Isomorphisms of directed $\vec{P}_3$-graphs *

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Abstract

The directed $\vec{P}_k$-graph of a digraph D is obtained by representing the directed paths on k vertices of D by vertices. Two such vertices are joined by an arc whenever the corresponding directed paths in D form a directed path on $k+1$ vertices or a directed cycle on k vertices in D. In this paper, we give a necessary and sufficient condition for two digraphs with isomorphic $\vec{P}_3$-graphs. This improves a previous result, where some additional conditions were imposed.

Keywords: digraph, directed path graph, line digraph, isomorphism.

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1 Introduction

In [4] directed path graphs were introduced as a generalization of (undirected) path graphs and line digraphs. Following [2], first we define a directed graph or digraph D to be a pair $(V(D), A(D))$, where $V(D)$ is a finite nonempty set of elements called *vertices*, and $A(D)$ is a (finite) set of distinct ordered pairs of distinct elements of $V(D)$ called *arcs*. For convenience we shall denote an arc (v, w) (where $v, w \in V(D)$) by vw, and say that v is *adjacent to* w, and w is *adjacent from* v. Let $\vec{P}_k$ and $\vec{C}_k$ be a directed path and directed cycle on k vertices, respectively. In this paper our digraphs do not have multiple arcs or loops.

Let k be a positive integer, and D be a digraph containing at least one $\vec{P}_k$. Denote by $\vec{\Pi}_k(D)$ the set of all $\vec{P}_k$'s of D. Then the *directed $\vec{P}_k$-graph* of D,

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denoted by $\vec{P}_k(D)$, is the digraph with vertex set $\vec{\Pi}_k(D)$; pq is an arc of $\vec{P}_k(D)$ if and only if there is a $\vec{P}_{k+1}$ or $\vec{C}_k : v_1 v_2 \cdots v_{k+1}$ in D (with $v_1 = v_{k+1}$ in the case of a $\vec{C}_k$) such that $p = v_1 v_2 \cdots v_k$ and $q = v_2 \cdots v_k v_{k+1}$.

For a graph transformation, there are two general problems, which are formulated by Grünbaum [7]. We state them here for directed path graphs.

Characterization Problem. Characterize those digraphs that are directed $\vec{P}_k$-graphs of some digraphs.

Determination Problem. Determine which digraphs have a given digraph as their directed $\vec{P}_k$-graphs.

For $k = 2$, i.e., line digraphs, the first problem of characterization was solved in different ways [5, 6], and the determination problem was completely solved by Ouyang and Ouyang [11]. For (undirected) path graphs we have two similar problems. For the case $k = 3$, Broersma and Hoede [3] characterized the graphs that are P_3-graphs; a problem about the characterization was found and resolved by Li and Lin [8]. In [9, 10] Li showed that the P_3-transformation is one-to-one to all connected graphs of minimum degree at least 3. Later, Aldred, Ellingham, Hemminger and Jipsen [1] characterized that two graphs with isomorphic P_3-graphs are either isomorphic or part of three exceptional families, and thus the determination problem for $k = 3$ was completely solved with no degree or connectedness constraints on two original graphs. Along this line, Prisner also did a lot of work on k-path graphs and line digraphs, see [12, 13].

For the directed $\vec{P}_3$-graphs, Broersma and Li [4] got a result on the problem of determination. However, they imposed an additional condition on the original digraphs, and so they did not solve this problem in general. The aim of this paper is to completely solve the determination problem of directed $\vec{P}_3$-graphs.

2 Preliminaries

Now, we recall some old and establish some new terminology and notations which will be used in the sequel. First, we introduce some terminology concerning isomorphisms. Let D and D' be two digraphs. An *isomorphism* of D onto D' is a bijection $f : V(D) \to V(D')$ such that $uv \in A(D)$ if and only if $f(u)f(v) \in A(D')$. To stress the head-to-tail adjacency, for two arcs $a, b \in A(D)$, define that *a hits b* if $a = vw$ and $b = wz$. An *arc-isomorphism* of D onto D' is a bijection $f : A(D) \to A(D')$ such that $a \in A(D)$ hits $b \in A(D)$ if and only if $f(a) \in A(D')$ hits $f(b) \in A(D')$. A *$\vec{P}_3$-isomorphism* of D onto D' is an isomorphism of $\vec{P}_3(D)$ onto $\vec{P}_3(D')$. We say that a $\vec{P}_3$-isomorphism f of D onto D' is *induced by an arc-isomorphism* of D onto D' if there exists an arc-isomorphism f^* of D onto D' such that $f(uvw) = f^*(uv)f^*(vw)$ for each

$\vec{P}_3 = uvw$ of D.

For any digraph D and any vertex v in D, let $N^-(v)$ denote the in-neighborhood of v and $N^+(v)$ the out-neighborhood of v in D. Let $d^-(v)$ denote the in-degree of v and $d^+(v)$ the out-degree of v. For $\alpha \in \vec{\Pi}_3(D)$, define $N^-(\alpha)$, $N^+(\alpha)$, $d^-(\alpha)$ and $d^+(\alpha)$ in $\vec{P}_3(D)$, similarly. A vertex v of D is called a *source* or *sink* if $d^-(v) = 0$ or $d^+(v) = 0$, respectively. We call two vertices $\{u, v\} \subseteq V(D)$ a $\vec{C}_2$-pair if both uv and vu are in $A(D)$. Let $\{u, v\}$ be a $\vec{C}_2$-pair, then v is called a *pseudo-source with respect to* u if $N^-(v) = \{u\}$ and $d^+(v) \geq 2$, v is called a *pseudo-sink with respect to* u if $N^+(v) = \{u\}$ and $d^-(v) \geq 2$, and v is called an *end* if $N^+(v) = N^-(v) = \{u\}$.

If $\vec{P}_3 = uvw$, then v is called the *middle vertex* of $\vec{P}_3$. Denote by $S(v)$ the set of all the $\vec{P}_3$-paths with a common middle vertex v, and then any subset of $S(v)$ is called a *star* at v. If a $\vec{P}_3$-path α in D corresponds to an isolated vertex in $\vec{P}_3(D)$, then we call α an *isolated* $\vec{P}_3$ in D. Let $S_2(D)$ be the set of all the isolated $\vec{P}_3$'s in D, and denote by $S_1(v)$ the set of all the non-isolated $\vec{P}_3$'s in $S(v)$. Set $D^* = D \backslash Iso(D)$, where $Iso(D)$ denote the set of all the isolated vertices in D. A mapping $\sigma : \vec{\Pi}_3(D) \to \vec{\Pi}_3(D^{'})$ is called star-preserving or partially star-preserving if the set $\sigma(S(v))$ or $\sigma(S_1(v))$ is a star in $D^{'}$ for every vertex v of D with $S(v) \neq \emptyset$ or $S_1(v) \neq \emptyset$, respectively.

3 Operations on digraphs

In [4] Broersma and Li showed that $\vec{P}_3(D) \cong \vec{P}_3(D^{'})$ "almost always" implies $D \cong D^{'}$, see the following.

Theorem 3.1 *Let D and $D^{'}$ be two connected digraphs. If for each arc $a = uv \in A(D) \cup A(D^{'})$ there exist arcs $b = xu$ and $c = vy$ in the same digraph with $x \neq v$ and $y \neq u$, then every $\vec{P}_3$-isomorphism of D onto $D^{'}$ is induced by an arc-isomorphism of D onto $D^{'}$.*

In this paper, we will consider the problem of $\vec{P}_3$-isomorphisms on two general digraphs D and $D^{'}$ with no restriction. First, we give four simple situations below, in which two digraphs D and $D^{'}$ with isomorphic $\vec{P}_3$-graphs maybe nonisomorphic.

(1) If we delete any arc not contained in a $\vec{P}_3$ from D to get $D^{'}$, then $\vec{P}_3(D) \cong \vec{P}_3(D^{'})$.

(2) For an arc $uv \in A(D)$, if u is a source or pseudo-source with respect to v and the out-neighborhood of v consists of sinks or/and pseudo-sinks with respect to v (if v is a sink or pseudo-sink with respect to u and the in-neighborhood of u consists of sources or\and pseudo-sources with respect to u), then we see that the arc uv is just in the isolated $\vec{P}_3$'s of D. So, if $D^{'}$ is

a digraph obtained from D by deleting this arc uv and adding some isolated $\vec{P}_3$'s to D such that $|S_2(D^{'})| = |S_2(D)|$, then $\vec{P}_3(D) \cong \vec{P}_3(D^{'})$.

Now, following [4], we introduce an operation which will be used in our third situation and later. An operation *Splitting Vertices* on a digraph D was defined as follows: Let $v \in V(D)$ be a source with out-arcs $vu_1, \cdots, vu_k$. First replace v by two (or more) new vertices v_1, v_2, and then split the out-arcs $vu_1, \cdots, vu_k$ into two (or more) disjoint (nonempty) sets $v_1u_1, \cdots, v_1u_{k_1}$, $v_2u_{k_1+1}, \cdots, v_2u_k$. Similar operation can be defined to apply it on a sink of D.

(3) Let $D^{'}$ be a digraph by doing the above operation "Splitting Vertices" at sources or sinks of D. Obviously, this operation preserves the $\vec{P}_3$-structure of D.

(4) If there is a $\vec{C}_2$-pair $\{u, v\}$ with v an end in a digraph D, then we let $D^{'}$ be a digraph obtained from D by replacing v by two new vertices v_1, v_2 and splitting the arcs uv, vu into uv_1, v_2u. It is clear that $\vec{P}_3(D \cup \vec{P}_3) \cong \vec{P}_3(D^{'})$.

So, from the discussion above, in the rest of this section we can assume that the digraph D satisfying the following properties:

(a) D has no end.

(b) All sources and sinks of D are of degree 1.

(c) For any arc $uv \in A(D)$, there exists a $\vec{P}_3$-path wxu or vyz in D with $x \neq v$ and $y \neq u$.

Now we take the three properties (a), (b) and (c) as one *property* $\mathcal{P}$.

In the following, we will examine two complicated cases, for each of which doing some operations on D also preserves the $\vec{P}_3$-structure of D if the number $|S_2(D)|$ is not changed.

Before discussion, we introduce some additional notations. Denote by S_v and T_v the sets of all sources adjacent to, and sinks adjacent from a vertex v, respectively; and by X_v and Y_v the sets of all pseudo-sources adjacent to, and pseudo-sinks adjacent from a vertex v, respectively.

By the definition of pseudo-sources and pseudo-sinks, if a vertex $u \in X_v$ or Y_v, then u is both the in-neighbor and the out-neighbor of v. In the first case, we mainly deal with the vertices whose in-neighbors or out-neighbors contain sources, sinks, pseudo-sources or pseudo-sinks.

Operation A

1. We consider the vertices whose in-neighborhoods consist of sources, pseudo-sources or/and pseudo-sinks. Here we will distinguish the following three sub-cases.

(i) The in-neighborhood of a vertex consists of only sources.

Let $v \in V(D)$ with $N^-(v) = S_v$, and $d^-(v) = m$. Then we have $X_v = Y_v = \emptyset$, and $T_v = \emptyset$ by property (c) of $\mathcal{P}$. Let $u_1, \cdots, u_k$ $(k \geq 1)$ be all out-neighbors of v. So there exists a vertex $w_i \in N^+(u_i) \backslash \{v\}$ such that vu_iw_i is a $\vec{P}_3$-path in D, for $i = 1, \cdots, k$. If $k \geq 2$, then we do the operation at v as follows:

First, delete all in-neighbors of v, then v is a new source with out-arcs $vu_1, \cdots, vu_k$. Second, we do the operation "Splitting Vertices" at v: replace v by k new vertices $v_1, \cdots, v_k$ and then split the out-arcs $vu_1, \cdots, vu_k$ into k arcs $v_1u_1, \cdots, v_ku_k$. Finally, for every vertex v_i $(i = 1, 2, \cdots, k)$ add m independent new vertices such that each vertex is adjacent to v_i.

(ii) The in-neighborhood of a vertex consists of sources and pseudo-sources.

Let $v \in V(D)$ with $N^-(v) = S_v \cup X_v$, and $|S_v| = m$ $(m \geq 0)$. Then we have $Y_v = \emptyset$, and $T_v = \emptyset$ by property (c) of $\mathcal{P}$. Now, let $X_v = \{x_1, \cdots, x_r\}$ $(r \geq 1)$, and $u_1, \cdots, u_k$ $(k \geq 0)$ be all out-neighbors of v except $x_1, \cdots, x_r$. Note that $k + r \geq 2$. Otherwise if $k + r = 1$, then $k = 0$ and $r = 1$. Thus we have $N^+(v) = \{x_1\}$, and so x_1r is not contained in any $\vec{P}_3$ of D, which is impossible by property (c) of $\mathcal{P}$. Hence $k + r \geq 2$, then we do the operation at v as follows:

First, delete all vertices in S_v and arcs $x_1v, \cdots, x_rv$, then v is a new source with out-arcs $vu_1, \cdots, vu_k, vx_1, \cdots, vx_r$. Second, if $k \geq 1$, then we do the operation "Splitting Vertices" at v: replace v by two new vertices v_1, v_2 and then split the arcs $vu_1, \cdots, vu_k, vx_1, \cdots, vx_r$ into $v_1u_1, \cdots, v_1u_k$, $v_2x_1, \cdots, v_2x_r$. Third, add $m + r$ independent new vertices such that each vertex is adjacent to v_1, and add other $m + r - 1$ independent new vertices such that each vertex is adjacent to v_2 (with $v_2 = v$ in the case of $k = 0$). Then we see that v_1 and v_2 satisfy the condition of (i). Finally, if $k \geq 2$ or $r \geq 2$, then continue to do the operation at v_1 and v_2 as (i).

(iii) The in-neighborhood of a vertex consists of sources, pseudo-sources and pseudo-sinks.

Let $v \in V(D)$ with $N^-(v) = S_v \cup X_v \cup Y_v$, and $X_v = \{x_1, \cdots, x_r\}$ and $Y_v = \{y_1, \cdots, , y_s\}$, where $r \geq 0$ and $s \geq 1$.

If there is a vertex $u \in N^+(v) \backslash (X_v \cup Y_v \cup T_v)$, then there exists a $\vec{P}_3$-path vuw in D. Thus we do the following: delete the arcs $vy_1, \cdots, vy_s$ from D, and then add $s - 1$ independent new vertices such that each vertex is adjacent from v.

Otherwise, we have $N^+(v) = X_v \cup Y_v \cup T_v$. If $X_v \neq \emptyset$, then do the following: delete the arcs $x_1v, \cdots, x_rv$ and $vy_1, \cdots, vy_s$, and then add $r - 1$ independent new vertices such that each vertex is adjacent to v, and add other $s - 1$ independent new vertices such that each vertex is adjacent from v. If $X_v = \emptyset$, then $N^+(v) = Y_v \cup T_v$. So, we can do the operation at v similar to (ii).

2. We deal with the vertices whose out-neighborhoods consist of sinks, pseudo-sources or/and pseudo-sinks. Then, we can do the operation at v similar to 1.

Remark. Let D' be a digraph resulting from D by doing the operation A. For a vertex $v \in V(D')$, if $X_v \neq \emptyset$ or $Y_v \neq \emptyset$, then there are two $\vec{P}_3$-paths xuv and vyz in D' such that $u \notin Y_v$ and $y \notin X_v$. Otherwise, the in-neighborhood of v

satisfies the condition of 1, or the out-neighborhood of v satisfies the condition of 2.

Now, we give a pair of nonisomorphic digraphs without sources, sinks, pseudo-sources, or pseudo-sinks, but their $\vec{P}_3$-graphs are also isomorphic. The first digraph $2\vec{C}_n$ consists of two disjoint $\vec{C}_n$'s. The second one denoted by F_n is obtained from a cycle C_n by replacing each edge uv of C_n by two arcs uv and vu, $n \geq 3$. So, in the second case, we will consider a connected digraph D which contains $\vec{C}_2$-pairs $\{x_i, x_{i+1}\}$ for $i = 1, 2, \cdots, k-1$, where $x_1, \cdots, x_{k-1}$ are all distinct vertices, and the in-neighborhood of x_i consists of x_{i-1}, x_{i+1} and sources, and the out-neighborhood of x_i consists of x_{i-1}, x_{i+1} and sinks in D, for $i = 2, \cdots, k-1$. (S_{x_i} or T_{x_i} may be an empty set for $i = 2, \cdots, k-1$.)

Operation B

Then, according to the in-neighbors and out-neighbors of x_1 and x_k, we will distinguish the following subcases.

1. Let $x_1 = x_k$, then $k \geq 3$.

If the in-neighborhood of x_1 consists of x_2, x_{k-1} and sources, and the out-neighborhood of x_1 consists of x_2, x_{k-1} and sinks, then take two copies of D, and let D_1 and D_2 be two digraphs by deleting the arcs $x_1x_2, \cdots, x_{k-1}x_1$ from one copy of D, and deleting the arcs $x_1x_{k-1}, \cdots, x_2x_1$ from another copy of D, respectively. Finally, we replace D by D_1 and D_2. Note that S_{x_1} or T_{x_1} may also be an empty set.

Now, in the following subcases, we can suppose that $x_1 \neq x_k$, and then $k \geq 2$.

2. If the in-neighborhood of x_i consists of x_{i+1}, sources, and pseudo-sources with respect to x_i, for $i = 1, k$, (with $x_{k+1} = x_{k-1}$ in the case of $i = k$), then do the following:

First, for the case $k \geq 3$, we delete the arcs x_1x_2, x_2x_1, $x_{k-1}x_k$ and x_kx_{k-1} from D, and thus there are at least two components. Denote by D_i the component containing x_i, for $i = 1, 2, k$. Note that $D_2 \neq D_1, D_k$, but D_1 and D_k may be the same component. Now, let H be a digraph which consists of D_2, two additional vertices u and v, and arcs ux_2, x_2u, $x_{k-1}v$ and vx_{k-1}. Then, take two copies of H, and denote by H_1 and H_2 the digraphs obtained by deleting the arcs $ux_2, x_2x_3, \cdots, x_{k-2}x_{k-1}, x_{k-1}v$ from one copy of H, and deleting the arcs $vx_{k-1}, x_{k-1}x_{k-2}, \cdots, x_3x_2, x_2u$ from another copy of H, respectively. Let $|S_{x_i}| = n_i$ and $|X_{x_i}| = r_i$ for $i = 1, k$. Then, we add $n_k + r_k$ independent new vertices such that each vertex is adjacent to v in H_1. Similarly, add $n_1 + r_1$ independent new vertices such that each vertex is adjacent to u in H_2. Finally, in order to preserve the $\vec{P}_3$-structure of D, we identify the vertex x_1 in D_1 to the vertex u in H_1, and identify the vertex x_k in D_k to the vertex v in H_2. At last, observe the out-neighborhoods of x_1 and x_k. If the out-neighborhoods of x_1 and x_k consist of sinks and pseudo-sources with respect to x_1 and x_k, respectively, then do the operation at x_1 and x_k similar to operation A. In Figure 1, we give an example for this situation.

For the case $k = 2$, it is similar to the case $k \geq 3$.

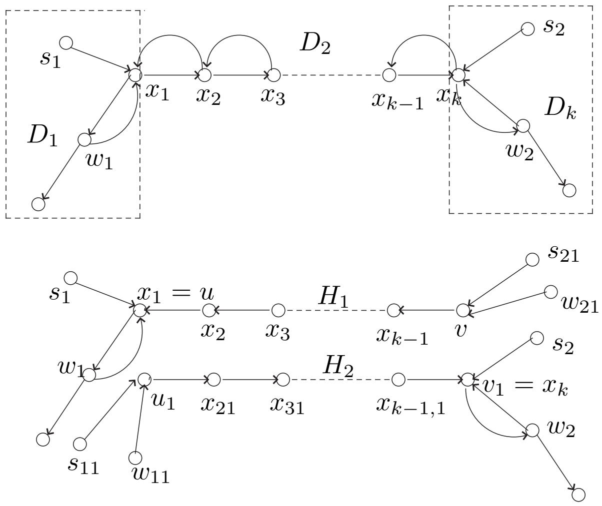

Figure 1: An example for the subcase 2 of the operation B.

3. The out-neighborhood of x_i consists of x_{i+1}, sinks, and pseudo-sinks with respect to x_i, for $i = 1, k$, (with $x_{k+1} = x_{k-1}$ in the case of $i = k$).
4. Without loss of generality, we consider the in-neighborhood of x_1 consists of x_2, sources, and pseudo-sources with respect to x_1, and the out-neighborhood of x_k consists of x_{k-1}, sinks, and pseudo-sinks with respect to x_k.

For the subcases 3 and 4, we can do the operations similar to 2 as above.
Remark. Let $D^{'}$ be a digraph obtained from D by doing the operation B. If there are $\vec{C}_2$-pairs $\{x_i, x_{i+1}\}$ in $D^{'}$ for $i = 1, \cdots, k-1$ $(k \geq 2)$, and the in-neighborhood of x_i consists of x_{i-1}, x_{i+1} and sources, and the out-neighborhood of x_i consists of x_{i-1}, x_{i+1} and sinks, for $i = 2, \cdots, k-1$, then there exist $\vec{P}_3$-paths $u_1v_1x_1$ and $x_1y_1z_1$, or $u_kv_kx_k$ and $x_ky_kz_k$ in $D^{'}$ with $v_1 \neq y_1$ and $v_k \neq y_k$.

Now we denote by $C(D)$ the digraph resulting from D by doing the operations A and B, and satisfying the property $\mathcal{P}$. It is clear that $\vec{P}_3(D)^* \cong \vec{P}_3(C(D))^*$. For two given digraphs D and $D^{'}$ with isomorphic $\vec{P}_3$-graphs, we have $\vec{P}_3(C(D))^* \cong \vec{P}_3(C(D^{'}))^*$. If necessary, add one or more isolated $\vec{P}_3$'s to either $C(D)$ or $C(D^{'})$ to obtain digraphs $\tilde{C}(D)$ and $\tilde{C}(D^{'})$ such that $|S_2(\tilde{C}(D))| = |S_2(\tilde{C}(D^{'}))|$, and then $\vec{P}_3(\tilde{C}(D)) \cong \vec{P}_3(\tilde{C}(D^{'}))$.

4 Main results

Before stating our main result, we give the following lemmas.

Lemma 4.1 *Let σ be a $\vec{P}_3$-isomorphism from D to D' and there are $\vec{P}_3$-paths $x_0x_1x_2$, $x_2x_1y_0$, $x_{i-1}x_ix_{i+1}$ and $x_{i+1}x_ix_{i-1}$ in D with $x_0 \neq y_0$, for $i = 2, \cdots, k-1$. If $\sigma(x_0x_1x_2)$ and $\sigma(x_2x_1y_0)$ have a common middle vertex, then $\sigma(x_{i-1}x_ix_{i+1})$ and $\sigma(x_{i+1}x_ix_{i-1})$ have a common middle vertex, for $i = 2, \cdots, k-1$.*

Proof. Assume, to the contrary, that i is the smallest index in $\{2, \cdots, k-1\}$ such that $\sigma(x_{i-1}x_ix_{i+1})$ and $\sigma(x_{i+1}x_ix_{i-1})$ have no common middle vertex. Let $\sigma(x_{i-1}x_ix_{i+1}) = x'_{i-1}x'_ix'_{i+1}$ and $\sigma(x_{i+1}x_ix_{i-1}) = x''_{i+1}x''_ix''_{i-1}$, then $x'_i \neq x''_i$. For $i \geq 2$, since $x_{i-2}x_{i-1}x_i$ is adjacent to $x_{i-1}x_ix_{i+1}$ and σ is a $\vec{P}_3$-isomorphism from D to D', we know that $\sigma(x_{i-2}x_{i-1}x_i)$ is adjacent to $\sigma(x_{i-1}x_ix_{i+1})$. So $\sigma(x_{i-2}x_{i-1}x_i)$ and $\sigma(x_{i-1}x_ix_{i+1})$ have the arc $x'_{i-1}x'_i$ in common, and thus $\sigma(x_{i-2}x_{i-1}x_i) \in S(x'_{i-1})$. By a similar argument as above, we can also know that $\sigma(x_ix_{i-1}x_{i-2}) \in S(x''_{i-1})$ for the case $i \geq 3$. Then, the choice of i implies that $x'_{i-1} = x''_{i-1}$ for $i \geq 3$. Now, for the case $i = 2$, it is easy to deduce that $\sigma(x_2x_1y_0) \in S(x''_1)$. From the condition, we know that $x'_1 = x''_1$. Thus $x'_{i-1} = x''_{i-1}$ for $i \geq 2$. So, there is a $\vec{P}_3$-path $x''_ix'_{i-1}x'_i$ in D' which is adjacent from $x''_{i+1}x''_ix''_{i-1}$ and adjacent to $x'_{i-1}x'_ix'_{i+1}$. Since σ^{-1} is also a $\vec{P}_3$-isomorphism, there exist a member $\alpha \in \vec{\Pi}_3(D)$ such that $\sigma^{-1}(x''_ix'_{i-1}x'_i) = \alpha$. Then α is adjacent from $x_{i+1}x_ix_{i-1}$ and adjacent to $x_{i-1}x_ix_{i+1}$, which is impossible. ∎

Lemma 4.2 *Let D and D' be two digraphs with each component of D non-isomorphic to F_n. Let σ be a $\vec{P}_3$-isomorphism from D to D'. Then σ is star-preserving if and only if for any arc $uv \in A(D)$*

(i) if $x_1, \cdots, x_r$ are all in-neighbors of u except v, then $\sigma(x_1uv), \cdots, \sigma(x_ruv)$ have a common middle vertex.

(ii) if $y_1, \cdots, y_s$ are all out-neighbors of v except u, then $\sigma(uvy_1), \cdots, \sigma(uvy_s)$ have a common middle vertex.

Proof. The condition is clearly necessary. Now we assume that (i) and (ii) hold for D. Let v be a vertex with $S(v) \neq \emptyset$ in D, uvw and $u'vw'$ be any two $\vec{P}_3$-paths in $S(v)$. If uvw and $u'vw'$ have a common arc, then by (i) or (ii), $\sigma(uvw)$ and $\sigma(u'vw')$ have a common middle vertex. Next we suppose that uvw and $u'vw'$ have no common arc. To show that $\sigma(uvw)$ and $\sigma(u'vw')$ have a common middle vertex, we distinguish the following three cases.

Case 1. $u \neq w'$ or $w \neq u'$, say $w \neq u'$. Then from condition (i), we know that $\sigma(uvw)$ and $\sigma(u'vw)$ have a common middle vertex; and from condition (ii), $\sigma(u'vw)$ and $\sigma(u'vw')$ have a common middle vertex. Thus $\sigma(uvw)$ and $\sigma(u'vw')$ have a common middle vertex.

Case 2. $u = w^{'}$ and $w = u^{'}$, but $S(v) \setminus \{uvw, u^{'}vw'\} \neq \emptyset$. Then there exists a $\vec{P}_3$-path xvy in $S(v)$, such that $\sigma(uvw)$ and $\sigma(xvy)$ have a common middle vertex, so do $\sigma(u^{'}vw')$ and $\sigma(xvy)$, and $\sigma(uvw)$ and $\sigma(u^{'}vw^{'})$.

Case 3. $u = w^{'}$ and $w = u^{'}$ with $S(v) = \{uvw, u^{'}vw'\}$. Since each component of D is nonisomorphic to F_n, we can assume that there are $\vec{P}_3$-paths $x_0x_1x_2$, $x_2x_1y_0$, $x_{i-1}x_ix_{i+1}$ and $x_{i+1}x_ix_{i-1}$ in D with $x_0 \neq y_0$, for $i = 2, \cdots, k-1$. Without loss of generality, let $u = x_{i-1}$, $v = x_i$ and $w = x_{i+1}$ for some $i \in \{2, \cdots, k-1\}$. By Case 1, we know that $\sigma(x_0x_1x_2)$ and $\sigma(x_2x_1y_0)$ have a common middle vertex. Then by Lemma 4.1, we know that $\sigma(uvw)$ and $\sigma(u^{'}vw^{'})$ have a common middle vertex. ∎

Lemma 4.3 *Let D and $D^{'}$ be two digraphs, σ be a $\vec{P}_3$-isomorphism from $\tilde{C}(D)$ to $\tilde{C}(D^{'})$. Then σ and σ^{-1} are partly star-preserving.*

Proof. It is clear that each component of $\tilde{C}(D)$ and $\tilde{C}(D^{'})$ satisfies the conditions of Lemma 4.2. So in order to prove that σ is partly star-preserving, we only need to show that for any arc uv in $\tilde{C}(D)$

(i) if $x_1, \cdots, x_r$ are all in-neighbors of u with $x_iuv \in S_1(u)$ for $i = 1, \cdots, r$, then $\sigma(x_1uv), \cdots, \sigma(x_ruv)$ have a common middle vertex.

(ii) if $y_1, \cdots, y_s$ are all out-neighbors of v with $uvy_j \in S_1(v)$ for $j = 1, \cdots, s$, then $\sigma(uvy_1), \cdots, \sigma(uvy_s)$ have a common middle vertex.

Case 1. We consider an arc uv in $\tilde{C}(D)$, where $x_1, \cdots, x_r$ are all in-neighbors of u except v, and $y_1, \cdots, y_s$ are all out-neighbors of v except u, and $r \geq 1$, $s \geq 1$. Then, it is clear that $x_iuv \in S_1(u)$ and $uvy_j \in S_1(v)$ for $i = 1, \cdots, r$, $j = 1, \cdots, s$.

Subcase 1.1 Since $x_1uv, \cdots, x_ruv$ are adjacent to uvy_1 and σ is a $\vec{P}_3$-isomorphism, we have that $\sigma(x_1uv), \cdots, \sigma(x_ruv)$ are adjacent to $\sigma(uvy_1)$. Let $\sigma(uvy_1) = u^{'}v^{'}y_1^{'}$, then $u^{'}v^{'}$ is the common arc of $\sigma(x_1uv), \cdots, \sigma(x_ruv)$. Thus $\sigma(x_1uv), \cdots, \sigma(x_ruv)$ have a common middle vertex.

Subcase 1.2 By a similar proof as Subcase 1.1, we can obtain that $\sigma(uvy_1)$, $\cdots, \sigma(uvy_s)$ have a common middle vertex.

Case 2. We consider an arc uv in $\tilde{C}(D)$ with v is a sink or pseudo-sink with respect to u, where $x_1, \cdots, x_r$ are all in-neighbors of u with $x_iuv \in S_1(u)$ for $i = 1, \cdots, r$. Let $X = \{x_1, \cdots, x_r\}$ and $r \geq 2$. If $r = 1$, then it is a trivial case. We shall show that $\sigma(x_1uv), \cdots, \sigma(x_ruv)$ have a common middle vertex.

Since $x_iuv \in S_1(u)$, there is a vertex $p_i \in N^-(x_i)\backslash\{u\}$ such that p_ix_iu is a $\vec{P}_3$-path in $\tilde{C}(D)$, for $i = 1, \cdots, r$. By the operation A, the situation that the out-neighborhood of u consists of only sinks and pseudo-sinks with respect to u can not occur in $\tilde{C}(D)$. Now, we denote $W = N^+(u)\backslash(T_u \cup Y_u)$, and let $W = \{w_1, \cdots, w_m\}$. Thus $m \geq 1$ and there is a vertex $q_j \in N^+(w_j)\backslash\{u\}$ such that uw_jq_j is a $\vec{P}_3$-path in $\tilde{C}(D)$ for $j = 1, \cdots, m$.

Subcase 2.1 There is a vertex $w \in W$ but $w \notin X$.

By Subcase 1.1, we have that $\sigma(x_1uw), \cdots, \sigma(x_ruw)$ have a common middle vertex. We also know that $\sigma(x_iuv)$ and $\sigma(x_iuw)$ have a common middle vertex by Subcase 1.2, for $i \in \{1, \cdots, r\}$. Hence, $\sigma(x_1uv), \cdots, \sigma(x_ruv)$ have a common middle vertex.

So, in the following subcases, we can assume that $W \subseteq X$. Without loss of generality, let $w_1 = x_r, w_2 = x_{r-1}, \cdots, w_m = x_{r-m+1}$.

Subcase 2.2 $W \subseteq X$ and $r > m \geq 2$.

By Subcase 2.1, we know that $\sigma(x_1uv), \cdots, \sigma(x_{r-2}uv), \sigma(x_{r-1}uv)$ have a common middle vertex since $w_1 \notin \{x_1, \cdots, x_{r-1}\}$, and so do $\sigma(x_1uv), \cdots, \sigma(x_{r-2}uv), \sigma(x_ruv)$ since $w_2 \notin \{x_1, \cdots, x_{r-2}, x_r\}$. Thus $\sigma(x_1uv), \cdots, \sigma(x_ruv)$ have a common middle vertex.

Subcase 2.3 $W \subseteq X$ and $r = m = 2$.

By Subcase 1.2, it is easy to see that $\sigma(x_1uv)$ and $\sigma(x_1ux_2)$, $\sigma(x_2uv)$ and $\sigma(x_2ux_1)$ have a common middle vertex, respectively. So in order to prove (i), we only need to show that $\sigma(x_1ux_2)$ and $\sigma(x_2ux_1)$ have a common middle vertex.

(1) If v is a pseudo-sink with respect to u, then there is a vertex $z \in N^-(v)\backslash\{u\}$ such that zvu is a $\vec{P}_3$-path in $\tilde{C}(D)$. Then, by Subcase 1.2, we have that $\sigma(vux_1)$ and $\sigma(vux_2)$ have a common middle vertex. We also know that $\sigma(x_1ux_2)$ and $\sigma(vux_2)$, $\sigma(x_2ux_1)$ and $\sigma(vux_1)$ have a common middle vertex by Subcase 1.1, respectively. Therefore $\sigma(x_1ux_2)$ and $\sigma(x_2ux_1)$ have a common middle vertex.

(2) If v is a sink, then we can suppose that there is no pseudo-sink with respect to u in $\tilde{C}(D)$. Otherwise, we obtain that $\sigma(x_1ux_2)$ and $\sigma(x_2ux_1)$ have a common middle vertex by Subcase 2.3(1). Now, we can assume that the out-neighborhood of u consists of x_1, x_2 and sinks. Since $W = X$, there is no pseudo-source with respect to u. Thus the in-neighborhood of u consists of x_1, x_2 and sources.

Then, without loss of generality, we can suppose that there are $\vec{P}_3$-paths $l_{i-1}l_il_{i+1}$ and $l_{i+1}l_il_{i-1}$ in $\tilde{C}(D)$ such that the in-neighborhood of l_i consists of l_{i-1}, l_{i+1} and sources, and the out-neighborhood of l_i consists of l_{i-1}, l_{i+1} and sinks, for $i = 2, \cdots, k-1$, where $l_1, l_2, \cdots, l_k$ are all distinct vertices. Now, let $x_1 = l_{i-1}$, $u = l_i$ and $x_2 = l_{i+1}$ for some $i \in \{2, \cdots, k-1\}$. By the operation B, for $i = 1$ or k, there must be two $\vec{P}_3$-paths $z_it_il_i$ and $l_i\bar{t_i}\bar{z_i}$ in $\tilde{C}(D)$ with $t_i \neq \bar{t_i}$, say $i = 1$. Then by Lemma 4.1, in order to show that $\sigma(x_1ux_2)$ and $\sigma(x_2ux_1)$ have a common middle vertex, we only need to show that $\sigma(t_1l_1l_2)$ and $\sigma(l_2l_1\bar{t_1})$ have a common middle vertex. By Subcases 1.1 and 1.2, we get that $\sigma(l_2l_1\bar{t_1})$ and $\sigma(t_1l_1\bar{t_1})$, $\sigma(t_1l_1l_2)$ and $\sigma(t_1l_1\bar{t_1})$ have a common middle vertex, respectively. Thus $\sigma(t_1l_1l_2)$ and $\sigma(l_2l_1\bar{t_1})$ have a common middle vertex.

Subcase 2.4 $W \subseteq X$ and $m = 1$.

Since $w_1 \notin \{x_1, \cdots, x_{r-1}\}$ and by Subcase 2.1, we know that $\sigma(x_1uv)$, $\cdots, \sigma(x_{r-1}uv)$ have a common middle vertex. And by subcase 1.2, it is easy to see that $\sigma(x_1uv)$ and $\sigma(x_1ux_r)$ have a common middle vertex. Thus, we only need to show that $\sigma(x_1ux_r)$ and $\sigma(x_ruv)$ have a common middle vertex.

Now, we see that the out-neighborhood of u consists of x_r, sinks and pseudo-sinks with respect to u. Then, without loss of generality, we can assume that there are $\vec{C}_2$-pairs $\{l_i, l_{i+1}\}$ in $\tilde{C}(D)$ for $i = 1, \cdots, k-1$ $(k \geq 2)$, and the in-neighborhood of l_i consists of l_{i-1}, l_{i+1} and sources, and the out-neighborhood of l_i consists of l_{i-1}, l_{i+1} and sinks, for $i = 2, \cdots, k-1$, where $l_1, l_2, \cdots, l_k$ are all distinct vertices. Let $u = l_1$ and $x_r = l_2$. By the operation B, there must be two $\vec{P}_3$-paths $l_kt_kz_k$ and $\bar{z_k}\bar{t_k}l_k$ in $\tilde{C}(D)$ with $t_k \neq \bar{t_k}$. Then by Subcase 1.1 and 1.2, it is easy to see that $\sigma(l_{k-1}l_kt_k)$ and $\sigma(\bar{t_k}l_kt_k)$, $\sigma(\bar{t_k}l_kl_{k-1})$ and $\sigma(\bar{t_k}l_kt_k)$ have a common middle vertex, respectively. So $\sigma(l_{k-1}l_kt_k)$ and $\sigma(\bar{t_k}l_kl_{k-1})$ have a common middle vertex for $k \geq 2$. Then, by Lemma 4.1, $\sigma(ux_rl_3)$ and $\sigma(l_3x_ru)$ have a common middle vertex if $k \geq 3$.

Now assume, to the contrary, that $\sigma(x_1ux_r)$ and $\sigma(x_ruv)$ have no common middle vertex. Let $\sigma(x_1ux_r) = x_1^{'}u^{'}x_r^{'}$ and $\sigma(x_ruv) = x_r^{''}u^{''}v^{''}$, then $u^{'} \neq u^{''}$. If $k \geq 3$, then it is easy to see that $\sigma(l_3x_ru)$ is adjacent to $\sigma(x_ruv)$. So $\sigma(l_3x_ru)$ and $\sigma(x_ruv)$ have the arc $x_r^{''}u^{''}$ in common, and then $\sigma(l_3x_ru) \in S(x_r^{''})$. Similarly, we can also know that $\sigma(x_1ux_r)$ is adjacent to $\sigma(ux_rl_3)$, then $u^{'}x_r^{'}$ is the common arc of $\sigma(x_1ux_r)$ and $\sigma(ux_rl_3)$. Thus $\sigma(ux_rl_3) \in S(x_r^{'})$. So, we obtain $x_r^{'} = x_r^{''}$. If $k = 2$, i.e., $x_r = l_k$, then we can get $\sigma(ux_rt_2) \in S(x_r^{'})$ and $\sigma(\bar{t_2}x_ru) \in S(x_r^{''})$ by a similar proof of the case $k \geq 3$. Hence $x_r^{'} = x_r^{''}$. Then, there is a $\vec{P}_3$-path $u^{'}x_r^{'}u^{''}$ in $\tilde{C}(D^{'})$ which is adjacent from $x_1^{'}u^{'}x_r^{'}$ and adjacent to $x_r^{'}u^{''}v^{''}$. Since σ^{-1} is also a $\vec{P}_3$-isomorphism, there is a $\vec{P}_3$-path β in $\tilde{C}(D)$ such that $\sigma^{-1}(u^{'}x_r^{'}u^{''}) = \beta$. Then, β is adjacent from x_1ux_r and adjacent to x_ruv, which is impossible.

Case 3. We consider an arc uv in $\tilde{C}(D)$ with u is a source or pseudo-source with respect to v, where $y_1, \cdots, y_s$ are out-neighbors of v with $uvy_j \in S_1(v)$ for $j \in \{1, \cdots, s\}$. This case is similar to Case 2.

Since σ^{-1} preserves the same properties as σ, σ^{-1} is also partly star-preserving. The proof is thus complete. ∎

Theorem 4.4 *Let D and $D^{'}$ be two digraphs with $|S_2(D)| = |S_2(D^{'})|$. Then $\vec{P}_3(D) \cong \vec{P}_3(D^{'})$ if and only if $C(D) \cong C(D^{'})$.*

Proof. By the definition of $C(D)$, it is clear that $\vec{P}_3(D)^* \cong \vec{P}_3(C(D))^*$ and $\vec{P}_3(D^{'})^* \cong \vec{P}_3(C(D^{'}))^*$. If $C(D) \cong C(D^{'})$, then $\vec{P}_3(D)^* \cong \vec{P}_3(D^{'})^*$. And as $|S_2(D)| = |S_2(D^{'})|$, we have that $\vec{P}_3(D) \cong \vec{P}_3(D^{'})$.

Now we assume that $\vec{P}_3(D) \cong \vec{P}_3(D^{'})$, and thus $\vec{P}_3(C(D))^* \cong \vec{P}_3(C(D^{'}))^*$. Let a and b be the number of isolated $\vec{P}_3$'s in $C(D)$ and $C(D^{'})$, respectively. Without loss of generality, say $a \geq b$. Then we add $a - b$ isolated $\vec{P}_3$'s to

$C(D^{'})$ to obtain a digraph $\tilde{C}(D^{'})$, and so $\vec{P}_3(C(D)) \cong \vec{P}_3(\tilde{C}(D^{'}))$. Note that if $v^{'}$ is a vertex in $\tilde{C}(D^{'})$ with $S_1(v^{'}) \neq \emptyset$, then $v^{'}$ is also in $C(D^{'})$. Let σ be a $\vec{P}_3$-isomorphism from $C(D)$ to $\tilde{C}(D^{'})$. For each vertex $v \in V(C(D))$ with $S_1(v) \neq \emptyset$, there is a vertex $v^{'}$ in $C(D^{'})$ such that $\sigma(S_1(v)) \subseteq S_1(v^{'})$ by Lemma 4.3. In fact, $v^{'}$ is uniquely determined by v. Now we construct a mapping $f : \{v \in V(C(D)) : S_1(v) \neq \emptyset\} \to \{v^{'} \in V(C(D^{'})) : S_1(v^{'}) \neq \emptyset\}$ such that $f(v) = v'$ if $\sigma(S_1(v)) \subseteq S_1(v^{'})$, and then we will show that f is a bijection. If $f(v) = v^{'}$ and $f(v) = u^{'}$, then $\sigma(S_1(v)) \subseteq S_1(v^{'})$ and $\sigma(S_1(v)) \subseteq S_1(u^{'})$. So $\sigma(S_1(v)) \subseteq S_1(v^{'}) \cap S_1(u^{'})$. Since $S_1(v) \neq \emptyset$ and σ is a $\vec{P}_3$-isomorphism, there is a $\vec{P}_3$-path $uvw \in S_1(v)$ such that $\sigma(uvw) \in S_1(v^{'}) \cap S_1(u^{'})$. Thus the middle vertex of $\sigma(uvw)$ is $v^{'}$ and $u^{'}$ which implies $v^{'} = u^{'}$, and so f is really a mapping. If $f(v) = v^{'}$ and $f(u) = v^{'}$, then $\sigma(S_1(v)) \subseteq S_1(v^{'})$ and $\sigma(S_1(u)) \subseteq S_1(v^{'})$, and so $\sigma(S_1(v)) \cup \sigma(S_1(u)) \subseteq S_1(v^{'})$. Since $S_1(v) \neq \emptyset$ and $S_1(u) \neq \emptyset$, and σ is a $\vec{P}_3$-isomorphism, we have that $S_1(v^{'}) \neq \emptyset$. By Lemma 4.3, there is a vertex $w \in V(C(D))$ such that $\sigma^{-1}(S_1(v^{'})) \subseteq S_1(w)$. Since σ^{-1} is also a $\vec{P}_3$-isomorphism, $S_1(v) \cup S_1(u) \subseteq S_1(w)$ which implies $v = w = u$, i.e., $v = u$. So f is an injection, and as σ^{-1} preserves the same properties as σ, then f is a surjection. Hence f is a bijection such that $\sigma(S_1(v)) = S_1(f(v))$.

Claim 1. For an arc pq in $C(D)$, if there are arcs xp and qy with $x \neq q$ and $y \neq p$, then $f(p)f(q)$ is in $C(D^{'})$.

It is clear that $xpq \in S_1(p)$ and $pqy \in S_1(q)$. Since σ is a $\vec{P}_3$-isomorphism, $\sigma(S_1(p)) = S_1(f(p))$ and $\sigma(S_1(q)) = S_1(f(q))$, we obtain that $\sigma(xpq)$ is adjacent to $\sigma(pqy)$, and $\sigma(xpq) \in S_1(f(p))$ and $\sigma(pqy) \in S_1(f(q))$. Thus $f(p)f(q)$ is the common arc of $\sigma(xpq)$ and $\sigma(pqy)$, and so $f(p)f(q)$ is in $C(D^{'})$.

Now, we will consider $u \in V(C(D))$ with $S_1(u) = \emptyset$, then u is a source or sink of $C(D)$. First we construct a one to one mapping between all the sources of $C(D)$ and $C(D^{'})$. Let w be an out-neighbor of a source in $C(D)$, and then denote by $s_1, \cdots, s_m$ ($m \geq 1$) the all sources adjacent to w. By property (c) of $\mathcal{P}$, there is a $\vec{P}_3$-path wxy in $C(D)$ such that $wxy \in S_1(x)$ and $s_iwx \in S_1(w)$ for $i = 1, \cdots m$. So, let $\sigma(s_iwx) = s_i^{'}w^{'}x^{'}$, where $w^{'} = f(w)$ and $x^{'} = f(x)$ by Claim 1, for $i = 1, \cdots, m$. Then, we extend f by defining $f(s_i) = s_i^{'}$ for every $i \in \{1, \cdots, m\}$. It can be similarly done for the case of sinks of $C(D)$. Let z be an in-neighbor of a sink in $C(D)$, and let $t_1, \cdots, t_n$ be all out-neighbors of z. By property (c) of $\mathcal{P}$, there is a $\vec{P}_3$-path $\bar{x}\bar{y}z$ in $C(D)$. Now, by Claim 1, we can let $\sigma(\bar{y}zt_j) = \bar{y}^{'}z^{'}t_j^{'}$, where $\bar{y}^{'} = f(\bar{y})$ and $z^{'} = f(z)$, for $j = 1, \cdots, n$. Then we extend f by defining $f(t_j) = t_j^{'}$ for every $j \in \{1, \cdots, n\}$ (and still denote the resulting function by f), and then we conclude that f determines a mapping $f : V(C(D)) \to V(C(D^{'}))$ such that $\sigma(S_1(v)) = S_1(f(v))$ for all $v \in V(C(D))$. Hence f is a bijection.

Next we will prove that f preserves adjacency and non-adjacency.

Claim 2. For an arc pq in $C(D)$, if p is a source or q is a sink, then $f(p)f(q)$ is in $C(D^{'})$. This can be easily seen by the definition of f.

Claim 3. For an arc pq in $C(D)$, if p is a pseudo-source with respect to q or q is a pseudo-sink with respect to p, then $f(p)f(q)$ is in $C(D^{'})$.

Without loss of generality, let $x_1, \cdots, x_r$ $(r \geq 1)$ be all pseudo-sources with respect to q, and let $p = x_1$. Then we will show that $f(x_i)f(q)$ is in $C(D^{'})$ for $i = 1, \cdots, r$. Let $N^-(q) \backslash (S_q \cup X_q \cup Y_q) = \{u_1, \cdots, u_k\}$, then $k \geq 1$ by the operation A and there is a $\vec{P}_3$-path $v_j u_j q$ in $C(D)$ for $j = 1, \cdots k$. Let $|S_q| = m$ and $|Y_q| = s$, where $m \geq 0$ and $s \geq 0$. Similarly, there is a $\vec{P}_3$-path qwy in $C(D)$ with $w \notin X_q$ by the operation A. Since x_i is a pseudo-source with respect to q, there is a vertex $y_i \in N^+(x_i) \backslash \{q\}$ such that qx_iy_i is a $\vec{P}_3$-path in $C(D)$, for $i = 1, \cdots, r$. Now, suppose that there is an index $i \in \{1, \cdots, r\}$ such that $f(x_i)f(q)$ is not in $C(D^{'})$. Let $\alpha = qwy$ and $\beta = qx_iy_i$. Obviously, $x_1qw \in S_1(q)$, $\alpha \in S_1(w)$, $u_1qx_i \in S_1(q)$ and $\beta \in S_1(x_i)$, and thus we have $\sigma(\alpha) = \alpha^{'} = f(q)f(w)y^{'}$ and $\sigma(\beta) = \beta^{'} = f(q)f(x_i)y_i^{'}$ by Claim 1. If $w \notin \{u_1, \cdots, u_k\}$, then $d^-(\alpha^{'}) = d^-(\alpha) = k + m + r + s$. Since $f(x_i)f(q)$ is not in $C(D^{'})$, we have $d^-(\beta^{'}) \geq d^-(\alpha^{'}) = k + m + r + s$. But in fact $d^-(\beta^{'}) = d^-(\beta) = k + m + r + s - 1$, a contradiction. If $w = u_j$ for some $j \in \{1, \cdots, k\}$, then $d^-(\alpha^{'}) = d^-(\alpha) = k + m + r + s - 1$. For the arc wq, as there are arcs v_jw and qx_1 with $v_j \neq q$ and $x_1 \neq w$, then we have $f(w)f(q)$ is in $C(D)$ by Claim 1. Since $f(x_i)f(q)$ is not in $C(D^{'})$, we have that $d^-(\beta^{'}) > d^-(\alpha^{'}) = k + m + r + s - 1$, also a contradiction.

Since σ^{-1} enjoys the same properties as σ, f also preserves non-adjacency. Then $C(D) \cong C(D^{'})$, and thus $a = b$. The proof is now complete. ∎

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