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Note

Graphs with the second largest number of maximal independent sets☆

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Abstract

Let G be a simple undirected graph. Denote by $\mathrm{mi}(G)$ (respectively, $\mathrm{xi}(G)$) the number of maximal (respectively, maximum) independent sets in G. Erdős and Moser raised the problem of determining the maximum value of $\mathrm{mi}(G)$ among all graphs of order n and the extremal graphs achieving this maximum value. This problem was solved by Moon and Moser. Then it was studied for many special classes of graphs, including trees, forests, bipartite graphs, connected graphs, (connected) triangle-free graphs, (connected) graphs with at most one cycle, and recently, (connected) graphs with at most r cycles. In this paper we determine the second largest value of $\mathrm{mi}(G)$ and $\mathrm{xi}(G)$ among all graphs of order n. Moreover, the extremal graphs achieving these values are also determined.

Keywords: Maximal (maximum) independent set; Extremal graph

1. Introduction

Let G be a simple undirected graph. The *neighborhood* $N(x)$ of a vertex x in G is the set of vertices adjacent to x. The *closed neighborhood* is defined to be the set $N[x] = N(x) \cup \{x\}$. Denote by $d(x) = |N(x)|$ the *degree* of x in G. Let $\delta(G) = \min\{d(x) \mid x \in V(G)\}$ and $\Delta(G) = \max\{d(x) \mid x \in V(G)\}$. For notation and terminology not defined here, we refer the reader to [1].

An *independent set* is a subset S of $V(G)$ such that no two vertices in S are adjacent in G. A *maximal* independent set is an independent set that is not a proper subset of any other independent set. A *maximum* independent set is an independent set of maximum size among all independent sets of G. Note that a maximum independent set is maximal but the converse does not always hold. Denote by $\mathrm{mi}(G)$ (respectively, $\mathrm{xi}(G)$) the number of maximal (respectively, maximum) independent sets in G.

Erdős and Moser raised the problem of determining the maximum value of $\mathrm{mi}(G)$ among all graphs of order n and the extremal graphs achieving the maximum value. This problem was solved by Moon and Moser [18]. Later researchers focused on the problem for special classes of graphs: for connected graphs this was done independently

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Fig. 1. The graph $K_4 * K_4$.

by Füredi [5] and Griggs et al. [7]; for trees independently by Meir and Moon [17], Sagan [19], and Wilf [21]; for forests by Griggs and Grinstead [6] and Jou and Chang [12]; for (connected) graphs with at most one cycle by Jou and Chang [12]; for bipartite graphs by Liu [16]; for triangle-free graphs by Hujter and Tuza [10] and for connected triangle-free graphs by Chang and Jou [2]. Recently, Sagan and Vatter [20] and Ying et al. [22] solved the problem for graphs with at most r cycles. For other related, including algorithmic, results on $\mathrm{mi}(G)$, see [4,8,11,13,14]. Unlike the case for parameter $\mathrm{mi}(G)$, there are few results for the parameter $\mathrm{xi}(G)$; see [3,9,15,23].

In this paper we determine the second largest values of $\mathrm{mi}(G)$ and $\mathrm{xi}(G)$ among all graphs of order n. Extremal graphs achieving these values are also determined.

2. Preliminary

In this section we present some notation and preliminary results.

Proposition 2.1 (*[10]*). *For any vertex x in a graph G,*

(1) $\mathrm{mi}(G) \le \mathrm{mi}(G - x) + \mathrm{mi}(G - N[x])$;
(2) *if x is a leaf adjacent to y, then* $\mathrm{mi}(G) = \mathrm{mi}(G - N[x]) + \mathrm{mi}(G - N[y])$.

Proposition 2.2 (*[5]*). *If $n \ge 6$, then* $\mathrm{mi}(C_n) = \mathrm{mi}(C_{n-2}) + \mathrm{mi}(C_{n-3})$.

Proposition 2.3 (*[10]*). *For any two vertex disjoint graphs G and H,* $\mathrm{mi}(G \cup H) = \mathrm{mi}(G)\mathrm{mi}(H)$.

Here for two vertex disjoint graphs G and H, we denote by $G \cup H$ the union of G and H. For an integer $n \ge 2$, define the graph $G(n)$ as follows:

$$G(n) = \begin{cases} sK_3, & \text{if } n = 3s; \\ K_4 \cup (s-1)K_3 \text{ or } 2K_2 \cup (s-1)K_3, & \text{if } n = 3s+1; \\ K_2 \cup sK_3, & \text{if } n = 3s+2. \end{cases}$$

Let $g(n) = \mathrm{mi}(G(n))$. From the preceding propositions, we have

$$g(n) = \begin{cases} 3^s, & \text{if } n = 3s; \\ 4 \cdot 3^{s-1}, & \text{if } n = 3s+1; \\ 2 \cdot 3^s, & \text{if } n = 3s+2. \end{cases}$$

For any graph of order n, we have the following result.

Theorem 2.4 (*[18]*). *If G is a graph of order $n \ge 2$, then* $\mathrm{mi}(G) \le g(n)$. *Furthermore, the equality holds if and only if $G \cong G(n)$.*

Denote by $K_m * K_n$ the graph obtained from $K_m \cup K_n$ by connecting a single vertex of one component to a single vertex of the other. For example, see the graph $K_4 * K_4$ as illustrated in Fig. 1.

For $n \ge 6$, define the graph $H(n)$ as follows:

$$H(n) = \begin{cases} (K_3 * K_3) \cup (s-2)K_3, \text{ or } 3K_2 \cup (s-2)K_3, & \\ \text{or } K_4 \cup K_2 \cup (s-2)K_3, & \text{if } n = 3s; \\ (K_4 * K_3) \cup (s-2)K_3, & \text{if } n = 3s+1; \\ (K_3 * K_3) \cup (s-2)K_3 \cup K_2, \text{ or } 4K_2 \cup (s-2)K_3, & \\ \text{or } K_4 \cup 2K_2 \cup (s-2)K_3, \text{ or } 2K_4 \cup (s-2)K_3, & \text{if } n = 3s+2. \end{cases}$$

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Let $h(n) = \mathrm{mi}(H(n))$. From the preceding propositions, we have

$$h(n) = \begin{cases} \frac{11}{12} g(n), & \text{if } n = 3s+1; \\ \frac{8}{9} g(n), & \text{otherwise.} \end{cases}$$

3. Main result

Theorem 3.1. *Let G be a graph of order $n \geq 3$ and $G \ncong G(n)$. Then $\mathrm{mi}(G) \leq h(n)$. Furthermore, the equality holds if and only if $G \cong H(n)$.*

Proof. It is easy to see that the equality holds for any graph $G \cong H(n)$. We prove the theorem by induction on n. Since $g(3) = 3$, $g(4) = 4$ and $g(5) = 6$, it is easy to see that for any graph $G \ncong G(n)$ of order n $(3 \leq n \leq 5)$, we have $\mathrm{mi}(G) < \frac{8}{9} g(n)$. Suppose that the theorem holds for all graphs of order less than n. Now we consider a graph G of order $n \geq 6$. First, we have the following remarks.

Remark 1. Suppose that G is connected and $\delta(G) = 1$. Choose $x \in V(G)$ with $N(x) = \{y\}$. Since $n \geq 3$, $d_G(y) \geq 2$. From Proposition 2.1, by the induction hypothesis we have

$$\begin{aligned} \mathrm{mi}(G) &= \mathrm{mi}(G - x - y) + \mathrm{mi}(G - N[y]) \\ &\leq g(n-2) + g(n-3) \\ &\leq \begin{cases} \frac{7}{9} g(n), & \text{if } n \equiv 0 \pmod 3; \\ \frac{5}{6} g(n), & \text{otherwise.} \end{cases} \\ &< h(n). \end{aligned}$$

Remark 2. Suppose that $G \cong C_n$ and $n \geq 6$. From Proposition 2.2, by the induction hypothesis we have

$$\begin{aligned} \mathrm{mi}(G) &= \mathrm{mi}(C_{n-2}) + \mathrm{mi}(C_{n-3}) \\ &\leq \begin{cases} \frac{11}{12} g(n-2) + \frac{8}{9} g(n-3), & \text{if } n \equiv 0 \pmod 3; \\ \frac{8}{9} g(n-2) + \frac{11}{12} g(n-3), & \text{if } n \equiv 1 \pmod 3; \\ \frac{8}{9} g(n-2) + \frac{8}{9} g(n-3), & \text{if } n \equiv 2 \pmod 3; \end{cases} \\ &\leq \begin{cases} \frac{19}{27} g(n), & \text{if } n \equiv 0 \pmod 3; \\ \frac{3}{4} g(n), & \text{if } n \equiv 1 \pmod 3; \\ \frac{20}{27} g(n), & \text{if } n \equiv 2 \pmod 3; \end{cases} \\ &< h(n). \end{aligned}$$

Next, we distinguish the following cases to complete the proof.

Case 1. $n = 3s$.

If G is disconnected, let $G = G_1 \cup G_2$ where G_1 and G_2 are of order n_1 and n_2, respectively. Then, without loss of generality, we have $n_1 = 3s_1$ and $n_2 = 3s_2$, or $n_1 = 3s_1 + 1$ and $n_2 = 3s_2 + 2$.

If $n_1 = 3s_1$ and $n_2 = 3s_2$, since $G \ncong G(n)$, we have $G_1 \ncong G(n_1)$ or $G_2 \ncong G(n_2)$. Then

$$\begin{aligned}\mathrm{mi}(G) &= \mathrm{mi}(G_1)\mathrm{mi}(G_2)\\ &\le \frac{8}{9}g(n_1)g(n_2)\\ &= \frac{8}{9}g(n).\end{aligned}$$

Equality holds if and only if $G_1 \cong G(n_1)$ and $G_2 \cong H(n_2)$, or $G_1 \cong H(n_1)$ and $G_2 \cong G(n_2)$, i.e., $G \cong H(n)$.

Now let $n_1 = 3s_1 + 1$ and $n_2 = 3s_2 + 2$. If $s_1 \ge 1$, then

$$\begin{aligned}\mathrm{mi}(G) &= \mathrm{mi}(G_1)\mathrm{mi}(G_2)\\ &\le g(n_1)g(n_2)\\ &\le \frac{8}{9}g(n).\end{aligned}$$

Equality holds if and only if $G_1 \cong G(n_1)$ and $G_2 \cong G(n_2)$, i.e., $G \cong H(n)$.

If $s_1 = 0$, then we have

$$\begin{aligned}\mathrm{mi}(G) &= \mathrm{mi}(G_2) \le g(n-1)\\ &= \frac{2}{3}g(n) < \frac{8}{9}g(n).\end{aligned}$$

So we assume that G is connected. From Remarks 1 and 2, we have $\delta(G) \ge 2$ and $\Delta(G) \ge 3$. Choose $y \in V(G)$ with $d(y) = \Delta(G)$.

If $d(y) \ge 4$, then we have

$$\begin{aligned}\mathrm{mi}(G) &\le \mathrm{mi}(G-y) + \mathrm{mi}(G-N[y])\\ &\le g(n-1) + g(n-5)\\ &= \frac{22}{27}g(n) < \frac{8}{9}g(n).\end{aligned}$$

So assume that $d(y) = 3$. Then

$$\begin{aligned}\mathrm{mi}(G) &\le \mathrm{mi}(G-y) + \mathrm{mi}(G-N[y])\\ &\le g(n-1) + g(n-4)\\ &= \frac{8}{9}g(n).\end{aligned}$$

Equality holds if and only if $G - y \cong G(n-1)$ and $G - N[y] \cong G(n-4)$. Since $d(y) = 3$, by simple analysis we have $G - y \cong G(n-1)$ and $G - N[y] \cong G(n-4)$ if and only if $G \cong H(n)$.

Case 2. $n = 3s + 1$.

If G is disconnected, say, G is the vertex disjoint union of two graphs G_1 and G_2 of order n_1 and n_2, respectively. Then, without loss of generality, we have $n_1 = 3s_1 + 1$ and $n_2 = 3s_2$, or $n_1 = 3s_1 + 2$ and $n_2 = 3s_2 + 2$.

If $n_1 = 3s_1 + 1$ and $n_2 = 3s_2$, since $G \ncong G(n)$, we have $G_1 \ncong G(n_1)$ or $G_2 \ncong G(n_2)$. When $s_1 = 0$, we have

$$\begin{aligned}\mathrm{mi}(G) &= \mathrm{mi}(G_1)\mathrm{mi}(G_2)\\ &\le g(n_2) = \frac{3}{4}g(n)\\ &< \frac{11}{12}g(n).\end{aligned}$$

When $s_1 \ne 0$, we have

$$\begin{aligned}\mathrm{mi}(G) &= \mathrm{mi}(G_1)\mathrm{mi}(G_2)\\ &\le \frac{11}{12}g(n_1)g(n_2)\\ &= \frac{11}{12}g(n).\end{aligned}$$

Equality holds if and only if $G_1 \cong H(n_1)$ and $G_2 \cong H(n_2)$, i.e., $G \cong H(n)$.

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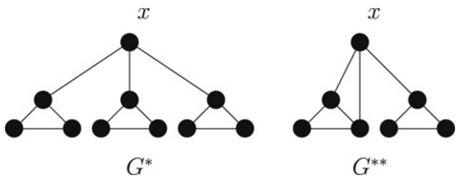

Fig. 2. The graphs G^* and G^{**}.

If $n_1 = 3s_1 + 2$ and $n_2 = 3s_2 + 2$, since $G \ncong G(n)$, we have $G_1 \ncong G(n_1)$ or $G_2 \ncong G(n_2)$. Then

$$\begin{aligned}\mathrm{mi}(G) &= \mathrm{mi}(G_1)\mathrm{mi}(G_2)\\ &\le \frac{8}{9}g(n_1)g(n_2)\\ &= \frac{8}{9}g(n) < \frac{11}{12}g(n).\end{aligned}$$

So we assume that G is connected. We distinguish the following subcases.

Subcase 2.1 $\delta(G) \ge 2$ and $\Delta(G) = 3$.

Choose $x \in V(G)$ with $d(x) = \Delta(G)$. If $G - x \ncong G(n-1)$ and $G - N[x] \ncong G(n-4)$, then by the induction hypothesis we have

$$\begin{aligned}\mathrm{mi}(G) &\le \mathrm{mi}(G-x) + \mathrm{mi}(G-N[x])\\ &\le \frac{8}{9}(g(n-1) + g(n-4))\\ &= \frac{8}{9}g(n) < \frac{11}{12}g(n).\end{aligned}$$

Now assume that $G-x \cong G(n-1)$. Since $G \ncong G(n)$ and $d(x) = 3$, $G \cong G^* \cup (s-3)K_3$ or $G \cong G^{**} \cup (s-2)K_3$. The graphs G^* and G^{**} are illustrated in Fig. 2.

Then by simple enumeration we have

$$\mathrm{mi}(G^* \cup (s-3)K_3) = \frac{3}{4}g(n) \quad \text{and} \quad \mathrm{mi}(G^{**} \cup (s-2)K_3) = \frac{3}{4}g(n).$$

This implies that, if $G - x \cong G(n-1)$, then $\mathrm{mi}(G) < \frac{11}{12}g(n)$.

Then we can assume that for any vertex $v \in V(G)$ with $d(v) = 3$, $G - v \ncong G(n-1)$ and $G - N[v] \cong G(n-4)$.

Let u be a vertex in G with $N(u) = \{a, b, c\}$. Since $G-u \ncong G(n-1)$ and $\Delta(G) = 3$, $G[\{a, b, c\}] \ncong K_3$. Without loss of generality, we assume $ab \notin E(G)$. Thus, since $\delta(G) \ge 2$, $\Delta(G) = 3$ and $G \ncong G(n)$, G must contain exactly one of the graphs illustrated in Fig. 3 as a component, while any other component is K_3.

By simple enumeration we have

$$\begin{aligned}&\mathrm{mi}(H_1 \cup (s-3)K_3) = \frac{3}{4}g(n), \quad \mathrm{mi}(H_2 \cup (s-2)K_3) = \frac{3}{4}g(n),\\ &\mathrm{mi}(H_3 \cup (s-2)K_3) = \frac{3}{4}g(n), \quad \mathrm{mi}(H_4 \cup (s-1)K_3) = \frac{3}{4}g(n).\end{aligned}$$

This implies that $\mathrm{mi}(G) < \frac{11}{12}g(n)$.

Subcase 2.2 $\delta(G) \ge 2$ and $\Delta(G) \ge 4$.

Choose $x \in V(G)$ with $d(x) = \Delta(G)$. Then we have

$$\begin{aligned}\mathrm{mi}(G) &\le \mathrm{mi}(G-x) + \mathrm{mi}(G-N[x])\\ &\le g(n-1) + g(n-5)\\ &= h(n),\end{aligned}$$

with equality if and only if $G-x \cong G(n-1)$ and $G-N[y] \cong G(n-5)$, which implies that $G \cong (K_4 {*} K_3) \cup (s-2)K_3 \cong H(n)$, as desired.

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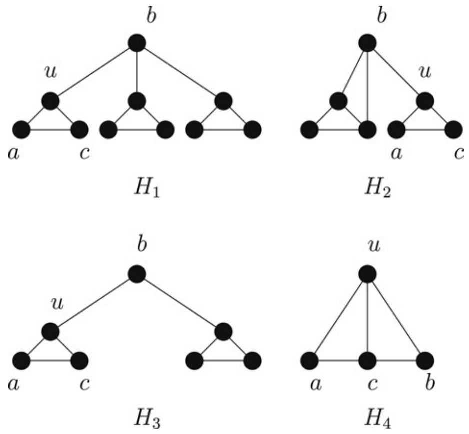

Fig. 3. The graphs H_1, H_2, H_3 and H_4.

Case 3. $n = 3s + 2$.

If G is disconnected, say, G is the vertex disjoint union of two graphs G_1 and G_2 of order n_1 and n_2, respectively. Then, without loss of generality, we have $n_1 = 3s_1 + 2$ and $n_2 = 3s_2$, or $n_1 = 3s_1 + 1$ and $n_2 = 3s_2 + 1$.

If $n_1 = 3s_1 + 2$ and $n_2 = 3s_2$, since $G \ncong G(n)$, we have $G_1 \ncong G(n_1)$ or $G_2 \ncong G(n_2)$. Then

$$\begin{aligned} \mathrm{mi}(G) &= \mathrm{mi}(G_1)\mathrm{mi}(G_2) \\ &\le \frac{8}{9} g(n_1) g(n_2) \\ &= \frac{8}{9} g(n). \end{aligned}$$

Equality holds if and only if $G_1 \cong G(n_1)$ and $G_2 \cong H(n_2)$, or $G_1 \cong H(n_1)$ and $G_2 \cong G(n_2)$, i.e., $G \cong H(n)$.

If $n_1 = 3s_1 + 1$ and $n_2 = 3s_2 + 1$, then

$$\begin{aligned} \mathrm{mi}(G) &= \mathrm{mi}(G_1)\mathrm{mi}(G_2) \\ &\le g(n_1) g(n_2) \\ &\le \frac{8}{9} g(n). \end{aligned}$$

Equality holds if and only if $s_1 \ge 1$, $s_2 \ge 1$, $G_1 \cong G(n_1)$ and $G_2 \cong G(n_2)$, i.e., $G \cong H(n)$.

So we assume that G is connected. From Remarks 1 and 2, we have $\delta(G) \ge 2$ and $\Delta(G) \ge 3$. Choose $x \in V(G)$ with $d(x) = \Delta(G)$.

If $d(x) \ge 4$, then we have

$$\begin{aligned} \mathrm{mi}(G) &\le \mathrm{mi}(G - x) + \mathrm{mi}(G - N[x]) \\ &\le g(n-1) + g(n-5) \\ &= \frac{5}{6} g(n) < \frac{8}{9} g(n). \end{aligned}$$

So, assume that $d(x) = 3$. If $G - x \ncong G(n-1)$, then by the induction hypothesis we have

$$\begin{aligned} \mathrm{mi}(G) &\le \mathrm{mi}(G - x) + \mathrm{mi}(G - N[x]) \\ &\le \frac{8}{9} g(n-1) + g(n-4) \\ &= \frac{22}{27} g(n) < \frac{11}{12} g(n). \end{aligned}$$

If $G - N[x] \ncong G(n-4)$, then by the induction hypothesis we have

$$\mathrm{mi}(G) \le \mathrm{mi}(G - x) + \mathrm{mi}(G - N[x])$$

$$\leq g(n-1) + \frac{8}{9} g(n-4)$$
$$= \frac{70}{81} g(n) < \frac{11}{12} g(n).$$

So, without loss of generality, for any vertex x with $d(x) = 3$, we can assume that $G - x \cong G(n-1)$ and $G - N[x] \cong G(n-4)$.

If $G - x \cong G(n-1) \cong K_4 \cup (s-1)K_3$, then since $\Delta(G) = 3$, x is not adjacent to any of the vertices that constitute the K_4 component of $G - x$. Thus we must have $G - N[x] \cong K_4 \cup (s-2)K_3$, so $G \cong 2K_4 \cup (s-2)K_3 \cong H(n)$, as desired.

If $G - x \cong G(n-1) \cong 2K_2 \cup (s-1)K_3$, since $\delta(G) \geq 2$, we have $d(x) \geq 4$, a contradiction, which completes the proof. ■

4. Concluding remarks

Note that an independent set of $H(n)$ is maximal if and only if it is maximum, i.e., $\mathrm{xi}(H(n)) = \mathrm{mi}(H(n))$. So, by Theorem 3.1, we have the following result.

Theorem 4.1. *Let G be a graph of order $n \geq 3$ and $G \not\cong G(n)$. Then $xi(G) \leq h(n)$. Furthermore, the equality holds if and only if $G \cong H(n)$.* ■

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