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Heterochromatic tree partition numbers for complete bipartite graphs☆

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Abstract

An r-edge-coloring of a graph G is a surjective assignment of r colors to the edges of G. A *heterochromatic tree* is an edge-colored tree in which any two edges have different colors. The *heterochromatic tree partition number* of an r-edge-colored graph G, denoted by $t_r(G)$, is the minimum positive integer p such that whenever the edges of the graph G are colored with r colors, the vertices of G can be covered by at most p vertex-disjoint heterochromatic trees. In this paper we give an explicit formula for the heterochromatic tree partition number of an r-edge-colored complete bipartite graph $K_{m,n}$.

MSC: 05C05; 05C15; 05C70; 05C75

Keywords: Edge-coloring; Complete bipartite graph; Heterochromatic tree; Partition number

1. Introduction

An r-edge-coloring of a graph G is a surjective assignment of r colors to the edges of G. A *monochromatic* (*heterochromatic*) *tree* is an edge-colored tree in which any two edges have the same (different) color(s). The (*monochromatic*) *tree partition number* of an r-edge-colored graph G is defined to be the minimum positive integer p such that whenever the edges of G are colored with r colors, the vertices of G can be covered by at most p vertex-disjoint monochromatic trees. The (*monochromatic*) *cycle partition number* and the (*monochromatic*) *path partition number* are defined similarly. Erdös et al. [2] proved that the (monochromatic) tree partition number and the (monochromatic) cycle partition number of K_n is at most $cr^2 \ln r$ for some constant c, and conjectured that the (monochromatic) cycle partition number of K_n is r and the (monochromatic) tree partition number is $r-1$. Almost solving one of the two conjectures, Haxell and Kohayakawa [5] proved that the (monochromatic) tree partition number of K_n is at most r provided that n is large enough with respect to r. Haxell [4] proved that the (monochromatic) cycle partition number of the complete bipartite graph $K_{n,n}$ is also independent of n, which answered a question in [2]. Kaneko et al. [6] gave an explicit expression for the (monochromatic) tree partition number of a 2-edge-colored complete multipartite graph. In particular, let $n_1, n_2, \ldots, n_k$ $(k \geqslant 2)$ be integers such that $1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$ and let $n = n_1 + n_2 + \cdots + n_{k-1}$, $m = n_k$, they [6]

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proved that

$$t'_2(K_{n_1,n_2,\ldots,n_k}) = \left\lfloor \frac{m-2}{2^n} \right\rfloor + 2,$$

where $t'_r(K_{n_1,n_2,\ldots,n_k})$ denotes the (monochromatic) tree partition number of the r-edge-colored graph $K_{n_1,n_2,\ldots,n_k}$. Other related partition problems can be found in [1,3,7,8].

Analogous to the monochromatic tree partition case, we define the *heterochromatic tree partition number* of an r-edge-colored graph G, denoted by $t_r(G)$, to be the minimum positive integer p such that whenever the edges of the graph G are colored with r colors, the vertices of G can be covered by at most p vertex-disjoint heterochromatic trees. In this paper we consider an r-edge-colored complete bipartite graph $K_{m,n}$.

Since it is almost trivial to get the heterochromatic tree partition number of the graph $K_{1,n}$, and $t_r(K_{1,n}) = n - r + 1$ for $1 \leqslant r \leqslant n$, in this paper we may assume that $2 \leqslant m \leqslant n$, and $|X| = m$, $|Y| = n$, when we consider the complete bipartite graph $K_{m,n}$ with bipartition (X, Y).

In order to prove our main result, we introduce the following notations. Let ϕ be an r-edge-coloring of a graph G. Denote by $t_r(G, \phi)$ the minimum positive integer p such that under the r-edge-coloring ϕ, the vertices of G can be covered by at most p vertex-disjoint heterochromatic trees. Clearly, $t_r(G) = \max_\phi t_r(G, \phi)$, where ϕ runs over all r-edge-colorings of the graph G. Let ϕ be an r-edge-coloring of the graph G and let F be a spanning forest of G whose every component is a heterochromatic tree. Then, F is called an *optimal heterochromatic tree partition* of the graph G with edge-coloring ϕ if F contains exactly $t_r(G, \phi)$ components. A tree consisting of a single vertex is also regarded as a heterochromatic tree. As usual, $\phi(e)$ denotes the color of an edge e, $d^\phi(v)$ denotes the color degree of a vertex v (the number of colors presenting at v), and $\phi(H)$ denotes the set of colors appearing in a subgraph or an edge-set H of G.

The paper is organized as follows. In Section 2, we first define a special r-edge-coloring ϕ^* of $K_{m,n}$ for every r, $1 \leqslant r \leqslant mn$, and then give an explicit formula for the heterochromatic tree partition number of $K_{m,n}$ under the r-edge-coloring ϕ^*. In Section 3, we prove that the coloring ϕ^* is a worst coloring, which means that for any $1 \leqslant r \leqslant mn$, $t_r(K_{m,n}) = t_r(K_{m,n}, \phi^*)$.

2. The r-edge-coloring ϕ^* of $K_{m,n}$ for every r

In this section, we give a special r-edge-coloring ϕ^* of $K_{m,n}$ for every r, $1 \leqslant r \leqslant mn$. Then we give the heterochromatic tree partition number of $K_{m,n}$ under the r-edge-coloring ϕ^*. This coloring will serve as a worst coloring, which means that among all edge-colorings of $K_{m,n}$ this coloring will force us to use a maximum number of vertex-disjoint heterochromatic trees to cover the vertex-set of $K_{m,n}$.

Definition 1. Suppose $2 \leqslant m \leqslant n$, $1 \leqslant r \leqslant mn$ and $K_{m,n}$ is a complete bipartite graph with bipartition (X, Y), where $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$.

- If $1 \leqslant r \leqslant m$, let ϕ^* be an r-edge-coloring of $K_{m,n}$ such that if $1 \leqslant i \leqslant r$, then all the edges incident with the vertex x_i are in color C_i, and if $r \leqslant i \leqslant m$, then all the edges incident with the vertex x_i are in color C_r.
- If $m = n$ and $r = n^2 - 2n + 2$, let ϕ^* be an r-edge-coloring of $K_{m,n}$ such that the subgraph $K_{m,n}[V(K_{m,n}) \backslash \{x_n, y_n\}]$ is heterochromatic with colors $C_1, C_2, \ldots, C_{n^2-2n+1}$, and all the remaining edges in $K_{m,n}$ are in color C_{n^2-2n+2}.
- Otherwise, let ϕ^* be an r-edge-coloring of $K_{m,n}$ such that every color of $C_1, \ldots, C_{r-1}$ appears exactly once and the number of vertices in Y with color degree at least 2 is as small as possible. In other words, in this r-edge-coloring ϕ^*, if $(j-1)m + i \leqslant r$, then $\phi^*(x_i y_j) = C_{(j-1)m+i}$; if $(j-1)m + i \geqslant r$, then $\phi^*(x_i y_j) = C_r$.

Then we can easily get the following theorem on the heterochromatic tree partition number of $K_{m,n}$ under the r-edge-coloring ϕ^*.

Theorem 2.1. *If $2 \leqslant m \leqslant n$, $1 \leqslant r \leqslant mn$, the two parts of $K_{m,n}$ are X and Y, and the r-edge-coloring is defined as in Definition 1, then*

- *If $1 \leqslant r \leqslant m$, then $t_r(K_{m,n}, \phi^*) = n$.*
- *If $m(n-1) + 1 \leqslant r \leqslant mn$, then $t_r(K_{m,n}, \phi^*) = 1$.*

- *If* $m=n$ *and* $r=n^2-2n+2$, *then* $t_r(K_{m,n},\phi^*)=2$.
- *Otherwise*, $t_r(K_{m,n},\phi^*)=n-\lceil\frac{r-1}{m}\rceil$.

Notice that if $m=n$ and $r=n^2-2n+2$, then $t_r(K_{m,n},\phi^*)=2>n-\lceil\frac{r-1}{m}\rceil$. So we have that

$$t_r(K_{m,n},\phi^*)\geqslant n-\left\lceil\frac{r-1}{m}\right\rceil. \tag{1}$$

3. Heterochromatic tree partition number of an r-edge-colored $K_{m,n}$

In this section, we prove that for any $1\leqslant r\leqslant mn$, $t_r(K_{m,n})=t_r(K_{m,n},\phi^*)$.

First, we do some preparations.

Theorem 3.1. *Suppose* G *is a disconnected bipartite graph with bipartition* (X,Y), *and* $|X|=m$, $|Y|=n$, $2\leqslant m,n$. *If every component of* G *contains at least one vertex in* X, *then* $|E(G)|\leqslant mn-n$; *if some component of* G *contains no vertex of* X, *then* $|E(G)|\leqslant mn-m$.

Proof. Since G is disconnected, G has at least two connected components. Suppose the connected components of G are $S_1,S_2,\ldots,S_a,\{x_1\},\ldots,\{x_b\},\{y_1\},\ldots,\{y_c\}$ (as shown in Fig. 1), where every S_i has at least two vertices. For $i=1,\ldots,a$, denote $X_i=X\cap S_i$, $Y_i=Y\cap S_i$, then $X_i\neq\emptyset$, $Y_i\neq\emptyset$.

First, we consider the case when each component of G contains at least one vertex in X, i.e., $c=0$. Hence $a+b\geqslant 2$, this implies that for each vertex $y\in Y$, there exists a vertex $x\in X$ such that $yx\notin E(G)$. So $mn-|E(G)|\geqslant|Y|=n$.

Now we consider the case when $c>0$, i.e., some component of G contains exactly one vertex that is in Y. Hence for each vertex $x\in X$ there exists a vertex $y\in Y$ such that $xy\notin E(G)$. So $mn-|E(G)|\geqslant|X|=m$. □

From this theorem, we can easily get the following result.

Corollary 3.2. *For a bipartite graph* G *with bipartition* X *and* Y, *if* $|E(G)|>|X||Y|-\min\{|X|,|Y|\}$, *then* G *is connected.*

Now we can get some lemmas for the heterochromatic tree partition number of $K_{m,n}$ under an r-edge-coloring ϕ.

Lemma 3.3. *Suppose* $2\leqslant m\leqslant n$ *and* $r\geqslant 1$. *Then, for any* r*-edge-coloring* ϕ *of* $K_{m,n}$ *we have* $t_r(K_{m,n},\phi)\leqslant n$, *and especially, if* $r>m$, *then* $t_r(K_{m,n},\phi)\leqslant n-1$.

Proof. Suppose the bipartition of $K_{m,n}$ is (X,Y) with $X=\{x_1,\ldots,x_m\}$, $Y=\{y_1,y_2,\ldots,y_n\}$ and ϕ is an r-edge-coloring of $K_{m,n}$. Thus $\{x_1y_1\},\{x_2y_2\},\ldots,\{x_my_m\},\{y_{m+1}\},\ldots,\{y_n\}$ is a heterochromatic tree partition, and so $t_r(K_{m,n},\phi)\leqslant n$.

Now we consider the case when $r>m$ and $m<n$. Since $r>m$, there exists a vertex in X such that there are at least two different colors presenting at it. Suppose x_1 is a such vertex and x_1y_1, x_1y_2 are two such edges with different colors. Hence $\{y_1x_1,x_1y_2\},\{x_2y_3\},\ldots,\{x_my_{m+1}\},\{y_{m+2}\},\ldots,\{y_n\}$ is a heterochromatic tree partition of $K_{m,n}$, and so $t_r(K_{m,n},\phi)\leqslant n-1$.

Finally, we consider the case when $r>m$ and $m=n$. Therefore there exists a vertex in Y such that there are at least two different colors presenting at it. Suppose y_1 is a such vertex. Since $r>n$ and $d^{\phi}(y_1)\leqslant d(y_1)=n$, there exists an edge x_iy_j $(j\neq 1)$ whose color is different from any color presenting at y_1. Noticing that $d^{\phi}(y_1)\geqslant 2$, there is a vertex $x_{i'}\neq x_i$ such that $\phi(x_{i'}y_1)\neq\phi(x_iy_1)$. So $T_1\cup\{e:e\in M\}$ (where T_1 is a heterochromatic spanning tree

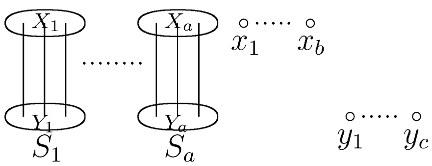

Fig. 1. A disconnected bipartite graph.

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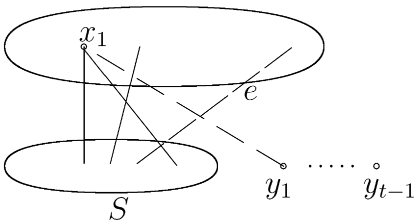

Fig. 2. An optimal heterochromatic tree partition of $(K_{m,n}, \phi)$.

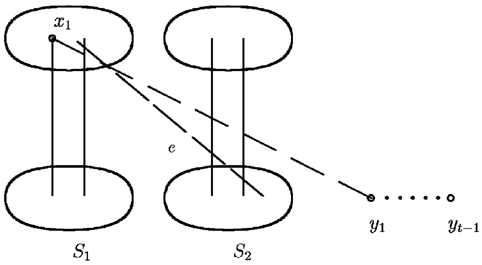

Fig. 3. Suppose $x_1 \in S_1$.

in $K_{m,n}[\{x_i, x_{i'}, y_1, y_j\}]$, and M is a perfect matching of $K_{m,n}\backslash\{x_i, x_{i'}, y_1, y_j\}$) is a heterochromatic tree partition of $K_{n,n}$, and so $t_r(K_{m,n}, \phi) \leqslant n-1$. This completes the proof. □

Lemma 3.4. *Suppose $2 \leqslant m \leqslant n$, $m < r \leqslant m(n-1)$, ϕ is an r-edge-coloring of $K_{m,n}$, and the bipartition of $K_{m,n}$ is (X, Y) with $|X| = m$, $|Y| = n$. If there is an optimal heterochromatic tree partition of $K_{m,n}$ under the coloring ϕ such that X is contained in a heterochromatic tree in the partition, then $t_r(K_{m,n}, \phi) \leqslant t_r(K_{m,n}, \phi^*)$.*

Proof. Suppose $T, \{y_1\}, \{y_2\}, \ldots, \{y_{t-1}\}$ is the optimal heterochromatic tree partition of $(K_{m,n}, \phi)$. Thus $X \subseteq V(T)$, $t = t_r(K_{m,n}, \phi)$. Let S be a heterochromatic spanning subgraph of $K_{m,n}[V(T)]$ that contains all the colors appearing in $K_{m,n}[V(T)]$ (as shown in Fig. 2). Hence it is obvious that $|E(S)| = r$. Denote $n_1 = |V(S) \cap Y|$. Therefore $t = n - n_1 + 1$, and $r \leqslant mn_1$ since $|E(S)| = r$. Now we have $n_1 \geqslant \lceil \frac{r}{m} \rceil$, and if $n_1 \geqslant \lceil \frac{r}{m} \rceil + 1$, then $t = n - n_1 + 1 \leqslant n - \lceil \frac{r}{m} \rceil \leqslant n - \lceil \frac{r-1}{m} \rceil \leqslant t_r(K_{m,n}, \phi^*)$ (the last inequality is because of Eq. (1)).

So we need only to consider the case when $n_1 = \lceil \frac{r}{m} \rceil = \lceil \frac{r-1}{m} \rceil$. In this case, we have $r - 1 > (n_1 - 1)m$, and $n_1 = \lceil \frac{r}{m} \rceil \leqslant \lceil \frac{m(n-1)}{m} \rceil = n - 1$. Thus $t = n - n_1 + 1 \geqslant 2$. Since $|E(S)| = r > m$, there must exist a vertex x_1 in X such that there are at least two edges in $E(S)$ incident with it. Noticing that $T, \{y_1\}, \{y_2\}, \ldots, \{y_{t-1}\}$ is an optimal heterochromatic tree partition of $(K_{m,n}, \phi)$, there is an edge $e \in E(S)$ such that $\phi(e) = \phi(x_1 y_1)$, and $(V(S), E(S)\backslash\{e\})$ is disconnected and has exactly two connected components S_1 and S_2. Without loss of generality, we may suppose $x_1 \in V(S_1)$, see Fig. 3. Since $d_S(x_1) \geqslant 2$, we get $|V(S_1)| \geqslant 2$.

Now we distinguish the following two cases for t.

Case 1: $t = 2$.

Hence $n_1 = n - 1$. So we have $|E(S)\backslash\{e\}| \leqslant mn_1 - \min\{m, n_1\} = m(n-1) - \min\{m, n-1\}$ by Corollary 3.2.

If $m < n$, then $m \leqslant n - 1$. Therefore $|E(S)\backslash\{e\}| \leqslant m(n-1) - m = m(n-2)$, and $r = |E(S)| \leqslant m(n-2) + 1$. By Eq. (1) we can easily get that $t = 2 \leqslant t_r(K_{m,n}, \phi^*)$.

If $m = n$, then $|E(S)\backslash\{e\}| \leqslant m(n-1) - (n-1) = (n-1)^2$, and $r = |E(S)| \leqslant (n-1)^2 + 1$. Note that if $r = (n-1)^2 + 1 = n^2 - 2n + 2$, then $t_r(K_{m,n}, \phi^*) = 2 = t$ under the condition of this case. So we need only to consider the case when $r \leqslant n^2 - 2n + 1$. Thus $t_r(K_{m,n}, \phi^*) \geqslant n - \lceil \frac{r-1}{m} \rceil \geqslant 2 = t$ (the first inequality is because of Eq. (1) and the second inequality is because $m = n$).

Case 2: $t \geqslant 3$.

Hence we have $S_2 \cap Y \neq \emptyset$. Since otherwise S_2 contains exactly one vertex that is in X, denoting it by x_2. Now $T_1, \{x_2 y_2\}, \{y_3\}, \ldots, \{y_{t-1}\}$ (T_1 is a spanning tree in $(V(S_1) \cup \{y_1\}, E(S_1) \cup \{x_1 y_1\})$) is a heterochromatic tree partition of $(K_{m,n}, \phi)$, and only has $t - 1$ vertex-disjoint trees, a contradiction. So we have $|E(S)\backslash\{e\}| \leqslant m \cdot n_1 - m$, because of Theorem 3.1.

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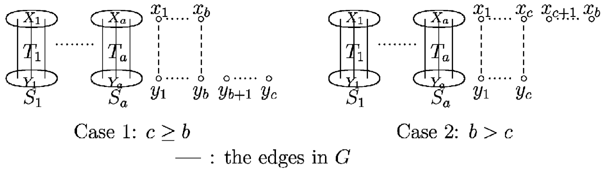

Fig. 4. A heterochromatic tree partition of $K_{m,n}$ under the r-edge-coloring ϕ.

Therefore $r = |E(S)| \leqslant mn_1 - m + 1 = m(n_1 - 1) + 1 = m(n - t) + 1$ (the last equality is because $t = n - n_1 + 1$). So $t \leqslant n - \lceil \frac{r-1}{m} \rceil \leqslant t_r(K_{m,n}, \phi^*)$ by Eq. (1). This completes the proof. □

Finally, we give our main results.

Theorem 3.5. *If* $2 \leqslant m \leqslant n$, *and* $1 \leqslant r \leqslant m$ *or* $m(n-1) + 1 \leqslant r \leqslant mn$, *then* $t_r(K_{m,n}) = t_r(K_{m,n}, \phi^*)$.

Proof. If $1 \leqslant r \leqslant m$, then $t_r(K_{m,n}, \phi^*) = n$. On the other hand, for any r-edge-coloring of $K_{m,n}$, we have $t_r(K_{m,n}, \phi) \leqslant n = t_r(K_{m,n}, \phi^*)$ by Lemma 3.3.

If $m(n-1) + 1 \leqslant r \leqslant mn$, then for any r-edge-coloring ϕ of $K_{m,n}$, the maximal spanning heterochromatic subgraph of $K_{m,n}$ under the r-edge-coloring ϕ must contain $r \geqslant m(n-1) + 1$ edges, and so it is connected by Corollary 3.2. Therefore $t_r(K_{m,n}, \phi) = 1 = t_r(K_{m,n}, \phi^*)$. This completes the proof. □

Theorem 3.6. *If* $2 \leqslant m \leqslant n$, $m < r \leqslant m(n-1)$, *then* $t_r(K_{m,n}) = t_r(K_{m,n}, \phi^*)$.

Proof. Suppose $t_r(K_{m,n}) > t_r(K_{m,n}, \phi^*)$ by contradiction. Choose an r-edge-coloring ϕ such that $t_r(K_{m,n}, \phi) = t_r(K_{m,n}) > t_r(K_{m,n}, \phi^*)$. Let E_0 be a subset of $E(K_{m,n})$ with r elements such that any two edges in E_0 have different colors. Denote $G = (V(K_{m,n}), E_0)$. It is obvious that G is heterochromatic. Suppose that the connected components of G are $S_1, S_2, \ldots, S_a, \{x_1\}, \ldots, \{x_b\}, \{y_1\}, \ldots, \{y_c\}$, where for any $i \in \{1, \ldots, a\}$, S_i contains at least two vertices. Suppose G is chosen so as to first minimize a and then minimize b. Denote $X_i = S_i \cap X$, $Y_i = S_i \cap Y$, $m_i = |X_i|$, $n_i = |Y_i|$. Without loss of generality, assume that the S_i's have been ordered so that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_a \geqslant 1$, if $c \geqslant b$, and so that $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_a \geqslant 1$, if $c < b$.

Suppose T_i is a spanning tree in S_i for $i = 1, 2, \ldots, a$; therefore if $c \geqslant b$, $T_1, \ldots, T_a, \{x_1 y_1\}, \ldots, \{x_b y_b\}, \{y_{b+1}\}, \ldots, \{y_c\}$ is a heterochromatic tree partition of $K_{m,n}$ under the r-edge-coloring ϕ, otherwise $T_1, \ldots, T_a, \{x_1 y_1\}, \ldots, \{x_c y_c\}, \{x_{c+1}\}, \ldots, \{x_b\}$ is a heterochromatic tree partition of $K_{m,n}$ under the r-edge-coloring ϕ, see Fig. 4. So we have $t_r(K_{m,n}, \phi) \leqslant a + \max\{b, c\}$. Now we distinguish the following two cases.

Case 1: $c \geqslant b$. Hence $t_r(K_{m,n}, \phi) \leqslant a + c$.

Since $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_a \geqslant 1$, $r = |E(G)| = \sum_{i=1}^{a} |E(S_i)| \leqslant \sum_{i=1}^{a} m_i n_i \leqslant n_1 (\sum_{i=1}^{a} m_i) \leqslant n_1 m$, and so $n_1 \geqslant \lceil \frac{r}{m} \rceil$.

If $n_2 \geqslant 2$, then $t_r(K_{m,n}, \phi) \leqslant a + c = n_1 + 2 + (a-2) + c - n_1 \leqslant n - n_1 \leqslant n - \lceil \frac{r}{m} \rceil \leqslant n - \lceil \frac{r-1}{m} \rceil \leqslant t_r(K_{m,n}, \phi^*)$, a contradiction. So we need only to consider the case when $n_2 = \cdots = n_a = 1$.

In this case, if $a + c > t_r(K_{m,n}, \phi)$, then $t_r(K_{m,n}, \phi) \leqslant a + c - 1 = n_1 + (a-1) + c - n_1 = n - n_1 \leqslant n - \lceil \frac{r}{m} \rceil \leqslant t_r(K_{m,n}, \phi^*)$, a contradiction. So we can assume that $a + c = t_r(K_{m,n}, \phi)$. Therefore $n - n_1 + 1 = t_r(K_{m,n}, \phi) \leqslant n - 1$ (the last inequality is because of Lemma 3.3). This implies that $n_1 \geqslant 2$. On the other hand, $n - n_1 + 1 = a + c = t_r(K_{m,n}, \phi) \geqslant t_r(K_{m,n}, \phi^*) + 1$. Noticing that $t_r(K_{m,n}, \phi^*) \geqslant n - \lceil \frac{r-1}{m} \rceil$, we have $2 \leqslant n_1 \leqslant \lceil \frac{r-1}{m} \rceil$. Therefore $\lceil \frac{r}{m} \rceil \leqslant n_1 \leqslant \lceil \frac{r-1}{m} \rceil$, and it remains only to consider the case that $n_1 = \lceil \frac{r}{m} \rceil = \lceil \frac{r-1}{m} \rceil$.

If there are at most $m_1 n_1 - m_1 + 1$ edges in S_1, then we have

$$\begin{aligned} r &= |E(S)| = |E(S_1)| + |E(S_2)| + \cdots + |E(S_a)| \\ &\leqslant (m_1 n_1 - m_1 + 1) + m_2 + \cdots + m_a \leqslant m_1 n_1 - m_1 + 1 + (m - m_1) \\ &\leqslant m_1(n_1 - 2) + m + 1. \end{aligned} \tag{2}$$

Therefore, $n_1 \geqslant \frac{r-m-1}{m_1} + 2 \geqslant \frac{r-m-1}{m-1} + 2 = \frac{r-2}{m-1} + 1 > \frac{r-2}{m} + 1 \geqslant \lceil \frac{r-1}{m} \rceil$ (the second inequality is because $m_1 < m$ since otherwise Lemma 3.4 completes the proof), a contradiction.

So we can assume S_1 contains at least $m_1 n_1 - m_1 + 2$ edges. Thus (by Theorem 3.1) we can see that removing any edge from S_1 results in either a connected graph or a graph one of whose connected components is a single vertex from

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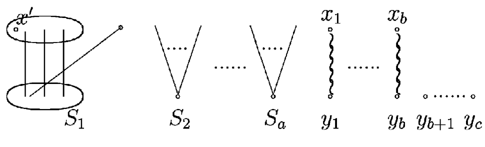

Fig. 5. Figure for Case 1 of Theorem 3.6.

X. Let x' be a vertex of X_1 with degree at least 2 (possible since $r \geqslant m+1$, $n_i = 1$ for $i \geqslant 2$). Note that no matter what edge we delete from S_1, x' is always in the connected component that contains all remaining edges, see Fig. 5.

If $a > 1$, then add the edge e between x' and S_2, and remove the edge e' from G of the same color; denote the new graph by G'. Since $t_r(K_{m,n}, \phi) = a + c$, $(V(S_1) \cup V(S_2), E(S_1) \cup E(S_2) \cup \{e\} \backslash \{e'\})$ must have exactly two components. This implies that $e' \in E(S_1) \cup E(S_2)$. If $e' \in E(S_1)$, then $(V(S_1), E(S_1)\backslash\{e'\})$ has two connected components, and one component is a single vertex in X_1, thus $(V(S_1) \cup V(S_2), E(S_1) \cup E(S_2) \cup \{e\}\backslash\{e'\})$ has a component which is a single vertex in X_1. If $e' \in E(S_2)$, by noticing that S_2 is a star centered at the only vertex in Y_2, then $(V(S_1) \cup V(S_2), E(S_1) \cup E(S_2) \cup \{e\}\backslash\{e'\})$ has a component which is a singlevertex in X_2. Therefore the number of connected components in G' which have at least two vertices is $a-1$, a contradiction to the choice of G. Thus $a = 1$.

If $c \geqslant b+2$, then add the edge between x' and y_c, and remove the edge e' from G of the same color, denote the resulting graph by G'. Since $t_r(K_{m,n}, \phi) = a + c$ and $|E(S_1)| \geqslant m_1 n_1 - m_1 + 2$, the graph $(V(S_1) \cup \{y_c\}, E(S_1) \cup \{x' y_c\}\backslash\{e'\})$ has exactly two components, and one component is a single vertex x'' from X, therefore T_1', $\{x'' y_{c-1}\}$, $\{x_1 y_1\}, \ldots, \{x_b y_b\}$, $\{y_{b+1}\}, \ldots, \{y_{c-2}\}$ is a heterochromatic tree partition of $(K_{m,n}, \phi)$, where T_1' is a spanning tree in $(V(S_1) \cup \{y_c\}\backslash\{x''\}, E(S_1) \cup \{x' y_c\}\backslash\{e'\})$, and has $a+c-1$ vertex-disjoint trees, a contradiction to $t_r(K_{m,n}, \phi) = a + c$. Thus $c \leqslant b+1$.

Furthermore, if S_1 is 2-edge-connected, then add an edge e between x_b and S_1 (by Lemma 3.4 x_b must exist), and delete the edge e' of the same color, denote the new graph by G'. Since S_1 is 2-edge-connected, $(V(S_1) \cup \{x_b\}, E(S_1) \cup \{e\}\backslash\{e'\})$ is connected, hence G' has exactly one connected components with at least two vertices, c components with exactly one vertex which is in Y, and $b-1$ components with exactly one vertex which is in X; this contradicts that b was minimized in our choice of G. Thus by Corollary 3.2 we conclude that S_1 has at most $m_1 n_1 - n_1 + 1$ edges, and hence that $n_1 < m_1$ (since we know there are at least $m_1 n_1 - m_1 + 2$ edges in S_1).

Since $m \leqslant n$, $a = 1$ and $b \leqslant c \leqslant b+1$, this implies $m = n$, $n_1 = m_1 - 1$ and $c = b+1$. Hence $r \leqslant m_1 n_1 - n_1 + 1 = m_1 n_1 - m_1 + 2$, implying $n_1 \geqslant \frac{r-2}{m_1} + 1 \geqslant \frac{r-2}{m-1} + 1$, which is greater than $\lceil \frac{r-1}{m} \rceil$, a contradiction. This completes the case $c \geqslant b$.

Case 2: $b > c$.

Therefore $b > 0$ and $t_r(K_{m,n}, \phi) \leqslant a + b$. It is easy to see that $r = |E(S)| = |E(S_1)| + \cdots + |E(S_a)| \leqslant \sum_{i=1}^{a} m_i n_i \leqslant m_1(\sum_{i=1}^{a} n_i) \leqslant m_1 n$, so we have $m_1 \geqslant \lceil \frac{r}{n} \rceil$.

If $m_1 = 1$, then $a + b = m$, $r \leqslant n$ and $m < n$, since $m_1 \geqslant \lceil \frac{r}{n} \rceil$ and $r > m$. If $n = m+1$, then $r = m+1$, and so $t_r(K_{m,n}, \phi) \leqslant a + b = m = n - 1 = n - \lceil \frac{r-1}{m} \rceil \leqslant t_r(K_{m,n}, \phi^*)$, a contradiction. If $n \geqslant m+2$, then $n - \lceil \frac{r-1}{m} \rceil \geqslant n - \lceil \frac{n-1}{m} \rceil \geqslant n - \frac{n+m-2}{m} = m + (n-m) - \frac{n+m-2}{m} = m + \frac{(m-1)n - m^2 - m + 2}{m} \geqslant m + \frac{(m-1)(m+2) - m^2 - m + 2}{m} = m \geqslant t_r(K_{m,n}, \phi)$, a contradiction. Thus $m_1 \geqslant 2$.

If $m_2 \geqslant 2$, then

$$\begin{aligned} t_r(K_{m,n}, \phi) &\leqslant a + b = a + m - \sum_{i=1}^{a} m_i \\ &\leqslant m - m_1 \leqslant m - \left\lceil \frac{r}{n} \right\rceil \\ &= n - \frac{r}{m} + (m-n) + \left(\frac{r}{m} - \frac{r}{n}\right) - \left(\left\lceil \frac{r}{n} \right\rceil - \frac{r}{n}\right) \\ &= n - \frac{r}{m} - \left(\left\lceil \frac{r}{n} \right\rceil - \frac{r}{n}\right) + (n-m)\left(\frac{r}{mn} - 1\right) \\ &\leqslant n - \frac{r}{m} < n - \frac{r-1}{m}. \end{aligned} \tag{3}$$

So we have that $t_r(K_{m,n}, \phi) \leqslant \lfloor n - \frac{r-1}{m} \rfloor = n - \lceil \frac{r-1}{m} \rceil \leqslant t_r(K_{m,n}, \phi^*)$, a contradiction. Thus, $m_i = 1$ for $2 \leqslant i \leqslant a$.

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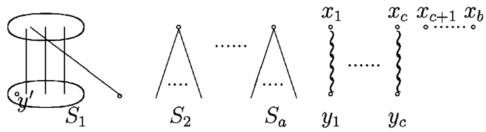

Fig. 6. Figure for Case 2 of Theorem 3.6.

If $m_1 \geqslant \lceil \frac{r}{n} \rceil + 1$, then $t_r(K_{m,n}, \phi) \leqslant m - m_1 + 1 \leqslant m - \lceil \frac{r}{n} \rceil \leqslant t_r(K_{m,n}, \phi^*)$ (the last inequality is because of Eq. (3)), a contradiction. Thus $m_1 = \lceil \frac{r}{n} \rceil$. If $t_r(K_{m,n}, \phi) \leqslant a + b - 1$, then $t_r(K_{m,n}, \phi) \leqslant a + b - 1 = m - m_1 = m - \lceil \frac{r}{n} \rceil$, a contradiction. Hence $t_r(K_{m,n}, \phi) = a + b$. Furthermore, if $r \leqslant n$, then $m_1 = \lceil \frac{r}{n} \rceil = 1$, a contradiction. Therefore $r \geqslant n + 1 \geqslant 2$.

If $r \equiv 1 \pmod n$, then $t_r(K_{m,n}, \phi) = a + b = m - m_1 + 1 = m - \lceil \frac{r}{n} \rceil + 1 = m - \frac{r-1}{n} = \frac{mn-r+1}{n} < \frac{mn-r+2}{m} = n - \frac{r-2}{m} \leqslant n - \lceil \frac{r-1}{m} \rceil + 1 \leqslant t_r(K_{m,n}, \phi^*) + 1$, a contradiction. So we can assume $r \not\equiv 1 \pmod n$. Thus $\lceil \frac{r}{n} \rceil \leqslant \frac{r+n-2}{n}$.

If there are at most $m_1 n_1 - n_1 + 1$ edges in S_1, then as in the previous case, we conclude $r \leqslant m_1 n_1 - n_1 + 1 + \sum_{i=2}^{a} n_i \leqslant m_1 n_1 - n_1 + 1 + (n - n_1)$, implying $m_1 \geqslant \frac{r-n-1}{n_1} + 2 \geqslant \frac{r-n-1}{n} + 2 = \frac{r+n-1}{n} > \frac{r+n-2}{n} \geqslant \lceil \frac{r}{n} \rceil$, a contradiction. Therefore we can assume S_1 has at least $n_1 m_1 - n_1 + 2$ edges. Thus (by Theorem 3.1) we see that removing any edge from S_1 results in either a connected graph, or a graph one of whose connected components is a single vertex from Y_1. Let y' be a vertex of Y_1 with degree at least 2 (possible since $r \geqslant n + 1$, and $m_i = 1$ for $2 \leqslant i \leqslant a$). Note no matter what edge we delete from S_1, y' is always in the connected component that contains all the remaining edges, see Fig. 6.

If $a > 1$, then add the edge e between y' and S_2, and remove the edge e' from G of the same color; denote the new graph by G'. Since $t_r(K_{m,n}, \phi) = a + b$ and S_2 is a star centered at the only vertex in X_2, the graph $(V(S_1) \cup V(S_2), E(S_1) \cup E(S_2) \cup \{e\} \backslash \{e'\})$ has exactly two components, and one component is a single vertex y'' from Y. Since $b > c$, T_1', $T_3, \ldots, T_a$, $\{x_1 y_1\}, \ldots, \{x_c y_c\}$, $\{x_b y''\}$, $\{x_{c+1}\}, \ldots, \{x_{b-1}\}$ (T_1' is a spanning tree in $(V(S_1) \cup V(S_2) \backslash \{y''\}$, $E(S_1) \cup E(S_2) \cup \{e\} \backslash \{e'\})$) is a heterochromatic treepartition of $(K_{m,n}, \phi)$ with $a + b - 1$ vertex disjoint trees, a contradiction. Thus $a = 1$.

Add the edge e between y' and x_b, remove the edge e' from G of the same color, and denote the new graph by G'. Since $(V(S_1) \cup \{x_b\}, E(S_1) \cup \{e\} \backslash \{e'\})$ is connected, or has two components such that one of them is a single vertex y'' of Y_1, G' contradicts the minimality of b in G. This completes the proof. □

From the above results, we can give an explicit formula for the heterochromatic tree partition number of an r-edge-colored complete bipartite graph.

Theorem 3.7. *If $2 \leqslant m \leqslant n$, $1 \leqslant r \leqslant mn$, then the heterochromatic tree partition number of an r-edge-colored $K_{m,n}$ is*

$$t_r(K_{m,n}) = t_r(K_{m,n}, \phi^*) = \begin{cases} n & \text{if } 1 \leqslant r \leqslant m, \\ 1 & \text{if } m(n-1) + 1 \leqslant r \leqslant mn, \\ 2 & \text{if } m = n \text{ and } r = n^2 - 2n + 2, \\ n - \left\lceil \dfrac{r-1}{m} \right\rceil & \text{otherwise.} \end{cases}$$

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