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On chromatic and flow polynomial unique graphs[☆]

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Abstract

It is known that the chromatic polynomial and flow polynomial of a graph are two important evaluations of its Tutte polynomial, both of which contain much information of the graph. Much research is done on graphs determined entirely by their chromatic polynomials and Tutte polynomials, respectively. Oxley asked which classes of graphs or matroids are determined by their chromatic and flow polynomials together. In this paper, we found several classes of graphs with this property. We first study which graphic parameters are determined by the flow polynomials. Then we study flow-unique graphs. Finally, we show that several classes of graphs, ladders, Möbius ladders and squares of n -cycle are determined by their chromatic polynomial and flow polynomial together. A direct consequence of our theorem is a result of de Mier and Noy [A. de Mier, M. Noy, On graphs determined by their Tutte polynomial, *Graphs Comb.* 20 (2004) 105–119] that these classes of graphs are Tutte polynomial unique.

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1. Introduction

The study of graph polynomials has been an active research topic for many years. It serves as a bridge between graph theory and traditional algebra. Since the coefficients of polynomials often contain rich combinatorial information, the study of graph polynomials provides new avenues to understand the complicated structures of graphs and graphic parameters. It is natural to ask what kind of graphs are determined by their polynomials. In particular, what kind of graphs have their structures entirely encoded in a graph polynomial? In other words, can we find families of graphs that are uniquely determined by a given polynomial? There is extensive research on graphs uniquely determined by their chromatic polynomials and more recently on their Tutte polynomials, but rather spotty research on graphs uniquely determined by their flow polynomials or the combination of both chromatic and flow polynomials. This article is an initiation of investigation on graphs uniquely determined by chromatic and flow polynomials and in the hope that this research will foster further research in this direction.

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Given $x \in \mathbb{N}^+$ and a graph G , the value $P_G(x)$ is the number of proper colorings $f : V(G) \rightarrow [x] = \{1, 2, \dots, x\}$, called the *chromatic polynomial* of G . Chromatic polynomials were introduced by Birkhoff [1] precisely in an attempt to solve the Four-Color Problem, since the problem is equivalent to the fact that $P_G(4) > 0$ for every planar graph G . The *chromatic number* of G is defined as $\min\{x : x \in \mathbb{N}^+ \text{ and } P_G(x) > 0\}$.

Let G be any bridgeless graph, D be the set of directed edges of G after implementing arbitrary orientation, and A be an Abelian group. An A -flow on G is a map $f : D \rightarrow A$ such that the total flow out of a vertex is equal to the total flow into the vertex, i.e., for each vertex $v \in V(G)$, $\sum_{w \in N^+(v)} f(vw) = \sum_{u \in N^-(v)} f(uv)$. An A -flow f is a *nowhere-zero flow* (NZF) if $f(uv) \neq 0$ for any $uv \in D$.

For an Abelian group A , the number of NZF A -flows on G is independent of the structure of A , and depends only of the order of A , $x = |A|$ (see [2]). This number is a polynomial of x , called the *flow polynomial* of G and denoted by $Q_G(x)$. The *flow number* of G is defined as $\min\{x : x \in \mathbb{N}^+ \text{ and } Q_G(x) > 0\}$.

Let G be a graph with vertex set V and edge set E . We assume that G has no isolated vertices, but loops and multiple edges are allowed. We use $d(v)$ to denote the number of edges incident with a vertex v in G , and use $\delta(G)$ to denote $\min\{d(v) : v \in V(G)\}$. We use $k(G)$ and $\lambda(G)$ to denote the number of components and the edge connectivity of G , respectively. The *rank* of a subset S of E is defined as the number of edges in the spanning forest of the subgraph induced by S in G , i.e., $r(S) = |V| - k(G[S])$, where $k(G[S])$ denotes the number of components of the spanning subgraph $G[S]$ induced by S in G . The *Tutte polynomial* of G is given by

$$T_G(x, y) = \sum_{S \subseteq E} (x-1)^{r(E)-r(S)} (y-1)^{|S|-r(S)}.$$

The Tutte polynomial was introduced in 1954 by Tutte [17] with the initial name dichromate polynomial. It is an important research tool and has many applications in graph theory, matroid theory and many other fields such as knot theory and statistical mechanics [5].

For the notation of matroids, we follow Oxley [15]. Let M be a matroid, the Tutte polynomial $T_M(x, y)$ of a matroid M is defined as

$$T_M(x, y) = \sum_{F \subseteq E(M)} (x-1)^{r(E)-r(F)} (y-1)^{|F|-r(F)}$$

and the *chromatic polynomial* of the matroid M (see [18]) is defined as

$$\chi_M(x) = \sum_{F \subseteq E(M)} (-1)^{|F|} (x-1)^{r(M)-r(F)}$$

where $r(F)$ denotes the rank of the subset F .

The chromatic polynomial of a matroid M is related to the Tutte polynomial of M by $\chi_M(x) = (-1)^{r(M)} T_M(1-x, 0)$ (see, for example, [18]). Moreover, $\chi_M(x) = 0$ if and only if M has a loop.

Clearly, any graph $G = (V, E)$ determines a unique matroid with ground set $E(G)$. This matroid is denoted by $M(G)$ and matroids of this kind are called *graphic matroids*. A base of $M(G)$ is a spanning forest and a circuit corresponds to a cycle in G . Moreover, $T_G(x, y) = T_{M(G)}(x, y)$.

It is shown that the chromatic polynomial of a graph G is a special evaluation of its Tutte polynomial and in addition, is closely related to the chromatic polynomial of $M(G)$:

$$P_G(x) = (-1)^{r(G)} x^{k(G)} T_{M(G)}(1-x, 0) \quad \text{and} \quad P_G(x) = x^{k(G)} \chi_{M(G)}(x).$$

For the flow polynomial, the following facts are well known:

- (i) The flow polynomial of graph G is also an evaluation of the Tutte polynomial. Indeed, $Q_G(x) = (-1)^{n(G)} T_G(0, 1-x)$, where $n(G) = |E(G)| - r(G)$;
- (ii) $Q_G(x) = \chi_{M^*(G)}(x)$, where $M^*(G)$ is the dual matroid of $M(G)$;
- (iii) $Q_G(x) = 0$ if G contains a bridge (coloop); and
- (iv) Let G be a graph obtained by sticking two vertex-disjoint graphs G_1 and G_2 at a vertex or by performing the direct sum of G_1 and G_2 . Then $Q_G(x) = Q_{G_1}(x) Q_{G_2}(x)$.

For a graph G , if a graph H having the same Tutte polynomial with G implies that $H \cong G$, then G is called *Tutte polynomial unique* (T -unique). Similarly, we say that a matroid M is *Tutte polynomial unique* (T -unique) if whenever

$T_M(x, y) = T_N(x, y)$ for a matroid N , we have that $M \cong N$. Recently, several classes of matroids and graphs are proven to be T -unique, see [4,11–13]. Two graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are *2-isomorphic* if there is a bijection $\psi : E_1 \rightarrow E_2$ such that the edge set $E'_1 \subseteq E_1$ is a cycle in H_1 if and only if $\psi(E'_1)$ is a cycle in H_2 . Two 2-isomorphic graphs have the same Tutte polynomial (see [15]), and thus have the same flow polynomial.

A *bond* of a graph is a minimal edge cut. A graph G is called *cosimple* if the dual matroid $M^*(G)$ is simple, i.e., it has no bonds of size less than three. Apparently, adding parallel edges to a graph will not change the chromatic polynomials and subdividing an edge will not change the flow polynomial. Therefore, graphs are assumed simple when chromatically unique (or χ -unique) graphs were defined in [6]. Here a simple graph G is χ -*unique* if whenever $P_G(x) = P_H(x)$ for any simple graph H , we have that $G \cong H$. Two simple graphs G and H are called χ -*equivalent* if $P_G(x) = P_H(x)$. Similarly, we assume that our graphs are both connected and cosimple when we define the flow-unique graphs next. A connected cosimple graph G is called *flow-unique* if whenever $Q_G(x) = Q_H(x)$ for any connected cosimple graph H , we have that $G \cong H$. Two connected cosimple graphs are called *flow-equivalent* if $Q_G(x) = Q_H(x)$. The problem of determining the χ -unique graphs was first introduced by Chao and Whitehead [6]. Since then much research is done on this subject. Many classes of graphs have been shown to be χ -unique, for example, the complete graphs K_n , cycles C_n , wheels W_n (where n is the number of spokes) for even n , and bipartite graphs $K_{p,q}$ for $p, q \geq 2$. For a comprehensive survey on this topic, the reader is referred to [9,10].

For flow-unique graphs, however, very little is known. In Section 2, we first study some properties of the flow-unique graphs. Using the properties, we show that several classes of graphs are flow-unique in Section 3.

Since both the chromatic and flow polynomials are evaluations of the Tutte polynomial, they contain less information of the graphs than that of the Tutte polynomial. Bonin pointed out (through personal communication with the second author [3]) that Oxley had asked the following question: when is a graph or matroid completely determined by its chromatic and flow polynomials together? In this paper, we focus on graphs only. A graph G (not necessarily simple or cosimple) is (P, Q) -*unique* if whenever $P_G(x) = P_H(x)$ and $Q_G(x) = Q_H(x)$, we have that $G \cong H$. Two graphs G and H are called (P, Q) -*equivalent* if both $P_G(x) = P_H(x)$ and $Q_G(x) = Q_H(x)$. Therefore, the (P, Q) -unique graphs are precisely those graphs determined by the chromatic and flow polynomials together. Clearly, a (P, Q) -unique graph is also T -unique. In Section 4, we will show that several classes of graphs, ladders, Möbius ladders and squares of n -cycles, are (P, Q) -unique. This result not only answers Oxley's question for these classes of graphs, but also implies the corresponding results of Mier and Noy [11]. The proof of the above result will appear in Section 4.

2. Preliminaries

In this section, we will prove some lemmas which will be needed in later sections. In proving χ -uniqueness, the main tool is the following lemma (see [9]).

Lemma 2.1. *Let G and H be two simple graphs such that $P_G(x) = P_H(x)$. Then*

- (i) $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$;
- (ii) G and H have the same girth, i.e., $g(G) = g(H)$. Furthermore, they have the same number of the shortest cycles;
- (iii) $t_1(G) = t_1(H)$ and $t_2(G) - 2t_3(G) = t_2(H) - 2t_3(H)$, where $t_1(G)$, $t_2(G)$ and $t_3(G)$ denote the number of triangles, the number of 4-element cycles without chords, and the number of K_4 's of G , respectively;
- (iv) G is connected if and only if H is connected;
- (v) G is 2-connected if and only if H is 2-connected;
- (vi) G and H have the same chromatic number.

Let M be a matroid with m elements. Suppose that f is an arbitrary bijection from $E(M)$ to $\{1, 2, \dots, m\}$. Let C be any circuit of M and c be an element of C such that $f(c) > f(e)$ for all $e \in C - c$. Then $C - c$ is called a *broken circuit* of M . (For brevity, we use $C - c$ instead of $C - \{c\}$.) Given a graph G , since both the chromatic and flow polynomials can be reduced to the chromatic polynomial of the matroids $M(G)$ and $M^*(G)$, respectively, the following result of Heron [7] is a very useful tool. It generalized the well-known Broken Cycle Theorem of Whitney [19] for chromatic polynomials of graphs.

Theorem 2.2. *Let M be a matroid with rank r . Then $\chi_M(x) = \sum_{i=0}^r (-1)^i b_i x^{r-i}$, where b_i is the number of i -subsets of $E(M)$ containing no broken circuits.*

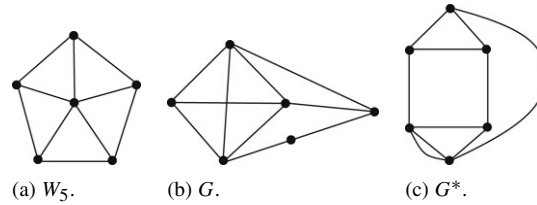


Fig. 1. An example of a cosimple graph χ -equivalent to a graph which is not cosimple.

It is easy to see that $b_0 = 1$ unless M has a loop. In the following corollary, we use c_g to denote the number of circuits of M with girth g .

Proposition 2.3. Let M be a matroid with m elements and girth $g(M) = g$. Suppose that $\chi_M(x) = \sum_{i=0}^r (-1)^i b_i x^{r-i}$. Then $g(M)$ is determined by $\chi_M(x)$ and m . In addition, if M is a simple binary matroid, then

(i) $b_{g-1} = \binom{m}{g-1} - c_g$;

(ii) c_g is determined by $\chi_M(x)$ and m .

Proof. A matroid M has a loop if and only if $\chi_M(x) = 0$. Thus if M has a loop, then g is clearly determined by $\chi_M(x)$. If M has parallel elements, then by Theorem 2.2, $b_1 < m$. Hence one can determine that $g = 2$ from $\chi_M(x)$ and m . When M is simple, in this case, by Theorem 2.2, $\binom{b_i}{i} = \binom{m}{i}$ for $0 \leq i \leq g-2$ and $b_{g-1} < \binom{m}{g-1}$. Hence g is determined by $\chi_M(x)$ and m .

Now suppose that M is a simple binary matroid. According to Theorem 2.2, b_{g-1} is the number of $(g-1)$ -subsets of $E(M)$ containing no broken circuits of $E(M)$. We need only show that M has exactly c_g broken circuits of size $g-1$. Let $C_1 - a$ and $C_2 - b$ be two broken circuits of M with size $g-1$ such that $C_1 \neq C_2$. We claim that $C_1 - a \neq C_2 - b$. Otherwise, since M is binary, the set $C_1 \Delta C_2$ contains a circuit. We conclude that M has a 2-element circuit $\{a, b\}$, a contradiction to the assumption that M is simple. Thus (i) holds. Moreover, from (i), $c_g = b_{g-1} + \binom{m}{g-1}$, which is determined by $\chi_M(x)$ and m . This completes the proof of the proposition. \square

As $Q_G(x) = \chi_{M^*(G)}(x)$, the next result follows from Theorem 2.2 and Proposition 2.3. It was also obtained by Jin and Zhang [8] using different proof techniques.

Corollary 2.4. Let G be a connected graph with n vertices, m edges and edge connectivity λ . Then $Q_G(x) = \sum_{i=0}^{n(G)} (-1)^i h_i x^{n(G)-i}$ is a polynomial of degree $n(G) = m - n + 1$. Moreover, when $\lambda \geq 3$, the following are true:

(i) $\binom{h_i}{i} = \binom{m}{i}$ for $0 \leq i \leq \lambda - 2$;

(ii) $h_{\lambda-1} = \binom{m}{\lambda-1} - c_\lambda$, where c_λ is the number of λ -element bonds.

Corollary 2.5. Let G be a graph with chromatic polynomial $P_G(x)$ and flow polynomial $Q_G(x)$. Then,

(i) Whether G is simple is determined by $P_G(x)$ and $|E(G)|$;

(ii) Whether G is cosimple is determined by $Q_G(x)$ and $|E(G)|$.

Proof. Recall that $P_G(x) = x^{k(G)} \chi_{M(G)}(x)$. From Proposition 2.3, $g(M(G))$ is determined by $\chi_{M(G)}(x)$ and $|E(M(G))|$. Since $g(M(G)) \geq 3$ if and only if G is simple, we conclude that whether G is simple is determined by $P_G(x)$ and $|E(G)|$.

Similarly, using Proposition 2.3 on $Q_G(x) = \chi_{M^*(G)}(x)$, we deduce that $g(M^*)$ is determined by $\chi_{M^*(G)}(x)$ and $|E(M^*(G))| = |E(G)|$, i.e., $\lambda(G)$ is determined by $Q_G(x)$ and $|E(G)|$. \square

It should be pointed out that cosimplicity of a graph cannot be deduced from its chromatic polynomial, and simplicity of a graph cannot be deduced from its flow polynomial either. For example, Chao and Whitehead (see [9]) proved that the graph G in Fig. 1(b) is χ -equivalent to the five-spoked wheel W_5 (Fig. 1(a)). Note that G is not cosimple. On the other hand, since both G and W_5 are planar, a geometric dual G^* of G (Fig. 1(c)) and $W_5^* = W_5$ have the same flow polynomial. As G^* is not simple, the simplicity of a graph cannot be deduced from the flow polynomial of the graph either.

3. Flow-unique graphs

In this section, we first study some properties of graphs which are determined by their flow polynomials. Then we use this information to prove that several classes of graphs are flow-unique. According to [14], there is no systematic study for flow-unique graphs yet.

Theorem 3.1. *Let G and H be connected cosimple graphs with at least one edge such that $Q_G(x) = Q_H(x)$. Then the following hold:*

- (i) $|E(G)| = |E(H)|$ and $|V(G)| = |V(H)|$;
- (ii) G is 2-connected and loopless if and only if H is 2-connected and loopless;
- (iii) $\lambda(G) = \lambda(H)$;
- (iv) G and H have the same number of minimum bonds;
- (v) The flow number of G is equal to the flow number of H .

Proof. Let $Q_G(x) = \sum_{i=0}^{n(G)} (-1)^i h_i(G) x^{n(G)-i}$ and $Q_H(x) = \sum_{i=0}^{n(H)} (-1)^i h_i(H) x^{n(H)-i}$.

(i) Note that $Q_G(x)$ and $Q_H(x)$ are polynomials of degree $n(G)$ and $n(H)$, respectively. As $Q_G(x) = Q_H(x)$, we deduce that $n(G) = n(H)$, i.e., $|E(G)| - |V(G)| + 1 = |E(H)| - |V(H)| + 1$. As both graphs are cosimple, we have $\lambda(G), \lambda(H) \geq 3$. By Corollary 2.4(i), $h_1(G) = |E(G)|$ and $h_1(H) = |E(H)|$. Since $Q_G(x) = Q_H(x)$, we have $h_1(G) = h_1(H)$, and hence $|E(G)| = |E(H)|$ and $|V(G)| = |V(H)|$.

(ii) Let $T_{M(G)}(x, y) = \sum t_{ij} x^i y^j$. From the deletion–contraction formula of the Tutte polynomial, it is easy to see that there is no constant term, i.e., $t_{00} = 0$, therefore y is a factor of $T_{M(G)}(0, y)$, and thus $1 - x$ is a factor of $Q_G(x) = (-1)^{n(G)} T_G(0, 1 - x)$. The graph G is 2-connected and loopless if and only if $M(G)$ is 2-connected. It is known that $M(G)$ is 2-connected if and only if $t_{10} = t_{01} > 0$ (see [5]). Moreover whether $t_{01} = 0$ is completely determined by $Q_G(x)$ (in $Q_G(x)$, let $x = 1 - y$, then we will get $(-1)^{n(G)} T(0, y)$ and t_{01} is the coefficient of y). In particular, $M(G)$ is not 2-connected if and only if $t_{01} = 0$, which is equivalent to the fact that $(x - 1)^2$ divides $Q_G(x)$. Thus $M(G)$ is 2-connected if and only if $M(H)$ is 2-connected. That is, G is 2-connected and loopless if and only if H is 2-connected and loopless.

(iii) By Proposition 2.3, $\lambda(G) = g(M^*(G))$ is determined by $Q_G(x)$ and $|E(G)|$. From (i), $|E(G)| = |E(H)|$ and hence $\lambda(G) = \lambda(H)$.

(iv) Again, by (i), (iii) and Corollary 2.4(ii), we see that $c_\lambda(G) = c_\lambda(H)$, or G and H have the same number of minimum bonds. Note that (v) is clearly true. This completes the proof of the theorem. \square

In the following, we use the information contained in the flow polynomial to prove the flow-uniqueness of several classes of graphs.

Let $\theta(k_1, k_2, \dots, k_s)$ denote the graph constructed by joining two vertices with s disjoint paths with length k_1, k_2, \dots, k_s , respectively. The graph $\theta(k_1, k_2, \dots, k_s)$ is called an s -bridge graph. If $k_1 = k_2 = k_3 = 1$ and $s = 3$, the graph is called a θ -graph. When $s = 3$, the graph is referred as a generalized θ -graph.

The generalized θ -graph $\theta_{(d,e,f)}$ is known to be χ -unique when at most one of d, e, f is one (see [9]). Let $\theta_{(d,e,f)}^*$ denote the dual graph of $\theta_{(d,e,f)}$. Next we prove the following result:

Proposition 3.2. *Suppose that at most one of d, e and f is one. Then $\theta_{(d,e,f)}^*$ is flow-unique.*

Proof. $\theta^*(d, e, f)$ is a graph constructed from a triangle by replacing every edge by multiple edges with multiplicities d, e, f , respectively. Since at most one of d, e and f is one, it is a connected cosimple graph. Suppose that G is a connected cosimple graph which is flow-equivalent to $\theta^*(d, e, f)$. By Theorem 3.1, $|V(G)| = 3$, $|E(G)| = d + e + f$ and G is a loopless 2-connected graph. Hence G is planar. Now as $Q_G(x) = Q_{\theta^*(d,e,f)}(x)$ we deduce that $P_{G^*}(x) = P_{\theta_{(d,e,f)}}(x)$. Since $\theta_{(d,e,f)}$ is χ -unique, $G^* \cong \theta_{(d,e,f)}$. Now it is straightforward to see that $G \cong \theta^*(d, e, f)$. \square

In the above result, the dual graph of a planar chromatically unique graph is flow-unique. It is natural to ask whether this is true in general. In other words, is a dual graph of a χ -unique planar graph always flow-unique? Unfortunately, it is not the case. It was noted in [10] that there are many χ -unique s -bridge graphs $\theta(k_1, k_2, \dots, k_s)$ ($s \geq 4$), where the set $\{k_1, k_2, \dots, k_s\}$ contains at least three distinct integers. (A reader may look at a special

case, for example, $\theta(4, 6, 7, 8)$.) Now we describe a class of graphs \mathcal{D} such that each graph in the class is a dual graph of $\theta(k_1, k_2, \dots, k_s)$. Let C be a cycle with s edges with labels $1, 2, \dots, s$, respectively. For any permutation (t_1, t_2, \dots, t_s) of (k_1, k_2, \dots, k_s) , we obtain a graph in \mathcal{D} by making the multiplicity of i in C to be t_i for $i \in \{1, 2, \dots, s\}$. It is straightforward to see that each graph in \mathcal{D} is a dual of $\theta(k_1, k_2, \dots, k_s)$. Moreover, all graphs in \mathcal{D} are clearly 2-isomorphic; thus they have the same flow (and Tutte) polynomial. Therefore, each graph in \mathcal{D} is not flow-unique, while it is a dual graph of a planar χ -unique graph.

We believe that the dual graphs of flow-unique planar graphs may not be χ -unique, although we have not found an example yet. (Indeed, we do not know many flow-unique graphs.)

Next we consider the important forbidden subgraphs of non-planar graphs, K_5 and $K_{3,3}$. We use the information contained in their flow polynomials to show that they are flow-unique. The following lemma will be useful in the paper. A graph G is called *super-edge-connected* if every minimum bond of G is trivial, i.e., it is a bond consisting of all edges incident to a single vertex. Clearly, if a connected graph G is super-edge-connected, then $\lambda(G) = \delta(G)$.

Lemma 3.3. *Let G be a connected, cosimple, and r -regular super-edge-connected graph for some $r \geq 3$. If H is also connected and cosimple, and G and H are flow-equivalent, then H is also an r -regular super-edge-connected graph.*

Proof. Since G is a connected r -regular super-edge-connected graph, we deduce that $\lambda(G) = r$. Hence, by Theorem 3.1, $\lambda(H) = r$ and G and H have the same number of vertices and edges, respectively. Furthermore, H is also r -edge-connected and the number of minimum bonds of H is $n = |V(G)|$.

Then $\delta(H) \geq \lambda(H) = r$. In addition, we have

$$r \cdot n \leq \delta(H) \cdot n \leq \sum_{v \in V(H)} d(v) = 2|E(H)| = 2|E(G)| = rn.$$

Therefore each inequality above becomes an equality, and hence $d(v) = r$ for all vertex $v \in V(H)$. Thus H is r -regular. As $\lambda(H) = r$, each vertex of H is corresponding to a minimum bond, and therefore H has at least n minimum bonds. By Theorem 3.1, G and H have the same number of minimum bonds. Therefore, H has only trivial minimum bonds, and hence H is also a super-edge-connected graph. \square

Theorem 3.4. *Both K_5 and $K_{3,3}$ are flow-unique.*

Proof. (i) Let G be a connected cosimple graph such that $Q_G(x) = Q_{K_5}(x)$. Then $|V(G)| = 5$, $|E(G)| = 10$, G is loopless, 2-connected, 4-edge-connected and the number of 4-element bonds is 5. Lemma 3.3 yields that G is 4-regular and every 4-element bond is trivial. In addition, G has no multiple edge. Otherwise, it is easy to see that G has a non-trivial bond of size at most four. Thus, G is simple and it is clear that $G \cong K_5$.

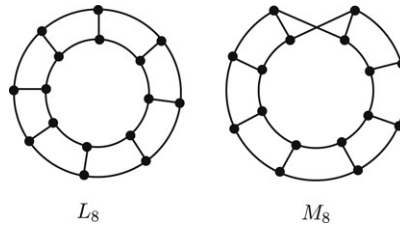
(ii) Let G be a cosimple connected graph such that $Q_G(x) = Q_{K_{3,3}}(x)$. Then by Theorem 3.1, $|V(G)| = 6$, $|E(G)| = 9$, G is loopless, 2-connected, 3-edge-connected and the number of 3-element bonds is 6. By Lemma 3.3, we deduce that G is 3-regular and every 3-element bond is trivial. Moreover, G is simple (otherwise, G would have a bond of size at most two).

We see that $g(G) \geq 4$ as G has no non-trivial 3-element bonds. On the other hand, since G is simple, 2-connected and is 3-regular, we deduce that $g(G) \leq 4$. Thus $g(G) = 4$. Let $C = v_1 v_2 v_3 v_4$ be a 4-cycle of G and v_5 and v_6 be the other two vertices. If $v_5 v_6 \notin E(G)$, then v_5 is incident to three vertices of C , a contradiction to $g(G) = 4$. Hence $v_5 v_6 \in E(G)$. As $d(v_5) = 3$ and G is a simple graph with girth 4, v_5 is incident to exactly two non-adjacent vertices of C , say v_1 and v_3 ; similarly, v_6 is adjacent to v_2 and v_4 . We conclude that $G \cong K_{3,3}$. This completes the proof of the theorem. \square

It is known that the complete graphs K_n and the complete bipartite graphs $K_{m,n}$ ($m, n \geq 2$) are χ -unique (see [9]). However, for larger m, n , the flow-uniqueness of K_n and $K_{m,n}$ is still unknown.

4. (P, Q) -unique graphs

In this section, we study the graphs determined by their chromatic and flow polynomials together. It is natural to consider those classes of graphs which are either not known to be χ -unique yet, or not χ -unique. We focus on three such classes, ladders L_n , Möbius ladders M_n and square of an n -cycle C_n^2 ($n \geq 3$). All three classes are conjectured to be χ -unique [16]. We will show that all graphs in these classes are (P, Q) -unique. As a consequence, we deduce the corresponding results of de Mier and Noy [11] that all such graphs are T -unique. We start with the following lemma.

Fig. 2. The ladder L_8 and Möbius ladder M_8 .

Lemma 4.1. Let G and H be two graphs such that $P_G(x) = P_H(x)$ and $Q_G(x) = Q_H(x)$. Then

- (1) $|E(G)| = |E(H)|$ and $|V(G)| = |V(H)|$;
- (2) If G is simple, then H is simple; if G is cosimple, then H is also cosimple.

Proof. As $P_G(x) = P_H(x)$ and $Q_G(x) = Q_H(x)$, both $r(G)$ and $n(G)$ are determined by the two polynomials together. That is, $r(G) = r(H)$ and $n(G) = n(H)$. As $|E(G)| = r(G) + n(G)$, we conclude that $|E(G)| = |E(H)|$. Moreover, $P_G(x)$ determines the number of components of G (see, for example, [9]). As $r(G) = r(H)$, we deduce that $|V(G)| = |V(H)|$.

As $|E(G)| = |E(H)|$, by Corollary 2.5, we know that if G is simple, then H is simple; and if G is cosimple, then H is also cosimple. \square

From the previous lemma, it is clear that if a simple graph G is χ -unique, then it is also (P, Q) -unique. On the other hand, if a cosimple graph G is flow-unique, then it is also (P, Q) -unique. Note that a (P, Q) -unique graph, however, is not necessarily χ -unique or flow-unique.

Proposition 4.2. W_5 is (P, Q) -unique, but it is neither χ -unique nor flow-unique.

Proof. Fig. 1 shows that W_5 is neither χ -unique nor flow-unique, as we pointed out before. If G is (P, Q) -equivalent to W_5 , then G is a simple and cosimple graph with 6 vertices, 10 edges, and 5 triangles. Moreover, $\lambda(G) = 3$ and hence each vertex has degree at least three. Thus the degree sequence of G is either 3, 3, 3, 3, 3, 5 or 3, 3, 3, 3, 4, 4. Assume that the former is true and let u be the vertex of degree 5. Then each other vertex has degree two in $G - u$. As G is simple, we deduce that $G - u$ is a cycle and hence $G \cong W_5$.

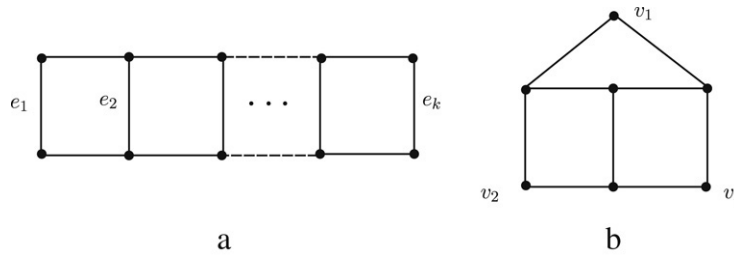
Now we assume that the latter is true, and G has two vertices u and v of degree four. If $uv \notin E(G)$, then as G has exactly 6 vertices, both u and v are adjacent to all four other vertices of G . Now, as $|E(G)| = 10$, it is easy to verify that G has only four triangles, a contradiction. So we may assume that $uv \in E(G)$. Let the neighbors of u be v, u_1, u_2 , and u_3 and the only other remaining vertex of G be v_1 . As $d(v) = 4$, v is adjacent to at least two vertices of u_1, u_2 and u_3 . If v is adjacent to all of u_1, u_2 , and u_3 , then as $d(v_1) = 3$, we deduce that v_1 is also adjacent to all of u_1, u_2, u_3 . Then G has exactly three triangles, a contradiction. Therefore, in the set $\{u_1, u_2, u_3\}$, the vertex v has exactly two neighbors, say, u_2 and u_3 (the other two neighbors are u and v_1). Now it is straightforward to see that $u_1v_1 \in E(G)$, and either $u_1u_2, v_1u_3 \in E(G)$, or $u_1u_3, v_1u_2 \in E(G)$. In either case, G has only 4 triangles, a contradiction. Therefore, W_5 is (P, Q) -unique. \square

The graph $L_n = C_n \times K_2$ is called a ladder (see [11], note that some authors call it a circular ladder). The Möbius ladder M_n is constructed from an even cycle C_{2n} by joining every pair of opposite vertices. Two examples are shown in Fig. 2. The square of an n -cycle, C_n^2 , is obtained from the cycle C_n by adding all the edges between vertices at distance two.

It is straightforward to verify the following lemma.

Lemma 4.3. If k is odd, the chromatic numbers of L_k and M_k are 3 and 2, respectively; if k is even, the chromatic numbers of L_k and M_k are 2 and 3, respectively.

The next two results show that the ladders and Möbius ladders are (P, Q) -unique. To prove the following theorem, we define H_k to be a bipartite graph obtained from a ladder L_k by removing two edges, as shown in Fig. 3(a). Here e_1 and e_k are called the end edges of the graph.

Fig. 3. The graphs H_k and \hat{H}_3 .

Theorem 4.4. The ladders L_n ($n \geq 3$) are (P, Q) -unique.

Proof. As L_3, L_4 and L_5 are all χ -unique (see [9]), they are also (P, Q) -unique. Now assume that $n \geq 6$. Let G be a graph with $P_G(x) = P_{L_n}(x)$ and $Q_G(x) = Q_{L_n}(x)$. By Lemmas 2.1, 4.1 and 4.3 and Theorem 3.1, G is both simple and cosimple (as L_n is both simple and cosimple), and the following hold:

- $|V(G)| = 2n, |E(G)| = 3n$;
- $g(G) = 4$ and the number of 4-cycles is n ;
- G is 2-connected;
- G is 3-edge-connected and the number of 3-element bonds is $2n$;
- if k is odd, the chromatic number of G is 3 and if k is even, the chromatic number of G is 2.

By Lemma 3.3, G is a 3-regular graph and every 3-element bond is trivial (that is, each such bond is associated with a vertex of degree three). Since there are n 4-cycles and $3n$ edges in G , there must exist an edge contained in at least two 4-cycles, say C_1 and C_2 .

Claim 1. For any two 4-cycles C_1 and C_2 in G , $|E(C_1) \cap E(C_2)| \leq 1$.

Suppose that there exist two cycles C_1 and C_2 such that $|E(C_1) \cap E(C_2)| \geq 2$. We have already shown that G is simple. Hence $|E(C_1) \cap E(C_2)| \leq 2$. Then $|E(C_1) \cap E(C_2)| = 2$ and therefore G contains a $K_{3,2}$ as a subgraph. Since G is 3-regular and $|V(G)| \geq 12$, the three edges joining the three 2-degree vertices of $K_{3,2}$ to other vertices of G form a non-trivial 3-element bond of G , a contradiction.

Using Claim 1, we deduce that for any 4-cycle C of G there are at most two 4-cycles that intersect with C and the intersecting edges are not incident. Otherwise, since G is 3-regular, G contains \hat{H}_3 as subgraph, as shown in Fig. 3(b). As G is 3-edge-connected, the three edges joining $\{v_1, v_2, v_3\}$ to $V(G) - V(\hat{H}_3)$ form a non-trivial 3-element bond, a contradiction. Moreover, any edge e of G is contained in at most two 4-cycles. Using this, we will show that G is a ladder. Since G contains n 4-cycles and $3n$ edges, by Claim 1, there are at least two 4-cycles sharing an edge. That is, G contains H_3 as a subgraph. Suppose that $k = \max\{i : H_i \text{ is a subgraph of } G\}$. Then $3 \leq k \leq n$ and H_k is a subgraph of G with end edges e_1 and e_k .

Claim 2. In H_k , either e_1 or e_k is contained in another 4-cycle.

Assume that the claim is false and that neither e_1 nor e_k is contained in any other 4-cycle. In $G - E(H_k)$, there are $n - k + 1$ 4-cycles left, and in $E(G) - E(H_k)$, there are at most $n - k + 1$ edges contained in two 4-cycles. We denote the number of edges contained in these 4-cycles by ξ . Then

$$\xi \geq 4(n - k + 1) - (n - k + 1) = 3n - 3k + 3.$$

On the other hand, there are only $3n - 3k + 2$ edges left in $G - H_k$, a contradiction.

By the choice of k , we conclude that $e_{k+1} = e_1$. Hence G has either a ladder or Möbius ladder as a subgraph. As G is connected and 3-regular, we deduce that $V(G) = V(H_k)$ and hence $k = n$. Therefore $G \cong L_k$ or M_k . By Lemma 4.3, we conclude that $G \cong L_n$. This completes the proof of the theorem. \square

Similarly, we can prove the next result.

Theorem 4.5. Möbius ladders M_n are (P, Q) -unique.

Proof. When $n = 2$, M_2 is just K_4 ; when $n = 3$, then $M_3 \cong K_{3,3}$. Both graphs are known to be χ -unique. The case for $n \geq 4$ is similar to the proof of Theorem 4.4. \square

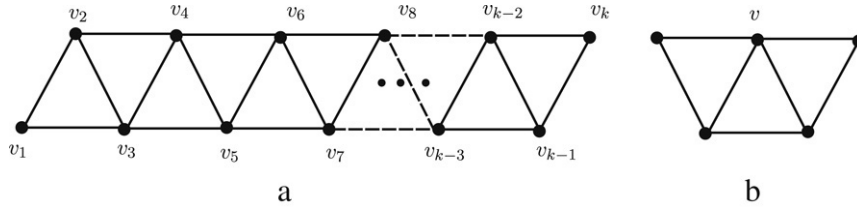


Fig. 4. The graphs R_k and R_5 .

It is known that C_n^2 is χ -unique for $n = 4, 5, 6, 7, 8, 9$ (see [9]). In [16], Read conjectured that for each $n \geq 4$, C_n^2 is χ -unique. This conjecture is still open.

We define a graph R_k recursively as follows: R_3 is a triangle with vertex set $\{v_1, v_2, v_3\}$; to obtain R_k , we add a new vertex v_k and two new edges $v_k v_{k-2}, v_k v_{k-1}$ to the graph R_{k-1} (see Fig. 4(a)). The edges $v_1 v_2$ and $v_{k-1} v_k$ are called the *verge edges* of R_k . Next we prove that C_n^2 is (P, Q) -unique.

Theorem 4.6. *Square of cycle C_n^2 is (P, Q) -unique for all $n \geq 3$.*

Proof. If $n \leq 9$, then C_n^2 is proved to be χ -unique (see [9]), and thus is also (P, Q) -unique. In the following, we assume that $n \geq 10$.

Let G be a graph which is (P, Q) -equivalent to C_n^2 . By Lemmas 2.1 and 4.1 and Theorem 3.1, G is both simple and cosimple (as C_n^2 is both simple and cosimple), and the following hold:

- $|V(G)| = n, |E(G)| = 2n$;
- $g(G) = 3$ and the number of C_3 's is n ;
- G is 2-connected;
- G is 4-edge-connected and the number of 4-element bonds is n .

By Lemma 3.3, G is a 4-regular graph and each 4-element bond is trivial.

Claim 1. G contains no K_4 or W_4 as a subgraph.

Suppose that G contains a subgraph with vertex set $\{v_1, v_2, v_3, v_4\}$ and isomorphic to K_4 . As $n \geq 6$ and G is a 4-regular graph, the edges joining $\{v_1, v_2, v_3, v_4\}$ to $V(G) - \{v_1, v_2, v_3, v_4\}$ form a non-trivial 4-element bond, a contradiction. Similarly, we can show that G contains no W_4 as a subgraph.

Claim 2. Every edge of G is contained in at most two triangles.

Since G is 4-regular, every edge of G is contained in at most three triangles. So it suffices to show that G contains no $K_{3,2}^+$ as a subgraph, where $K_{3,2}^+$ denotes the graph $K_{3,2}$ with an extra edge joining the two vertices in the partite set of order two. As G is 4-regular and contains neither K_4 nor W_4 as a subgraph, every vertex of G is contained in at most three triangles.

Denote $\eta = \{(v_i, t) : \text{vertex } v_i \text{ is contained in the triangle } t\}$. Clearly $|\eta| = 3n$.

Denote $\eta_i =$ the number of triangles containing $v_i, 0 \leq \eta_i \leq 3$.

Let k denote the number of subgraphs isomorphic to $K_{3,2}^+$, say $(K_{3,2}^+)^1, \dots, (K_{3,2}^+)^k$. Clearly, these $(K_{3,2}^+)^i$'s are edge disjoint, so in each $(K_{3,2}^+)^i$, there are two vertices contained in exactly three triangles and the other three vertices contained in at most two triangles. On the other hand, every vertex of G is contained in at most three triangles.

Thus $3n = |\eta| = \sum_{i=1}^n \eta_i \leq 3(2k) + 2(3k) + 3(n - 5k) = 3n - 3k$; hence $k = 0$. Moreover, every vertex is contained in exactly three triangles.

Since G contains no K_4 or $K_{3,2}^+$ as a subgraph, the three triangles containing a vertex v form a subgraph as shown in Fig. 4(b). Thus, G contains R_5 as a subgraph. Assume that $k = \max\{i : R_i \text{ is a subgraph of } G\}$, where $k \geq 5$ and $v_1 v_2$ and $v_{k-1} v_k$ are the verge edges of R_k , as shown in Fig. 4(a). Note that all of the vertices v_3, v_4, \dots, v_{k-2} have degree four in R_k . Consider the vertex v_{k-1} , which must be contained in another triangle T not in R_k . Clearly, v_k is also a vertex of T . Suppose that $V(T) = \{v_{k-1}, v_k, v_{k+1}\}$. As R_k is a maximal subgraph of G , we deduce that $v_{k+1} \in V(R_k)$. Hence $v_{k+1} = v_1$. Now in the graph R'_k obtained by adding two edges $v_1 v_k$ and $v_1 v_{k-1}$ to R_k , there are only two vertices having degrees less than four. Namely, v_2 and v_k , each of which has degree three. If $V(G) \neq V(R_k)$, then G would have a bond of size at most two, a contradiction as G is 4-edge-connected. We conclude that $V(G) = V(R_k)$ and therefore $k = n$. Since G is 4-regular, $v_2 v_k \in E(G)$, and hence $G \cong C_n^2$. \square

1 By now, we have proved that ladders, Möbius ladders and squares of n -cycle are determined by their chromatic
 2 and flow polynomials together. This generalizes the corresponding results of de Mier and Noy [11] that these classes
 3 of graphs are T -unique. Are there graphs which are not (P, Q) -unique, but T -unique? We agree with the comment
 4 from one of the referees that there must exist such graphs, although we have not found such an example yet.

5 In [11], the wheels, complete multipartite graphs and n -cubes are also proved to be T -unique. In addition, wheels
 6 with even number of spokes, and balanced complete multipartite graphs are χ -unique, and n -cubes are conjectured to
 7 be χ -unique (see [9]). It is natural to consider the (P, Q) -uniqueness of these three classes of graphs. It is interesting
 8 to see if more classes of graphs and matroids can be determined by their chromatic polynomial and flow polynomial
 9 together. It is also interesting to find more classes of flow-unique graphs.

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