,q},x)P[T(r,m,t),x] - xP^p(K_{1,q},x)P^r[T(m,t),x].$$

By Lemma 1.2(1)–(3), we get that

$$
\begin{aligned}
&P[T(p,q)\bullet T(r,m,t),x]\\
&\quad = x^{rm(t-1)+p(q-1)+r-1}(x^2-q)^{p-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}\{x^2[x^2-(p+q)]\\
&\qquad \cdot [x^2-(m+t)]+(x^2-q)[x^4-(m+t+r)x^2+rt]-x^2(x^2-q)[x^2-(m+t)]\}\\
&\quad = x^{rm(t-1)+p(q-1)+r-1}(x^2-q)^{p-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}\{(x^2-q)\\
&\qquad \cdot [x^4-(m+t+r)x^2+rt]-px^2[x^2-(m+t)]\}\\
&\quad = x^{rm(t-1)+p(q-1)+r-1}(x^2-q)^{p-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}\\
&\qquad \cdot \{x^6-(p+q+m+t+r)x^4+[(p+q)(m+t)+r(q+t)]x^2-qrt\}.
\end{aligned}
$$

Thus, this theorem is proved. □

Corollary 2.2. (1) *If* $r=1$, *then the characteristic polynomial of the tree* $T(p,q)\bullet T(r,m,t)=T(p,q)\ominus T(m,t)$ *of diameter* 5 *is* $P[T(p,q)\bullet T(r,m,t),x]=P([T(p,q)\ominus T(m,t),x]=x^{m(t-1)+p(q-1)}(x^2-q)^{p-1}(x^2-t)^{m-1}\{x^6-(p+q+m+t+1)x^4+[(p+q)(m+t)+q+t]x^2-qt\}$.

(2) *If* $m+t=q$, *then the characteristic polynomial of the tree* $T(p,q)\bullet T(r,m,t)$ *of diameter* 6 *is* $P[T(p,q)\bullet T(r,m,t),x]=x^{rm(t-1)+p(q-1)+r-1}(x^2-q)^{p+r-1}(x^2-t)^{r(m-1)}[x^4-(p+q+r)x^2+rt]$.

(3) *If* $p+q=t$, *then the characteristic polynomial of the tree* $T(p,q)\bullet T(r,m,t)$ *of diameter* 6 *is* $P[T(p,q)\bullet T(r,m,t),x]=x^{rm(t-1)+p(q-1)+r-1}(x^2-q)^{p-1}(x^2-t)^{r(m-1)+1}[x^2-(m+t)]^{r-1}[x^4-(m+t+r)x^2+qr]$.

(4) *If* $p=m$, $q=t$, *then the characteristic polynomial of the tree* $T(p,q)\bullet T(r,m,t)$ *of diameter* 6 *is* $P[T(m,t)\bullet T(r,m,t),x]=x^{m(r+1)(t-1)+r-1}(x^2-t)^{(r+1)(m-1)}[x^2-(m+t)]^{r-1}\{x^6-[2(m+t)+r]x^4+[(m+t)^2+2rt]x^2-rt^2\}$.

Proof. The proof follows directly from Theorem 2.1. □

Theorem 2.3. $P[K_{1,s}\bullet T(p,q)\bullet T(r,m,t),x]=x^{rm(t-1)+p(q-1)+r+s-1}(x^2-q)^{p-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}\{x^6-(p+q+s+m+t+r)x^4+[(p+q+s)(m+t)+r(q+t)+qs]x^2-q[s(m+t)+rt]\}$.

Proof. Assume that the centers u and v of $K_{1,s}$ and $T(p,q)\bullet T(r,m,t)$ are the roots of these trees, respectively, and let $G=T(p,q)\bullet T(r,m,t)$. By Lemma 1.2(4), we find

$$
\begin{aligned}
&P[K_{1,s}\bullet T(p,q)\bullet T(r,m,t),x]\\
&\quad = x^{s-1}\{xP[T(p,q)\bullet T(r,m,t),x]-sP^p(K_{1,q},x)P^r[T(m,t),x]\}.
\end{aligned}
$$

By Theorem 2.1 and Lemma 1.2(1)–(2), we obtain

$$
\begin{aligned}
&P[K_{1,s}\bullet T(p,q)\bullet T(r,m,t),x]\\
&\quad = x^{s-1}\{x\cdot x^{rm(t-1)+p(q-1)+r-1}(x^2-q)^{p-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}\{(x^2-q)\\
&\qquad \cdot [x^4-(m+t+r)x^2+rt]-px^2[x^2-(m+t)]\}-sx^{p(q-1)}(x^2-q)^p\cdot x^{rm(t-1)+r}\\
&\qquad \cdot (x^2-t)^{r(m-1)}[x^2-(m+t)]^r\}\\
&\quad = x^{rm(t-1)+p(q-1)+r+s-1}(x^2-q)^{p-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}\{(x^2-q)\\
&\qquad \cdot [x^4-(m+t+r)x^2+rt]-px^2[x^2-(m+t)]-s(x^2-q)[x^2-(m+t)]\}\\
&\quad = x^{rm(t-1)+p(q-1)+r+s-1}(x^2-q)^{p-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}\{x^6-(p+q+s+m\\
&\qquad +t+r)x^4+[(p+q+s)(m+t)+r(q+t)+qs]x^2-q[s(m+t)+rt]\}.
\end{aligned}
$$

Thus, this theorem is proved. □

As special cases we find:

Corollary 2.4. (1) (*Wang and Li* [23]) *If* $r=1$, *then the characteristic polynomial of the tree* $K_{1,s} \bullet T(p,q) \bullet T(r,m,t) = K_{1,s} \bullet T(p,q) \ominus T(m,t)$ *of diameter* 5 *is* $P[K_{1,s} \bullet T(p,q) \bullet T(r,m,t),x] = P[K_{1,s} \bullet T(p,q) \ominus T(m,t),x] = x^{m(t-1)+p(q-1)+s}(x^2-q)^{p-1}(x^2-t)^{m-1}\{x^6-(p+q+s+m+t+1)x^4+[(p+q+s)(m+t)+q+t+qs]x^2-q[s(m+t)+t]\}$.

(2) *If* $m+t=q$, *then the characteristic polynomial of the tree* $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ *of diameter* 6 *is* $P[K_{1,s} \bullet T(p,q) \bullet T(r,m,t),x] = x^{rm(t-1)+p(q-1)+r+s-1}(x^2-q)^{p+r-1}(x^2-t)^{r(m-1)}[x^4-(p+q+s+r)x^2+qs+rt]$.

3. Integral trees of diameter 6

In this section, we derive sufficient and necessary conditions for the trees $T(p,q) \bullet T(r,m,t)$ and $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6 to be integral. Some concrete sufficient conditions are also obtained by computer search. Some of these classes are infinite. They are different from those of [1,5,6,11–14,16,17,22,23,25–28]. For the case of $r=1$, such integral trees have investigated in [5] and [23], respectively. So we consider only the case of $r>1$ in the rest of the paper.

Theorem 3.1. *The tree* $T(p,q) \bullet T(r,m,t)$ *of diameter* 6 *is integral if and only if there exist natural numbers* a, b *and* c *such that* $x^6-(p+q+m+t+r)x^4+[(p+q)(m+t)+r(q+t)]x^2-qrt$ *can be factorized as* $(x^2-a^2)(x^2-b^2)(x^2-c^2)$, *and one of the following four conditions holds*: (i) $p>1$, $m>1$, q, t *and* $m+t$ *are perfect squares*, (ii) $p=1$, $m>1$, t *and* $m+t$ *are perfect squares*, (iii) $p>1$, $m=1$, q *and* $m+t$ *are perfect squares*, (iv) $p=m=1$, $m+t$ *is a perfect square*.

In particular, we have the following results for the tree $T(p,q) \bullet T(r,m,t)$ *of diameter* 6.

(1) *If* $m+t=q$, *then* $T(p,q) \bullet T(r,m,t)$ *is integral if and only if there exist natural numbers* a *and* b *such that* $x^4-(p+q+r)x^2+rt$ *can be factorized as* $(x^2-a^2)(x^2-b^2)$, *and one of the following two conditions holds*: (i) $m>1$, t *and* $q(=m+t)$ *are perfect squares*, (ii) $m=1$, $q(=1+t)$ *is a perfect square*.

(2) *If* $p+q=t$, *then* $T(p,q) \bullet T(r,m,t)$ *is integral if and only if there exist natural numbers* a *and* b *such that* $x^4-(m+t+r)x^2+qr$ *can be factorized as* $(x^2-a^2)(x^2-b^2)$, *and one of the following two conditions holds*: (i) $p>1$, t, q *and* $m+t$ *are perfect squares*, (ii) $p=1$, t *and* $m+t$ *are perfect squares*.

(3) *If* $p=m$, $q=t$, *then* $T(p,q) \bullet T(r,m,t) = T(m,t) \bullet T(r,m,t)$ *is integral if and only if there exist natural numbers* a, b *and* c *such that* $x^6-[2(m+t)+r]x^4+[(m+t)^2+2rt]x^2-rt^2$ *can be factorized as* $(x^2-a^2)(x^2-b^2)(x^2-c^2)$, *and one of the following two conditions holds*: (i) $m>1$, t *and* $m+t$ *are perfect squares*, (ii) $m=1$, $m+t$ *is a perfect square*.

Proof. By Theorem 2.1, we know that the tree $T(p,q) \bullet T(r,m,t)$ of diameter 6 is integral if and only if the equation $(x^2-q)^{p-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}\{x^6-(p+q+m+t+r)x^4+[(p+q)(m+t)+r(q+t)]x^2-qrt\}=0$ has only integral roots, where p, q, r, m and t are positive integers, and it hints $r>1$.

From the above representation the main statement directly follows.

From Theorem 2.1 or Corollary 2.2(2)–(4), it is not difficult to prove in a similar way that (1)–(3) are also true. □

Corollary 3.2. *Let* $p=m$, $q=t$, *and suppose that the tree* $T(m,t) \bullet T(r,m,t)$ *of diameter* 6 *is integral*, *such that* t *and* $m+t$ *are perfect squares*, *then for any positive integer* n *the tree* $T(mn^2,tn^2) \bullet T(rn^2,mn^2,tn^2)$ *of diameter* 6 *is integral*, *too*.

Proof. Because the tree $T(m,t) \bullet T(r,m,t)$ of diameter 6 is integral, t and $m+t$ are perfect squares, by Theorem 3.1(3) or Corollary 2.2(4), we find

$$P[T(m,t) \bullet T(r,m,t),x] = x^{m(r+1)(t-1)+r-1}(x^2-t)^{(r+1)(m-1)}[x^2-(m+t)]^{r-1}$$
$$\cdot\{x^6-[2(m+t)+r]x^4+[(m+t)^2+2rt]x^2-rt^2\}.$$
$$= x^{m(r+1)(t-1)+r-1}(x^2-t)^{(r+1)(m-1)}[x^2-(m+t)]^{r-1}(x^2-a^2)(x^2-b^2)(x^2-c^2),$$

where a, b and c are integers, t and $m+t$ are perfect squares.

Table 1
Integral trees $T(p,q)\bullet T(r,m,t)$ of diameter 6, where $m+t=q$, $m>1$

a	b	p	q	r	m	t	a	b	p	q	r	m	t
1	4	4	9	4	5	4	1	6	17	16	4	7	9
1	6	19	9	9	5	4	1	6	8	25	4	16	9
1	6	12	16	9	12	4	1	6	3	25	9	21	4
2	6	6	25	9	9	16	2	6	8	16	16	7	9
2	8	27	25	16	9	16	2	8	16	36	16	20	16
2	8	3	49	16	33	16	3	8	8	49	16	13	36
3	8	12	25	36	9	16	3	8	1	36	36	20	16
4	9	12	49	36	13	36	4	10	10	81	25	17	64
4	10	16	36	64	11	25	4	10	3	49	64	24	25
6	12	16	100	64	19	81	6	12	18	81	81	17	64

Table 2
Integral trees $T(p,q)\bullet T(r,m,t)$ of diameter 6, where $m+t=q$, $m=1$

a	b	p	q	r	m	t	a	b	p	q	r	m	t
1	3	3	4	3	1	3	1	4	6	9	2	1	8
1	6	21	4	12	1	3	2	4	3	9	8	1	8
2	6	13	9	18	1	8	2	6	9	25	6	1	24
2	8	27	9	32	1	8	3	5	3	16	15	1	15
3	8	24	25	24	1	24	3	8	12	49	12	1	48
4	6	3	25	24	1	24	4	9	18	25	54	1	24
4	9	21	49	27	1	48	4	10	15	81	20	1	80
5	7	3	36	35	1	35	6	8	3	49	48	1	48
6	10	12	49	75	1	48	6	10	10	81	45	1	80
6	11	13	100	44	1	99	6	12	23	49	108	1	48

By Theorem 3.1(3) or Corollary 2.2(4), we obtain

$$\begin{aligned}
&P[T(mn^2,tn^2)\bullet T(rn^2,mn^2,tn^2),x]\\
&\quad = x^{mn^2(rn^2+1)(tn^2-1)+rn^2-1}(x^2-tn^2)^{(rn^2+1)(mn^2-1)}\\
&\qquad \cdot [x^2-(m+t)n^2]^{rn^2-1}\{x^6-[2(m+t)+r]n^2x^4+[(m+t)^2+2rt]n^4x^2-rt^2n^6\}.\\
&\quad = x^{mn^2(rn^2+1)(tn^2-1)+rn^2-1}(x^2-tn^2)^{(rn^2+1)(mn^2-1)}[x^2-(m+t)n^2]^{rn^2-1}\\
&\qquad \cdot (x^2-a^2n^2)(x^2-b^2n^2)(x^2-c^2n^2),
\end{aligned}$$

where a, b and c are integers, t and $m+t$ are perfect squares.

This proves the corollary. □

Theorem 3.3. *For the tree $T(p,q)\bullet T(r,m,t)$ of diameter* 6, *let a, b, c, p, q, r, m and t be as in Theorem* 3.1(1)–(3) *or Theorem* 3.1. *Then the following statements hold*:

(1) *For $m+t=q$, $m>1$ the tree $T(p,q)\bullet T(r,m,t)$ is integral if and only if $t=k^2$, $m=n^2+2nk$, $r=a^2b^2/k^2(>1)$, $q=(n+k)^2$, and $p=a^2+b^2-(n+k)^2-a^2b^2/k^2(\geqslant 1)$, where a, b, k and n are positive integers. Examples are presented in Table* 1. (*Table* 1 *is obtained by computer search*, *where* $1\leqslant a\leqslant 6$, $a\leqslant b\leqslant a+6$.)

(2) *For $m+t=q$, $m=1$ the tree $T(p,q)\bullet T(r,m,t)$ is integral if and only if $t=n^2+2n$, $m=1$, $r=a^2b^2/(n^2+2n)(>1)$, $q=(n+1)^2$, and $p=a^2+b^2-(n+1)^2-a^2b^2/(n^2+2n)(\geqslant 1)$, where a, b and n are positive integers. Examples are presented in Table* 2. (*Table* 2 *is obtained by computer search*, *where* $1\leqslant a\leqslant 6$, $a\leqslant b\leqslant a+6$.)

Table 3
Integral trees $T(p,q)\bullet T(r,m,t)$ of diameter 6, where $p+q=t$, $p>1$

a	b	p	q	r	m	t	a	b	p	q	r	m	t
1	8	32	4	16	13	36	1	8	21	4	16	24	25
1	8	12	4	16	33	16	1	8	5	4	16	40	9
2	9	27	9	36	13	36	2	9	16	9	36	24	25
2	9	7	9	36	33	16	3	14	95	49	36	25	144
3	14	72	49	36	48	121	3	14	51	49	36	69	100
3	14	32	49	36	88	81	3	14	15	49	36	105	64
3	16	84	16	144	21	100	3	16	65	16	144	40	81
3	16	48	16	144	57	64	3	16	33	16	144	72	49
3	16	20	16	144	85	36	3	16	9	16	144	96	25
4	18	133	36	144	27	169	4	18	108	36	144	52	144
4	18	85	36	144	75	121	4	18	64	36	144	96	100
4	18	45	36	144	115	81	4	18	28	36	144	132	64
4	18	13	36	144	147	49	5	14	51	49	100	21	100
5	14	32	49	100	40	81	5	14	15	49	100	57	64

Table 4
Integral trees $T(p,q)\bullet T(r,m,t)$ of diameter 6, where $p+q=t$, $p=1$

a	b	p	q	r	m	t	a	b	p	q	r	m	t
1	6	1	3	12	21	4	1	16	1	8	32	216	9
2	8	1	8	32	27	9	2	15	1	15	60	153	16
3	8	1	24	24	24	25	3	10	1	15	60	33	16
3	16	1	24	96	144	25	3	20	1	15	240	153	16
4	12	1	24	96	39	25	4	15	1	80	45	115	81
4	24	1	48	192	351	49	5	14	1	35	140	45	36

(3) *For $p+q=t$, $p>1$ the tree $T(p,q)\bullet T(r,m,t)$ is integral if and only if $t=k^2$, $m=n^2+2nk$, $r=a^2b^2/s^2(>1)$, $q=s^2$, and $p=k^2-s^2(\geqslant 1)$, where a, b, k, n and s are positive integers satisfying*

$$s^2(a^2+b^2)=s^2(n+k)^2+a^2b^2. \tag{1}$$

Examples are presented in Table 3. (Table 3 is obtained by computer search, where $1\leqslant a\leqslant 5$, $a\leqslant b\leqslant 2a+10$.)

(4) *For $p+q=t$, $p=1$ the tree $T(p,q)\bullet T(r,m,t)$ is integral if and only if $t=k^2$, $m=n^2+2nk$, $r=a^2b^2/(k^2-1)(>1)$, $q=k^2-1(\geqslant 1)$, and $p=1$, where a, b, k and n are positive integers satisfying*

$$(k^2-1)(a^2+b^2)=(k^2-1)(n+k)^2+a^2b^2. \tag{2}$$

Examples are presented in Table 4. (Table 4 is obtained by computer search, where $1\leqslant a\leqslant 5$, $a\leqslant b\leqslant a+20$.)

(5) *For $p=m$, $q=t$, suppose that $m>1$, t and $m+t$ are perfect squares. Let a,b,c,p,q,r,m and t be as in Theorem 3.1(3) and given in Table 5, then for any positive integer n the tree $T(mn^2,tn^2)\bullet T(rn^2,mn^2,tn^2)$ of diameter 6 is integral. (Table 5 is obtained by computer search, where $1\leqslant a\leqslant 10$, $a\leqslant b\leqslant 50$, $b\leqslant c\leqslant 200$.)*

(6) *For $m+t\neq q$, $p+q\neq t$, $(p,q)\neq(m,t)$, $p>1$, $m>1$, suppose that q, t and $m+t$ are perfect squares. Let a, b, c, p, q, r, m and t be as in Theorem 3.1 and given in Table 6, then $T(p,q)\bullet T(r,m,t)$ is integral. (Table 6 is obtained by computer search, where $1\leqslant a\leqslant 7$, $a\leqslant b\leqslant a+9$, $b\leqslant c\leqslant 20$.)*

(7) *For $m+t\neq q$, $p+q\neq t$, $(p,q)\neq(m,t)$, $p=1$, $m>1$, suppose that t and $m+t$ are perfect squares. Let a, b, c, p, q, r, m and t be as in Theorem 3.1 and given in Table 7, then $T(p,q)\bullet T(r,m,t)$ is integral. (Table 7 is obtained by computer search, where $1\leqslant a\leqslant 18$, $a\leqslant b\leqslant a+10$, $b\leqslant c\leqslant b+20$.)*

Proof. (1)–(2). When $q=m+t$, by Theorem 3.1(1), it follows that the tree $T(p,q)\bullet T(r,m,t)$ is integral if and only if a,b,p,q,r,t are positive integral solutions for Eqs. (3) and one of the following two conditions holds: (i) $m>1$, t

Table 5
Integral trees $T(mn^2, tn^2) \bullet T(rn^2, mn^2, tn^2)$, where t and $m+t$ are perfect squares, and n is a positive integer

a	b	c	r	m	t	a	b	c	r	m	t
1	6	15	100	72	9	1	7	27	441	160	9
1	19	75	3249	1344	25	1	22	54	1089	1120	36
1	37	147	12 321	5280	49	1	40	104	4225	4032	64
2	12	30	400	288	36	2	14	54	1764	640	36
2	38	150	12 996	5376	100	2	44	108	4356	4480	144
3	7	25	441	96	25	3	10	42	1225	288	36
3	13	49	1521	480	49	3	18	45	900	648	81
3	21	81	3969	1440	81	3	31	121	8649	3360	121
3	35	60	784	1800	225	3	43	169	16 641	6720	169
4	13	48	1521	420	64	4	24	60	1600	1152	144
4	28	108	7056	2560	144	4	49	192	21 609	8580	256
5	30	75	2500	1800	225	5	35	135	11 025	4000	225
6	14	50	1764	384	100	6	15	90	6561	800	100
6	20	84	4900	1152	144	6	26	98	6084	1920	196
6	33	49	484	1080	441	6	36	90	3600	2592	324
6	42	162	15 876	5760	324	7	42	105	4900	3528	441
7	49	189	21 609	7840	441	8	26	96	6084	1680	256
8	48	120	6400	4608	576	9	21	75	3969	864	225
9	30	126	11 025	2592	324	9	39	147	13 689	4320	441
10	21	165	23 716	1800	225	10	49	65	676	1800	1225

Table 6
Integral trees $T(p, q) \bullet T(r, m, t)$ of diameter 6, where $m+t \neq q$, $p+q \neq t$, $(p, q) \neq (m, t)$, and q, t, $m+t$ are perfect squares

a	b	c	p	q	r	m	t	a	b	c	p	q	r	m	t
1	4	9	33	4	36	16	9	1	4	9	28	9	36	21	4
1	4	15	17	4	100	112	9	1	4	15	12	9	100	117	4
1	7	15	48	9	49	144	25	1	7	15	32	25	49	160	9
1	7	20	81	4	196	144	25	1	7	20	60	25	196	165	4
1	9	20	60	25	36	325	36	1	9	20	49	36	36	336	25
1	10	14	135	16	25	72	49	1	10	14	102	49	25	105	16
2	5	18	70	9	225	33	16	2	5	18	63	16	225	40	9
2	6	20	24	16	144	231	25	2	6	20	15	25	144	240	16
2	8	18	132	16	144	64	36	2	8	18	112	36	144	84	16
2	11	20	210	25	121	105	64	2	11	20	171	64	121	144	25
3	8	15	68	81	100	33	16	3	12	20	220	64	100	88	81
3	12	20	203	81	100	105	64	4	10	18	88	144	144	39	25

Table 7
Integral trees $T(p, q) \bullet T(r, m, t)$ of diameter 6, where $m+t \neq q$, $p+q \neq t$, $(p, q) \neq (m, t)$, $p = 1$, $m > 1$, t and $m+t$ are perfect squares

a	b	c	p	q	r	m	t	a	b	c	p	q	r	m	t
8	9	20	1	75	108	105	256	14	16	27	1	252	252	100	576

and $q(=m+t)$ are perfect squares, (ii) $m = 1$, $q(=1+t)$ is a perfect square.

$$\begin{cases} a^2 + b^2 = p + q + r, \\ a^2b^2 = rt. \end{cases} \tag{3}$$

(1) Assume that $t = k^2$, $q(=m+t) = (n+k)^2$, *i.e.* $m = n^2 + 2nk$. Then by Eqs. (3), we get $r = a^2b^2/k^2(>1)$, $q = m + t = (n+k)^2$, and $p = a^2 + b^2 - (n+k)^2 - a^2b^2/k^2(\geqslant 1)$, where a, b and k are positive integers.

For the tree $T(m, t) \bullet T(r, m, t)$ of diameter 6 with $m + t = q$, by computer search, based on Theorem 3.1(1), we have found some positive integral solutions a, b, p, q, r, m and t for Eqs. (3) satisfying $t = k^2$, $m = n^2 + 2nk$,

$q(=m+t)=(n+k)^2$, $1 \leqslant a \leqslant 6$, $a \leqslant b \leqslant a+6$. So we can construct such integral trees $T(p,q) \bullet T(r,m,t)$ from Theorem 3.3(1). They are different from those in the existing literature.

(2) Assume that $m=1$, $t=n^2+2n$. Then by Eqs. (3), we find $r=a^2b^2/(n^2+2n)(>1)$, $q=1+t=(n+1)^2$, $p=a^2+b^2-(n+1)^2-a^2b^2/(n^2+2n)(\geqslant 1)$, where a, b and n are positive integers.

Similarly, we have found some positive integral solutions a, b, p, q, r, m and t for Eqs. (3) satisfying $m=1$, $t=n^2+2n$, $q(=m+t=1+t)=(n+1)^2$, $1 \leqslant a \leqslant 6$, $a \leqslant b \leqslant a+6$.

(3)–(7) These results follow from Theorem 3.1(2)–(3) and Corollary 3.2 or Theorem 3.1 by arguments similar to those used in Theorem 3.3(1) or (2). □

Remark 3.4. When $p+q=t$, by using Theorem 3.1(2), we can construct the integral tree $T(p,q) \bullet T(r,m,t)$ from any positive integral solution of the following Diophantine equations (4) satisfying one of the two conditions of Theorem 3.1(2). So, it is very important to find positive integral solutions for Eqs. (4) satisfying one of the two conditions of Theorem 3.1(2).

$$\begin{cases} a^2+b^2=m+t+r, \\ a^2b^2=qr. \end{cases} \tag{4}$$

For the case $p+q=t$, by Theorem 3.3(3)–(4) and examples in Tables 3 and 4, we know that there exist many such integral trees $T(p,q) \bullet T(r,m,t)$ of diameter 6 satisfying one of the two conditions of Theorem 3.1(2).

Hence, we raise the following question.

Question 3.5. *For $p+q=t$, are there infinitely many positive integral solutions for Eqs.* (4) *satisfying one of the two conditions of Theorem* 3.1(2)? *If yes, how can they be found, i.e. what are all positive integral solutions for Eq.* (1) *or Eq.* (2)?

Remark 3.6. In view of Theorem 3.1(3), we have to find positive integral solutions of the following Diophantine equations (5) satisfying one of the two conditions of Theorem 3.1(3).

$$\begin{cases} a^2+b^2+c^2=2(m+t)+r, \\ a^2b^2+b^2c^2+c^2a^2=(m+t)^2+2rt, \\ a^2b^2c^2=rt^2. \end{cases} \tag{5}$$

For the tree $T(m,t) \bullet T(r,m,t)$ of diameter 6 with $p=m$, $q=t$, by Corollary 3.2 and Theorem 3.3(5), there exist infinitely many such integral trees satisfying Condition (i) of Theorem 3.1(3). However, we have to mention that for the case (ii) of Theorem 3.1(3), when $m=1$, $m+t$ is a perfect square, $1 \leqslant a \leqslant 20$, $a \leqslant b \leqslant 5a+10$, $b \leqslant c \leqslant 5b+10$, we have not found positive integral solutions of Eqs. (5) by computer search.

Hence, we raise the following question.

Question 3.7. *For the tree $T(p,q) \bullet T(r,m,t)$ of diameter* 6 *with $p=m$, $q=t$, we ask*:

(1) *What are all positive integral solutions for Eqs.* (5) *satisfying Condition* (i) *of Theorem* 3.1(3)?
(2) *Are there positive integral solutions for Eqs.* (5) *satisfying Condition* (ii) *of Theorem* 3.1(3)? *If the answer is yes, how can we find all positive integral solutions of Eqs.* (5)?

Similarly, we have:

Remark 3.8. When $m+t \neq q$, $p+q \neq t$, and $(p,q) \neq (m,t)$, by Theorem 3.1, we have to find positive integral solutions for the following Diophantine equations (6) satisfying one of the four conditions of Theorem 3.1.

$$\begin{cases} a^2+b^2+c^2=p+q+m+t+r, \\ a^2b^2+b^2c^2+c^2a^2=(p+q)(m+t)+r(q+t), \\ a^2b^2c^2=qrt. \end{cases} \tag{6}$$

For the tree $T(p,q)\bullet T(r,m,t)$ of diameter 6 with $m+t\neq q$, $p+q\neq t$, and $(p,q)\neq(m,t)$, by Theorem 3.3(6)–(7), there exist many such integral trees satisfying Condition (i) or Condition (ii) of Theorem 3.1. However we also note that for the following cases: (iii) when $p>1$, $m=1$, q and $m+t$ are perfect squares, $1\leqslant a\leqslant 10$, $a\leqslant b\leqslant a+10$, $b\leqslant c\leqslant b+20$, (iv) when $p=m=1$, $m+t$ is a perfect square, $1\leqslant a\leqslant 10$, $a\leqslant b\leqslant a+10$, $b\leqslant c\leqslant b+20$, we have not found positive integral solutions of Eqs. (6) by computer search.

Hence, we raise the following question.

Question 3.9. *For the tree $T(p,q)\bullet T(r,m,t)$ of diameter 6 with $m+t\neq q$, $p+q\neq t$, and $(p,q)\neq(m,t)$, we ask*:

(1) *What are all positive integral solutions for Eqs.* (6) *satisfying Condition* (i) *or Condition* (ii) *of Theorem* 3.1?
(2) *Are there positive integral solutions for Eqs.* (6) *satisfying Condition* (iii) *or Condition* (iv) *of Theorem* 3.1? *If yes, how to find all positive integral solutions of Eqs.* (6)?

Theorem 3.10. *The tree $K_{1,s}\bullet T(p,q)\bullet T(r,m,t)$ of diameter 6 is integral if and only if there exist natural numbers a, b and c such that $x^6-(p+q+s+m+t+r)x^4+[(p+q+s)(m+t)+r(q+t)+qs]x^2-q[s(m+t)+rt]$ can be factorized as $(x^2-a^2)(x^2-b^2)(x^2-c^2)$, and one of the following four conditions holds*: (i) $m>1$, $p>1$, q, t *and* $m+t$ *are perfect squares*, (ii) $p=1$, $m>1$, t *and* $m+t$ *are perfect squares*, (iii) $m=1$, $p>1$, q *and* $m+t$ *are perfect squares*, (iv) $m=p=1$, $m+t$ *is a perfect square*.

In particular, if $m+t=q$, then the tree $K_{1,s}\bullet T(p,q)\bullet T(r,m,t)$ of diameter 6 is integral if and only if there exist natural numbers a and b such that $x^4-(p+q+s+r)x^2+qs+rt$ can be factorized as $(x^2-a^2)(x^2-b^2)$, and one of the following two conditions holds: (i) $m>1$, q *and* t *are perfect squares*, (ii) $m=1$, q *is a perfect square*.

Proof. Follows from Theorem 2.3, as similarly Theorem 3.3 did. □

Theorem 3.11. *For the tree $K_{1,s}\bullet T(m,t)\bullet T(r,m,t)$ of diameter* 6, *let a,b,c,s,p,q,r,m and t be as in Theorem* 3.10. *we have the following results*:

(1) *For $m+t=q$, $m>1$ the tree $K_{1,s}\bullet T(p,q)\bullet T(r,m,t)$ is integral if and only if $t=k^2$, $m=n^2+2nk$, $q=(n+k)^2$, and s, p, $q(=n+k)^2)$, $r(>1)$, $m(=n^2+2nk)$ and $t(=k^2)$ are positive integers satisfying*

$$\begin{cases} a^2+b^2=p+(n+k)^2+r+s,\\ a^2b^2=rk^2+s(n+k)^2. \end{cases} \tag{7}$$

Examples are presented in Table 8. (*Table* 8 *is obtained by computer search, where* $1\leqslant a\leqslant 3$, $a\leqslant b\leqslant a+4$.)
(2) *For $m+t=q$, $m=1$ the tree $K_{1,s}\bullet T(p,q)\bullet T(r,m,t)$ is integral if and only if $t=n^2+2n$, $m=1$, $q=(n+1)^2$, and s, p, $q(=n+1)^2)$, $r(>1)$, $m(=1)$ and $t(=n^2+2n)$ are positive integers satisfying*

$$\begin{cases} a^2+b^2=p+(n+1)^2+r+s,\\ a^2b^2=r(n^2+2n)+s(n+1)^2. \end{cases} \tag{8}$$

Examples are presented in Table 9. (*Table* 9 *is obtained by computer search, where* $1\leqslant a\leqslant 4$, $a\leqslant b\leqslant a+5$.)
(3) *For $m+t\neq q$, $p>1$, $m>1$, suppose that q, t and $m+t$ are perfect squares, and $a,b,c,s,p,q,r(>1),m,t$ are given in Table* 10, *then $K_{1,s}\bullet T(p,q)\bullet T(r,m,t)$ is integral*. (*Table* 10 *is obtained by computer search, where* $1\leqslant a\leqslant 4$, $a\leqslant b\leqslant a+9$, $b\leqslant c\leqslant 15$.)
(4) *For $m+t\neq q$, $p=1$, $m>1$, suppose that t and $m+t$ are perfect squares, and $a,b,c,s,p,q,r(>1),m,t$ are given in Table* 11, *then $K_{1,s}\bullet T(p,q)\bullet T(r,m,t)$ is integral*. (*Table* 11 *is obtained by computer search, where* $1\leqslant a\leqslant 9$, $a\leqslant b\leqslant 10$, $b\leqslant c\leqslant 20$.)

Proof. (1)–(2). When $m+t=q$, by the special case of Theorem 3.10, it follows that the tree $K_{1,s}\bullet T(p,q)\bullet T(r,m,t)$ is integral if and only if $a,b,s,p,q,r(>1),t$ are positive integral solutions for Eqs. (7) and one of the following two

Table 8
Integral trees $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6 , where $m+t=q, m>1$

a	b	s	p	q	r	m	t	a	b	s	p	q	r	m	t
1	4	3	6	4	4	3	1	1	4	2	3	4	8	3	1
1	5	1	12	9	4	5	4	1	5	5	12	4	5	3	1
1	5	4	9	4	9	3	1	1	5	3	6	4	13	3	1
1	5	2	3	4	17	3	1	1	5	2	8	9	7	8	1
2	5	4	5	16	4	7	9	2	5	8	5	9	7	5	4
2	5	5	3	16	5	12	4	2	5	6	3	16	4	15	1
2	6	12	10	9	9	5	4	2	6	8	5	9	18	5	4
2	6	8	12	16	4	12	4	2	6	7	9	16	8	12	4
2	6	15	7	9	9	8	1	2	6	6	6	16	12	12	4
2	6	5	3	16	16	12	4	3	6	18	7	16	4	7	9
3	6	4	2	25	14	9	16	3	6	19	5	16	5	12	4
3	6	18	2	16	9	12	4	3	6	20	5	16	4	15	1
3	6	12	2	25	6	21	4	3	7	6	7	36	9	11	25
3	7	18	7	16	17	7	9	3	7	1	6	25	26	9	16
3	7	17	12	25	4	21	4	3	7	11	6	36	5	27	9
3	7	10	3	36	9	27	9	3	7	27	6	16	9	15	1

Table 9
Integral trees $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6, where $m+t=q, m=1$

a	b	s	p	q	r	m	t	a	b	c	p	q	r	m	t
1	4	1	8	4	4	1	3	1	5	4	15	4	3	1	3
1	5	1	14	4	7	1	3	1	5	1	14	9	2	1	8
1	6	6	23	4	4	1	3	1	6	3	22	4	8	1	3
2	5	4	8	9	8	1	8	2	6	8	14	9	9	1	8
2	7	20	22	9	2	1	8	2	7	12	21	9	11	1	8
2	7	4	20	9	20	1	8	2	7	1	24	16	12	1	15
2	7	4	20	25	4	1	24	3	6	9	8	16	12	1	15
3	7	21	14	16	7	1	15	3	7	6	13	16	23	1	15
3	7	9	15	25	9	1	24	3	8	21	20	16	16	1	15
3	8	6	19	16	32	1	15	4	7	16	8	25	16	1	24
4	7	14	7	36	8	1	35	4	8	16	13	25	26	1	24
4	8	9	15	36	20	1	35	4	8	16	10	49	5	1	48
4	9	48	20	25	4	1	24	4	9	24	19	25	29	1	24
4	9	1	24	36	36	1	35	–	–	–	–	–	–	–	–

conditions holds:(i) $m>1$, $q(=m+t)$ and t are perfect squares, (ii) $m=1$, $q(=1+t)$ is a perfect square.

$$\begin{cases} a^2+b^2=p+q+s+r, \\ a^2b^2=qs+rt. \end{cases} \tag{9}$$

(1) Assume that $t=k^2$, $q(=m+t)=(n+k)^2$, *i.e.* $m=n^2+2nk$. Then Eqs. (9) can change into Eqs. (7). Thus, the result of Theorem 3.11(1) is true.

For the tree $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6, based on Theorem 3.10, by computer search, we have found 4483 solutions for Eqs. (7) satisfying $m+t=q$, $m=n^2+2nk>1$, $t=k^2$, $r>1$, $q=(n+k)^2$, $1 \leqslant a \leqslant 10$, $a \leqslant b \leqslant a+10$. So we can construct such integral trees $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ from Theorem 3.11(1). They are different from those in the existing literature.

(2) Assume that $m=1$, $t=n^2+2n$. Then Eqs. (9) can change into Eqs. (8). Thus, the result of Theorem 3.11(2) is true.

Similarly, we have also found 438 solutions for Eqs. (8) satisfying $m+t=q$, $m=1$, $t=n^2+2n$, $r>1$, $q=(n+1)^2$, $1 \leqslant a \leqslant 10$, $a \leqslant b \leqslant a+10$.

Table 10
Integral trees $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$, where $m+t \neq q$, $p>1$, $m>1$

a	b	c	s	p	q	r	m	t	a	b	c	s	p	q	r	m	t
1	3	10	11	30	4	49	15	1	1	3	10	3	6	4	33	63	1
1	3	14	21	60	4	105	15	1	1	3	14	5	12	4	121	63	1
1	4	8	5	12	4	11	48	1	1	4	9	12	33	4	24	24	1
1	4	10	10	27	4	40	35	1	1	4	12	2	63	64	28	3	1
1	4	12	21	60	4	51	24	1	1	4	12	8	21	4	64	63	1
1	4	13	12	33	4	88	48	1	1	4	14	19	54	4	100	35	1
1	4	14	7	18	4	84	99	1	1	5	8	8	21	4	8	48	1
1	5	12	7	48	9	57	48	1	1	6	12	8	56	4	49	48	16
1	6	12	2	50	16	49	60	4	1	6	13	13	96	9	39	48	1
1	6	13	23	66	4	49	63	1	1	6	14	15	48	4	66	96	4
1	6	14	17	48	4	64	99	1	1	6	15	10	72	9	90	80	1
1	7	10	19	54	4	9	63	1	1	7	14	37	108	4	33	63	1
1	7	15	1	48	25	80	117	4	1	7	15	3	48	25	78	120	1
1	7	15	7	48	9	42	168	1	1	8	14	31	90	4	36	99	1
1	10	13	4	108	25	12	105	16	1	10	14	10	135	16	15	120	1
1	10	15	9	136	4	56	40	81	1	10	15	4	150	25	26	105	16
1	10	15	45	136	4	20	112	9	1	10	15	20	156	9	20	117	4
2	5	9	18	16	9	18	48	1	2	5	11	7	18	100	9	15	1
2	5	12	28	30	9	57	45	4	2	5	12	10	18	16	65	60	4
3	5	14	31	66	36	81	7	9	3	5	14	27	78	100	9	15	1
3	6	14	9	15	16	80	85	36	–	–	–	–	–	–	–	–	–

Table 11
Integral trees $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$, where $m+t \neq q$, $p=1$, $m>1$

a	b	c	s	p	q	r	m	t	a	b	c	s	p	q	r	m	t
2	3	8	5	1	8	27	32	4	2	3	10	8	1	5	18	77	4
3	4	10	18	1	10	32	55	9	3	4	10	16	1	15	44	45	4
3	4	14	8	1	14	54	128	16	3	5	8	12	1	24	12	48	1
4	5	12	18	1	24	78	48	16	4	5	18	18	1	18	72	220	36
4	5	18	24	1	18	66	240	16	5	6	13	50	1	26	72	56	25
5	6	13	66	1	26	56	72	9	5	6	17	18	1	34	128	133	36
5	6	18	32	1	27	100	189	36	6	7	16	39	1	48	153	64	36
6	7	16	90	1	48	102	96	4	6	7	19	14	1	38	168	144	81
6	7	19	54	1	38	128	189	36	6	7	19	70	1	38	112	216	9
6	7	19	74	1	38	108	224	1	6	9	20	55	1	80	125	252	4
7	8	12	78	1	63	34	56	25	8	9	20	68	1	80	252	80	64
8	9	20	176	1	80	144	140	4	8	10	13	128	1	65	18	57	64
9	10	16	75	1	96	69	96	100	–	–	–	–	–	–	–	–	–

(3)–(4) These results follow from Theorem 3.10 by arguments similar to those used in Theorem 3.11(1) or (2). □

Remark 3.12. For the tree $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6 with $m+t=q$, by Theorem 3.11(1)–(2), we know that there exist many such integral trees. Hence it is important to find positive integral solutions of the Diophantine equations (7) and (8).

Based on Theorem 3.11(1)–(2), we raise the following question.

Question 3.13. *For $m+t=q$, are there infinitely many positive integral solutions for Eqs.* (7) *or Eqs.* (8) *in Theorem* 3.11? *If yes, how to find them*?

Remark 3.14. For $m + t \neq q$, in view of Theorem 3.11(3)–(4), we wish to find positive integral solutions of the following Diophantine equations (10) one of the four conditions of Theorem 3.10.

$$\begin{cases} a^2 + b^2 + c^2 = p + q + s + m + t + r, \\ a^2b^2 + b^2c^2 + c^2a^2 = (p + q + s)(m + t) + r(q + t) + qs, \\ a^2b^2c^2 = q[s(m + t) + rt]. \end{cases} \tag{10}$$

However, when $m + t \neq q$, we have not found positive integral solutions of Eqs. (10) by computer search in the cases: (iii) when $m = 1$, $p > 1$, q and $m + t$ are perfect squares, $1 \leqslant a \leqslant 7$, $a \leqslant b \leqslant 10$, $b \leqslant c \leqslant 20$, (iv) when $m = p = 1$, $m + t$ is a perfect square, $1 \leqslant a \leqslant 10$, $a \leqslant b \leqslant 10$, $b \leqslant c \leqslant 20$.

So, the following question arises.

Question 3.15. *For the tree* $K_{1,s} \bullet T(p, q) \bullet T(r, m, t)$ *of diameter* 6 *with* $m + t \neq q$, *we ask*:

(1) *What are all positive integral solutions for Eqs*. (10) *satisfying Condition* (i) *or Condition* (ii) *of Theorem* 3.10?
(2) *Are there positive integral solutions for Eqs*. (10) *satisfying Condition* (iii) *or Condition* (iv) *of Theorem* 3.10? *If yes*, *how to find them*?

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