KERNEL METHOD AND LINEAR RECURRENCE SYSTEM

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Abstract

Based on the kernel method, we present systematic methods to solve equation systems on generating functions of two variables. Using these methods, we get the generating functions for the number of permutations which avoid 1234 and $12k(k-1)\cdots 3$ and permutations which avoid 1243 and $12\cdots k$.

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1. Introduction

A lot of enumerative problems can be solved by setting up a recurrence relation or a system of recurrence relations, especially in the enumeration of permutations with forbidden patterns. After introducing a bivariate generating function $F(z,x)$, these recursions often become equations of the form

$$F(z,x) = A(z,x)F(z_0,x) + B(z,x), \tag{1}$$

where z_0 is a specific number, usually equal to either 0 or 1. Knuth [10] first introduce a significant method to solve such equations. It was turned into a method in [2], called the *kernel method*, and a collection of examples are provided by Prodinger [13]. The kernel method is widely used [2,5] and has various generalizations, such as the obstinate form [3,4,6,8].

One generalization of kernel method is applying it to equation systems. In [9], Firro and Mansour presented several applications of kernel method to equation systems, and the theorem which use to solve these equation systems can be formulated as follows. Let $\mathbf{P}(x,v) = (p_{ij}(x,v))_{1\le i,j\le \ell}$ and $\mathbf{Q}(x,v) = (q_{ij}(x,v))_{1\le i,j\le \ell}$ be any two $\ell\times\ell$ matrices of rational functions in x and v, and $\mathbf{b}(x,v) = (b_1(x,v),\ldots,b_\ell(x,v))^T$ be any vector of rational functions in x and v. Suppose $\mathbf{A}(x,v) = (A_1(x,v),\ldots,A_\ell(x,v))^T$ satisfies

$$\mathbf{P}(x,v)\mathbf{A}(x,v) = \mathbf{b}(x,v) + \mathbf{Q}(x,v)\mathbf{A}(x,1), \tag{2}$$

where $A_1(x,v),\ldots,A_\ell(x,v)$ are formal power series. For ease reference, we will denote (2) by $(\mathbf{P},\mathbf{b},\mathbf{Q})$. Using elementary linear algebra (Gaussian elimination) we can assume that the matrix $\mathbf{P} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix. To find a solution for the system $(\mathbf{D},\mathbf{b},\mathbf{Q})$ with diagonal matrix $\mathbf{D}$, we state the following theorem.

Theorem 1. (Firro and Mansour [9, Theorem 5]) *Let* $\mathbf{D}(x,v)\mathbf{A}(x,v) = \mathbf{b}(x,v) + \mathbf{Q}(x,v)\mathbf{A}(x,1)$ *be any linear system of functional equations with* ℓ *variables* $A_1(x,v),\ldots,$ $A_\ell(x,v)$ *of power series in* x *and* v *such that* $\mathbf{D} = diag(d_1(x,v),\ldots,d_\ell(x,v))$ *is a diagonal matrix, where* $d_i(x,v)\not\equiv 0$ *is a rational function for all* $i = 1,2,\ldots,\ell$. *Suppose there exists a formal power series* $u_i(x)$ *such that*

$d_i(x, u_i(x)) = 0$, $i = 1, 2, \ldots, \ell$, and such that $q_{ij}(x, u_i(x))$ is a formal power series for all i, j, where $\mathbf{Q}(x, v) = (q_{ij}(x, v))_{1\le i,j\le \ell}$. Then the system $(\mathbf{D}, \mathbf{b}, \mathbf{Q})$ has a unique solution of algebraic functions if and only if $\det(\mathbf{T}(x)) = \det((q_{ij}(x, u_i(x)))_{1\le i,j\le \ell}) \neq 0$.

The paper [9] presents that the above theorem is useful when the diagonal matrix $\mathbf{D}$ has distinct rational functions in the diagonal. However, when the diagonal of $\mathbf{D}$ contains at least two equal entries, the theorem fails to solve the equations, see [9, Subsection 4.1.4].

In this paper, we extend the kernel method to general equation systems. The paper is organized as follows. In the Section 2, we describe the principal of kernel method to solve the following equation system:

$$\mathbf{K}(x, y)\mathbf{F}(x, y) = \mathbf{A}(x, y)\mathbf{G}(x) + \mathbf{B}(x, y), \tag{3}$$

where $\mathbf{K}(x, y)$ and $\mathbf{A}(x, y)$ are $n \times n$ rational matrices, $\mathbf{B}(x, y)$ is a column vector, and $\mathbf{F}(x, y) = (F_i(x, y)), \mathbf{G}(x) = (G_i(x))$ are unknown column vectors. Our aim is to find all formal power series solutions $\mathbf{F}(x, y)$ and $\mathbf{G}(x)$. We present two methods to solve the system. The first one finds the explicit solutions by induction on the dimension n. We first transform $\mathbf{K}, \mathbf{A}, \mathbf{B}$ to polynomial matrices such that $\mathbf{K}$ is diagonal. Then we apply kernel method to the first equation to obtain a set of equations on $\mathbf{G}(x)$. Solving these equations, we may represent $G_1(x)$ as the linear combination of $G_2(x), \ldots, G_n(x)$: $G_1(x) = \sum c_j G_j(x)$. Thus the dimension is reduced by the substitution $G_1(x) = \sum c_j G_j(x)$. Using this reduction iteratively, we finally reach to an equation of form (1), which can be solved by the standard kernel method. Then the solutions to (3) are obtained by substituting back. The second method provides the algebraic equations that the solutions satisfy using polynomial elimination. As in the first method, we first transform $\mathbf{K}, \mathbf{A}, \mathbf{B}$ to polynomial matrices such that $\mathbf{K}$ is diagonal. For each row of $\mathbf{K}$, using polynomial elimination, we obtain an algebraic equation that $G_1(x), \ldots, G_n(x)$ satisfy. Finally, by polynomial elimination once again, we get equations that $G_i(x), 1 \le i \le n$ satisfies.

In Section 3, we provide a relative small system of equations to illustrate the useful of the method. We consider the number of permutations of length n that avoid 1234 and contain 1243 exactly once. By setting up an equation system of order 3 on the generating functions, we get the explicit formula. Furthermore, we notice that there is a bijection between the permutations that avoid 1234 and contain 1243 exactly once and the permutations that avoid 1243 and contain 1234 exactly once. Thus their generating function coincides.

In Section 4, we generalize the problems in Section 3 to the enumeration on the number of permutations that avoid both 1234 and $12k(k-1)\ldots 3$. We introduce the notation $g_n^{(\ell)}(i_1, i_2, \ldots, i_m)$ which denotes the number of permutations $\pi \in S_n(1234, \tau)$ such that $\pi_1\pi_2\ldots\pi_m = i_1 i_2 \ldots i_m$ together with

$$\pi_n^{-1} < \pi_{n-1}^{-1} < \cdots < \pi_{n+1-\ell}^{-1}.$$

By combinatorial discussion, we derive a system of recursions, which lead to a system of equations on the corresponding generating functions. Applying the method given in Section 2, we get the explicit formulas for the generating functions. In a similar way, we get the generating functions for the number of permutations that avoid both 1243 and $12\ldots k$.

2. General Problem

Kernel method is a powerful tool in solving equations of generating functions. The standard model deal with the case of a functional equation of the form

$$K(x, y)F(x, y) = A(x, y)G(x) + B(x, y),$$

where $F(x,y)$ and $G(x)$ are unknown functions. Suppose that there is a small branch $y = y_0(x)$ such that $K(x, y_0(x)) = 0$. Then

$$G(x) = -B(x, y_0(x))/A(x, y_0(x)),$$

and hence

$$F(x,y) = \frac{-A(x,y)B(x,y_0(x))/A(x,y_0(x)) + B(x,y)}{K(x,y)}.$$

While, in some enumeration problems, we encounter the following equations system of generating functions:

$$\mathbf{K}(x,y)\mathbf{F}(x,y) = \mathbf{A}(x,y)\mathbf{G}(x) + \mathbf{B}(x,y), \tag{4}$$

where $\mathbf{K}(x,y) = (K_{i,j}(x,y))$ and $\mathbf{A}(x,y) = (A_{i,j}(x,y))$ are $n \times n$ matrices, $\mathbf{B}(x,y) = (B_i(x,y))$ is a column vector, and $\mathbf{F}(x,y) = (F_i(x,y)), \mathbf{G}(x) = (G_i(x))$ are unknown column vectors. We also assume that all the entries of $\mathbf{K}, \mathbf{A}, \mathbf{B}$ are rational functions and $\mathbf{K}(x,y)$ is invertible as rational matrix.

Let $\mathbb{Z}$ and $\mathbb{C}$ be the set of integer numbers and complex numbers respectively. Let K be a field, the ring of formal power series and the field of Laurent polynomials on x over K are denoted by $K[[x]]$ and $K((x))$, respectively. We focus on the field of Laurent polynomials on y over $\mathbb{C}((x))$, denoted by $\mathbb{C}((x))((y))$ (see, [18]). Notice that the field of rational functions is a subfield of $\mathbb{C}((x))((y))$. Thus, for any $G_1(x), \dots, G_n(x) \in \mathbb{C}((x))$, since $\mathbf{K}$ is invertible, there are $F_1(x,y), \dots, F_n(x,y) \in \mathbb{C}((x))((y))$ such that (4) holds. However, we are only interested in those $\mathbf{G}(x)$ such that $G_1(x), \dots, G_d(x)$ and $F_1(x,y), \dots, F_n(x,y)$ are formal power series. Using the same idea of kernel method, we present systematic methods to find out such $\mathbf{G}(x)$ and the corresponding $\mathbf{F}(x,y)$.

It is convenience to work over the algebraic closed field

$$\mathbb{C}^{\text{fra}}((x)) = \left\{ \sum_{n \ge n_0} a_n x^{n/N} \;\middle|\; N, n_0 \in \mathbb{Z}, N \ge 1, a_n \in \mathbb{C} \right\}$$

and the field $\mathbb{C}^{\text{fra}}((x))((y))$, see [14, Chapter 6] and [18, Chapter 1]. We need notice that the substitution in $\mathbb{C}^{\text{fra}}((x))((y))$ dose not always make sense. Let ldeg $f(x)$ denote the lowest degree of $f(x)$ on x. Suppose $F(x,y) = \sum_{n \ge n_0} a_n(x) y^n \in \mathbb{C}^{\text{fra}}((x))((y))$ and $y_0(x) \in \mathbb{C}^{\text{fra}}((x))$. Then the substitution $F(x, y_0(x))$ is well-defined if and only if

$$\lim_{n \to \infty} \text{ldeg } a_n(x) + n \cdot \text{ldeg } y_0(x) = +\infty,$$

where ldeg $f(x)$ denotes the lowest degree of the nonzero terms in a Laurent series $f(x)$.

Before solving the system, let us first consider an equation of the form

$$K(x,y)F(x,y) = \sum_{i=1}^{n} a_i(x,y) g_i(x) + b_i(x,y),$$

where K, a_i, b_i are given polynomials. Since $\mathbb{C}^{\text{fra}}((x))$ is algebraic closed, we know that $K(x,y)$ has the following factorization:

$$K(x,y) = (y - y_1(x))^{m_1} (y - y_2(x))^{m_2} \cdots (y - y_r(x))^{m_r},$$

where $y_j(x) \in \mathbb{C}^{\text{fra}}((x))$ are distinct *roots* and $m_j \in \mathbb{Z}^+$ are the multiplicities of the roots. Suppose $F(x, y_j(x))$ is well-defined. Then $y_j(x)$ is an m_j-multiple root of

$$\sum_{i=1}^{n} a_i(x,y) g_i(x) + b_i(x,y) = 0.$$

Hence we have

$$\frac{\partial^\ell}{\partial y^\ell}\sum_{i=1}^n a_i(x,y_j(x))g_i(x)+b_i(x,y_j(x))=0,\quad \text{for } 0\le \ell\le m_j-1. \tag{5}$$

Now we are ready to solve the system (4).

2.1. **Recursive Method.**

Step 1. Multiplying both sides of (4) by $\mathbf{K}^{-1}(x,y)$ so that the equation becomes:

$$\mathbf{F}(x,y)=\mathbf{A}^{(1)}(x,y)\mathbf{G}(x)+\mathbf{B}^{(1)}(x,y).$$

Let $d_i(x,y)$ be the common denominator of the entries of $A^{(1)}_{i,1},\dots,A^{(1)}_{i,n}$ and $B^{(1)}_i$. Multiplying both sides by

$$\mathbf{D}(x,y)=\operatorname{diag}(d_1(x,y),\dots,d_n(x,y))$$

so that the equation becomes

$$\mathbf{D}(x,y)\mathbf{F}(x,y)=\mathbf{A}^{(2)}(x,y)\mathbf{G}(x)+\mathbf{B}^{(2)}(x,y), \tag{6}$$

where the entries of $\mathbf{D},\mathbf{A}^{(2)},\mathbf{B}^{(2)}$ are all polynomials.

Step 2. Solve the system (6) by induction on the dimension n.

Case $n=1$. Let $y_0(x)\in\mathbb{C}^{\text{fra}}(\!(x)\!)$ be a root of $d_1(x,y)$ with multiplicity m. By (5), we have a system of linear equations on $G_1(x)$:

$$H(x,y)\big|_{y=y_0(x)}=0,\ \left.\frac{\partial H(x,y)}{\partial y}\right|_{y=y_0(x)}=0,\ \dots,\ \left.\frac{\partial^{m-1}H(x,y)}{\partial y^{m-1}}\right|_{y=y_0(x)}=0, \tag{7}$$

where

$$H(x,y)=A^{(2)}_{1,1}(x,y)G_1(x)+B^{(2)}_1(x,y).$$

Suppose $G_1(x)=G^*(x)\in\mathbb{C}[[x]]$ is a solution to (7) such that $F_1(x,y)=H(x,y)/d_1(x,y)\in\mathbb{C}[[x,y]]$, it is one solution to the system. After trying all roots of $d_1(x,y)$, we finally get the desired solution set.

Case $n>1$. Let $y_0(x)\in\mathbb{C}^{\text{fra}}(\!(x)\!)$ be a root of $d_1(x,y)$ with multiplicity m. We also have the system (7) of linear equations on $G_1(x)$ with

$$H(x,y)=\sum_{j=1}^n A^{(2)}_{1,j}(x,y)G_j(x)+B^{(2)}_1(x,y).$$

Solving $G_1(x)$, we may represent $G_1(x)$ as linear combinations of $G_2(x),\dots,G_n(x)$:

$$G_1(x)=\sum_{j=2}^n c_{1,j}(x)G_j(x)=\cdots=\sum_{j=2}^n c_{r,j}(x)G_j(x),$$

which is equivalent to a linear system:

$$G_1(x)=\sum_{j=2}^n c_{1,j}(x)G_j(x), \tag{8}$$

$$\sum_{j=2}^n (c_{2,j}-c_{1,j})G_j(x)=0,\dots,\sum_{j=2}^n (c_{r,j}-c_{r-1,j})G_j(x)=0. \tag{9}$$

Now substituting (8) into (6) and deleting the first equation, we obtain a new equations system of dimensional $n-1$:

$$\mathbf{D}'(x,y)\mathbf{F}(x,y) = \mathbf{A}'(x,y)\mathbf{G}(x) + \mathbf{B}'(x,y), \tag{10}$$

where

$$D'(x,y) = \mathrm{diag}(d_2(x,y),\ldots,d_n(x,y))$$

is an $(n-1)\times(n-1)$ submatrix of $D(x,y)$. By induction, (10) is solvable. Let $G_2(x),\ldots,G_n(x)$ be a solution and $G_1(x)$ be given by (8). Suppose that they satisfy (9) and the corresponding $G_1(x)$ and $F_1(x,y)$ are formal power series. Then $(G_1(x),\ldots,G_n(x))$ is a solution to the system. After trying all the roots of $d_1(x,y)$, we get all possible solutions.

Step 3. Substitute $\mathbf{G}(x)$ into the original equation systems, we obtain the vector $\mathbf{F}(x,y)$.

We remark that, the method has no restriction on $y_0(x)$. While in most examples, we may assume that $y_0(x) \in x\mathbb{C}[[x]]$. In the enumerative problems on permutations with avoid patterns, we usually encounter with the equation of the form

$$\mathbf{F}(x,v) = \mathbf{A}(x,v)\mathbf{F}(xv,1) + \mathbf{B}(x,v),$$

where $F_i(x,v) = \sum_{n\geq 0} p_n^{(i)}(v)x^n$ with $p_n^{(i)}(v)$ being polynomials of degree less than or equal to n. By substituting $v = v/x$, we get

$$\mathbf{F}(x,v/x) = \mathbf{A}(x,v/x)\mathbf{F}(v,1) + \mathbf{B}(x,v/x).$$

Note that $\widetilde{\mathbf{F}}(x,v) = \mathbf{F}(x,v/x)$ and $\widetilde{\mathbf{G}}(v) = \mathbf{F}(v,1)$ are formal power series, we can use the above method to solve them.

Moreover, substituting $v = 1$, we get

$$\mathbf{F}(x,1) = \mathbf{A}(x,1)\mathbf{F}(x,1) + \mathbf{B}(x,1).$$

They are addition equations to the system which should be considered also. We have implemented the algorithm in Maple, which can be download from [11].

Example 2. The following equations are obtained by Prodinger on counting Knödel walks [13]:

$$\begin{aligned} F(z,x) &= 1 + zxF(z,x) + \frac{z}{x}F(z,x) + \frac{z}{x}f_0(z) + zG(z,x) + zxg_0(z) + z^2(1+x)f_0(z) \\ G(z,x) &= zF(z,x) + zxG(z,x) + \frac{z}{x}G(z,x) + \frac{z}{x}g_0(z) + z^2 f_0(z). \end{aligned}$$

Prodinger solved the system by finding equations on $F+G$ and $F-G$. Our method is fully automatic and obtains the same result as given by Prodinger:

$$\begin{aligned} f_0(z) &= \frac{3z^2 - (4 - 3\beta + 3\alpha)z - \beta + 1 + \alpha\beta - \alpha}{4z(3z^2 + 2z - 1)}, \\ g_0(z) &= -\frac{1}{2}\cdot\frac{3z^2 + 2z + \alpha\beta - 1}{3z^2 + 2z - 1}, \end{aligned}$$

where

$$\alpha = \sqrt{-3z^2 - 2z + 1}, \quad \text{and} \quad \beta = \sqrt{-3z^2 + 2z + 1}.$$

2.2. **Elimination Method.** Using recursive method, we can get the explicit formulas of the solutions. However, in some cases it is more useful to get the algebraic equations that the solutions satisfy. For such purpose, we provide the following elimination method.

Step 1. Same as the first step in recursive method, that is, transfer the system to

$$\mathbf{D}(x,y)\mathbf{F}(x,y) = \mathbf{A}^{(2)}(x,y)\mathbf{G}(x) + \mathbf{B}^{(2)}(x,y).$$

Step 2. Let $y_0(x) \in \mathbb{C}^{\text{fra}}((x))$ be a root of $d_i(x,y)$ with multiplicity m. We have a system of linear equations on $G_1(x), \ldots, G_n(x)$:

$$H(x,y)\Big|_{y=y_0(x)} = 0, \ \frac{\partial H(x,y)}{\partial y}\Big|_{y=y_0(x)} = 0, \ \ldots, \ \frac{\partial^{m-1} H(x,y)}{\partial y^{m-1}}\Big|_{y=y_0(x)} = 0, \tag{11}$$

where

$$H(x,y) = \sum_{j=1}^{n} A^{(2)}_{i,j}(x,y)G_j(x) + B^{(2)}_i(x,y).$$

Adding the new equation $d_i(x,y_0) = 0$ to the system (11), we obtain a polynomial system on $G_1(x), \ldots, G_n(x)$ and y_0. Using Gröbner basis theory [1, 7] or Wu's method [17], we can reduce the system by eliminating the variable y_0. Thus we obtain an algebraic equation $P_i(G_1, \ldots, G_n) = 0$ for each $1 \le i \le n$.

Step 3. Applying polynomial elimination to the system $P_1 = \cdots = P_n = 0$ to find out the algebraic equations that $G_i(x), 1 \le i \le n$ satisfies.

Example 3. We use the example given by Prodinger on counting Knödel walks once again. Our method provides the following equations:

$$(-9z^3 - 2z^2 + 4z^4 + z + 9z^7 + 30z^6 + 31z^5)f_0(z)^4 + (-9z^6 - 12z^5 + 11z^4 + 16z^3 - 3z^2 - 4z + 1)f_0(z)^3$$
$$+ (9z^5 + 9z^4 + 2z^3 + 10z^2 + 5z - 3)f_0(z)^2 + (6z^3 - 2z^2 - 6z + 2)f_0(z) + 4z = 0.$$
$$(2z + 3z^2 - 1)g_0(z)^2 + (2z + 3z^2 - 1)g_0(z) + z = 0$$

In the following sections, we only provide the explicit formulas given by the recursive method. The equations given by the elimination method can be found in [11].

3. Two applications on permutation patterns

In this section, our goal is to find an explicit formula of the generating functions (1) for the number of permutations that avoid 1234 and contain 1243 exactly once, (2) for the number of permutations that avoid 1243 and contain 1234 exactly once.

Example 4. (Avoiding 1234 and 1243) Let a_n be the number of permutations of length n that avoid both 1234 and 1243. More general, for given $i_1, i_2, \ldots, i_\ell$ we define $a_n(i_1, i_2, \ldots, i_\ell)$ to be the number of permutations π of length n that avoid 1234 and 1243 where $\pi_1\pi_2\ldots\pi_\ell = i_1 i_2 \ldots i_\ell$. By using the scanning-element algorithm as described in [9] it is not hard to see that

$$a_n(i) = a_{n-1}(1) + \cdots + a_{n-1}(i-1) + 2a_{n-1}(i), \ i = 1, 2, \ldots, n-2,$$

and $a_n(n-1) = a_n(n) = a_{n-1}$. If we define $A_n(v) = \sum_{i=1}^{n} a_n(i)v^{i-1}$, then the above recurrence relation can be written as

$$A_n(v) = \frac{1}{1-v}(A_{n-1}(v) - v^n A_{n-1}(1)) + A_{n-1}(v) - A_{n-2}(1)v^{n-2}, \ n \ge 2.$$

Let $A(x,v)=\sum_{n\geq0}A_n(v)x^n$, so rewriting the recurrence relation of $A_n(v)$ in terms of generating functions with using the initial conditions $A_0(v)=A_1(v)=1$ we obtain that

$$\left(1-\frac{x(2-v)}{(1-v)}\right)A(x,v)=1-x-\left(\frac{xv}{1-v}+x^2\right)A(xv,1).$$

Setting $v=v/x$, our algorithm obtains that $A(x,1)=\frac{3-x-\sqrt{1-6x+x^2}}{2}$, see [16].

Example 5. (Restricted the set of permutations $S_n(1234,1243)$**)** Let dw_n be the number of permutations π of length n that avoid both 1234 and 1243 and $\pi_{n-1}^{-1}>\pi_n^{-1}$. More general, for given $i_1,i_2,\dots,i_\ell$ we define $dw_n(i_1,i_2,\dots,i_\ell)$ to be the number of permutations $\pi\in S_n(1234,1243)$ such that $\pi_1\pi_2\dots\pi_\ell=i_1i_2\dots i_\ell$ and $\pi_{n-1}^{-1}>\pi_n^{-1}$. By using the scanning-element algorithm as described in [9] it can be proved that

$$dw_n(i)=dw_{n-1}(1)+\cdots+dw_{n-1}(i-1)+a_{n-1}(i),\ i=1,2,\dots,n-2,$$

$dw_n(n-1)=0$, and $dw_n(n)=a_{n-1}$. If we define $DW_n(v)=\sum_{i=1}^n dw_n(i)v^{i-1}$, then the above recurrence relation can be written as

$$DW_n(v)=\frac{v}{1-v}(DW_{n-1}(v)-v^{n-3}DW_{n-1}(1))+A_{n-1}(v)+A_{n-1}(1)v^{n-1},\ n\geq3.$$

Let $DW(x,v)=\sum_{n\geq0}DW_n(v)x^n$, so rewriting the recurrence relation of $DW_n(v)$ in terms of generating functions with using the initial conditions $DW_0(v)=0, DW_1(v)=1$ and $DW_2(v)=v$, we obtain that

$$\left(1-\frac{xv}{1-v}\right)DW(x,v)-xA(x,v)=-x-\frac{x}{v(1-v)}DW(xv,1)+xA(xv,1).$$

Thus, by Example 4 we get the following system of functional equations:

$$\begin{bmatrix}1-\frac{xv}{1-v} & -x\\ 0 & 1-\frac{x(2-v)}{(1-v)}\end{bmatrix}\begin{bmatrix}DW(x,v)\\ A(x,v)\end{bmatrix}=\begin{bmatrix}-x\\ 1-x\end{bmatrix}+\begin{bmatrix}-\frac{x}{v(1-v)} & x\\ 0 & -\frac{xv}{1-v}-x^2\end{bmatrix}\begin{bmatrix}DW(xv,1)\\ A(xv,1)\end{bmatrix}.$$

Setting $v=v/x$, our algorithm obtains $DW(x,1)=\frac{1+x-\sqrt{1-6x+x^2}}{4}$. Hence, the number of permutations $\pi\in S_n(1234,1243)$ such that $\pi_{n-1}^{-1}>\pi_n^{-1}$ is exactly the half of the number of permutations in $S_n(1234,1243)$, for all $n\geq2$.

Now we ready to find the generating functions for the number of permutations that avoid 1234 (resp. 1243) and contain 1243 (resp. 1234) exactly once.

Example 6. (Avoiding 1234 and Containing 1243 exactly once) In this example we find an explicit formula for t_n the number of permutations that avoid 1234 and contain 1243 exactly once. For given $i_1,i_2,\dots,i_\ell$ we define $t_n(i_1,i_2,\dots,i_\ell)$ to be the number of permutations π that avoid 1234 and contain 1243 exactly once such that $\pi_1\pi_2\dots\pi_\ell=i_1i_2\dots i_\ell$. By using the scanning-element algorithm as described in [9] it can be shown that

$$t_n(i)=t'_n(i)+2t_{n-1}(i)+dw_{n-1}(i),\ i=1,2,\dots,n-3,$$

$t_n(n-2)=t_n(n-1)=t_n(n)=t_{n-1}$, where $t'_n(i)=\sum_{j=1}^{i-1}t_n(i,j)$ satisfies the following relation

$$t'_n(i)=t'_{n-1}(1)+\cdots+t'_{n-1}(i)+t'_{n-1}(i),\ i=1,2,\dots,n-3,$$

with the initial condition $t'_n(n-2)=t_{n-1}-2t_{n-2}$. If we define $T_n(v)=\sum_{i=1}^{n}t_n(i)v^{i-1}$ and $T'_n(v)=\sum_{i=1}^{n-3}t'_n(i)v^{i-1}$, then the above recurrence relations can be written as

$$\begin{aligned}T_n(v)&=(1+v+v^2)v^{n-3}T_{n-1}(1)+T'_n(v)+2(T_{n-1}(v)-v^{n-3}(1+v)T_{n-2}(1))\\&\quad+DW_{n-1}(v)-A_{n-2}(1)v^{n-2},\\T'_n(v)&=\tfrac{1}{1-v}(T'_{n-1}(v)-v^{n-3}T'_{n-1}(1))+T'_{n-1}(v)+2v^{n-4}(T_{n-2}(1)-2T_{n-3}(1)),\end{aligned}$$

for all $n\geq 3$. Let $T(x,v)=\sum_{n\geq 0}T_n(v)x^n$ and $T'(x,v)=\sum_{n\geq 0}T'_n(v)x^n$, so rewriting the recurrence relation of $T_n(v)$ and $T'_n(v)$ in terms of generating functions with using the initial conditions $T_j(v)=T'_j(v)=0$, $j=0,1,2$, we obtain that

$$\begin{aligned}&(1-2x)T(x,v)-T'(x,v)-xDW(x,v)=\left(\tfrac{x(1+v+v^2)}{v^2}-\tfrac{2x^2(1+v)}{v}\right)T(xv,1)-x^2A(xv,1),\\&\left(1-x-\tfrac{x}{1-v}\right)T'(x,v)=\tfrac{2x^2(1-2xv)}{v^2}T(xv,1)-\tfrac{x}{v^2(1-v)}T'(xv,1).\end{aligned}$$

Hence, by Example 4 and Example 5 together with the above equations we obtain that

$$\begin{bmatrix}1-2x&-1&-x&0\\0&1-x\frac{2-v}{1-v}&0&0\\0&0&1-\frac{xv}{1-v}&-x\\0&0&0&1-x\frac{2-v}{1-v}\end{bmatrix}\begin{bmatrix}T(x,v)\\T'(x,v)\\DW(x,v)\\A(x,v)\end{bmatrix}$$
$$=\begin{bmatrix}0\\0\\-x\\1-x\end{bmatrix}+\begin{bmatrix}\frac{x(1+v+v^2)}{v^2}-\frac{2x^2(1+v)}{v}&0&0&-x^2\\\frac{2x^2(1-2xv)}{v^2}&-\frac{x}{v^2(1-v)}&0&0\\0&0&-\frac{x}{v(1-v)}&x\\0&0&0&-\frac{xv}{1-v}-x^2\end{bmatrix}\begin{bmatrix}T(xv,1)\\T'(xv,1)\\DW(xv,1)\\A(xv,1)\end{bmatrix}.$$

Our algorithm obtains

$$\begin{aligned}&T'(x,1)\\&=(2x-1)\frac{(8x^5-56x^4+97x^3-57x^2+13x-1)+(8x^4-32x^3+31x^2-10x+1)\sqrt{1-6x+x^2}}{8x^2},\end{aligned}$$

and

$$T(x,1)=\frac{(x-1)(4x^3-20x^2+9x-1)+(2x-1)(2x^2-5x+1)\sqrt{1-6x+x^2}}{8x^2}.$$

Note that we need an extra equation by the substitution $v=1$ in the first equation.

Up to now, we obtain the generating function for the number of permutations that avoid 1234 and contain 1243 exactly once. The following theorem shows that it is also the generating function for the number of permutations that avoid 1243 and contain 1234 exactly once.

Theorem 7. *There is a bijection between the permutations that avoid* 1234 *and contain* 1243 *exactly once and the permutations that avoid* 1243 *and contain* 1234 *exactly once.*

Proof. Suppose π is a permutation that avoids 1234 and contains 1243 exactly once. Suppose further that 1243 occurs at positions i_1,i_2,i_3,i_4. Let π' be the permutation defined by

$$\pi'_j=\begin{cases}\pi_j&\text{if } j\neq i_3,i_4\\\pi_{i_4}&\text{if } j=i_3\\\pi_{i_3}&\text{if } j=i_4\end{cases}$$

Since π avoids 1234 and contains 1243 exactly once, we know that its block decomposition (see [12] and references therein) is as follows, see Figure 1.

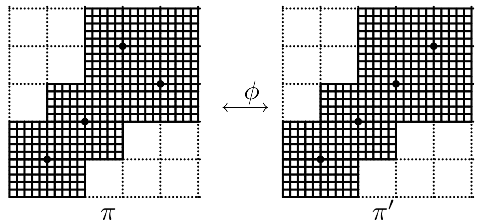

FIGURE 1. A bijection ϕ.

Suppose $\pi'_{j_1}\pi'_{j_2}\pi'_{j_3}\pi'_{j_4}$ is of pattern 1234. At least one of π'_{j_i} is not equal to π_{j_i}. Assume $i_4 \in \{j_1,\dots,j_4\}$. Since $\pi'_{i_4} = \pi_{i_3}$ is right-maximal, it forces $j_4 = i_4$. If $j_3 > i_3$, then $\pi_{j_1}\pi_{j_2}\pi_{j_3}\pi_{j_4}$ is of pattern 1234. If $j_3 < i_3$, then $\pi_{j_1}\pi_{j_2}\pi_{j_3}\pi_{i_3}$ is of pattern 1234. Therefore $j_3 = i_3$ and hence $j_1 = i_1, j_2 = i_2$. Assume $i_3 \in \{j_1,\dots,j_4\}$ but $i_4 \notin \{j_1,\dots,j_4\}$. Since $\pi'_{i_3} = \pi_{i_4}$ is right-maximal, it forces $j_4 = i_3$. But then $\pi_{j_1}\pi_{j_2}\pi_{j_3}\pi_{i_3}$ will be of pattern 1234, which is a contradiction. Thus, π' contains the pattern 1234 exactly once. Similar discussion shows that π' avoids the pattern 1243. Conversely, any permutation π' that avoids 1243 and contains 1234 exactly once can be mapped back to a permutation π which avoids 1234 and contains 1243 exactly once. This completes the proof. ∎

4. Two big systems of equaitons

In this section we present two series of equation systems which generalize the examples given in Section 3. Our first example is to enumerate the number of permutations that avoid both 1234 and $12k(k-1)\dots3$ (Subsection 4.1) and our second example is to enumerate the number of permutations that avoid both 1243 and $12\dots k$ (Subsection 4.2).

4.1. Avoiding 1234 and $12k(k-1)\dots3$.

In this subsection we present an exact formula of the generating function for the number of permutations of length n that avoid 1234 and $\tau = 12k(k-1)\dots3$. To do that we need the following notations. We denote by $g_n(i_1,i_2,\dots,i_m)$ the number of permutations $\pi \in S_n(1234,\tau)$ such that $\pi_1\pi_2\dots\pi_m = i_1i_2\dots i_m$. It is natural to extend g_n to the case $m=0$ by setting $g_n(\emptyset) = \#S_n(1234,\tau)$. More generally, we denote by $g_n^{(\ell)}(i_1,i_2,\dots,i_m)$ the number of permutations $\pi \in S_n(1234,\tau)$ such that $\pi_1\pi_2\dots\pi_m = i_1i_2\dots i_m$ together with

$$\pi_n^{-1} < \pi_{n-1}^{-1} < \cdots < \pi_{n+1-\ell}^{-1}. \tag{12}$$

It is natural to extend $g_n^{(\ell)}$ to the case $m=0$ as the number of permutations $\pi \in S_n(1234,\tau)$ that satisfy (12).

Lemma 8. *Let $k \geq 4$, $\ell = 1,2,\dots,k-3$, and $1 \leq j \leq n-\ell$. Then*

$$\begin{cases} g_n^{(\ell)}(j) = \sum_{i=1}^{j-1} g_{n-1}^{(\ell)}(i) + \sum_{i=\ell}^{\min\{k-3,n-j-1\}} g_{n-1}^{(i)}(j) + g_{n-1}^{(\max\{1,\ell-1\})}(j), \\ g_n^{(\ell)}(n-\ell+1) = \cdots = g_n^{(\ell)}(n-1) = 0, \\ g_n^{(\ell)}(n) = g_{n-1}^{(\max\{1,\ell-1\})}. \end{cases} \tag{13}$$

Proof. Let π be any permutation of length n that avoid both 1234 and τ, start at $\pi_1 = j$ and satisfy (12). Then, $g_n^{(\ell)}(j) = 0$ for all $j = n+1-\ell, n+2-\ell, \ldots, n-1$ and $g_n^{(\ell)}(n) = g_{n-1}^{(\ell-1)}$. For all $j = 1, 2, \ldots, n-\ell$,

$$\begin{aligned} g_n^{(\ell)}(j) &= \sum_{i=1}^{j-1} g_n^{(\ell)}(j,i) + \sum_{i=j+1}^{n+2-k} g_n^{(\ell)}(j,i) + \sum_{i=n+3-k}^{n-1} g_n^{(\ell)}(j,i) + g_n^{(\ell)}(j,n) \\ &= \sum_{i=1}^{j-1} g_{n-1}^{(\ell)}(i) + \sum_{i=\ell}^{\min\{k-3,n-j-1\}} g_{n-1}^{(i)}(j) + g_{n-1}^{(\max\{1,\ell-1\})}(j). \end{aligned}$$

The last equality materializes from the following facts:

(1) $\pi \in S_n(1234, \tau)$ that satisfy (12) with $\pi_1\pi_2 = ji$ and $j > i$ if and only if $\pi_2\pi_3 \ldots \pi_n$ is a permutation on the letters $1, 2, \ldots, j-1, j+1, \ldots, n$ that avoid both 1234 and τ where it satisfies (12) with $\pi_2 = i$, so in this case there are exactly $g_{n-1}^{(\ell)}(i)$ permutations.
(2) if $\pi \in S_n(1234, \tau)$ satisfies (12) with $\pi_1\pi_2 = ji$ and $j < i < n+3-k$, then π contains τ.
(3) $\pi \in S_n(1234, \tau)$ satisfies (12) with $\pi_1\pi_2 = ji$ and $j \le n+2-k < i \le n-\ell$ if and only if $\pi_1\pi_3 \ldots \pi_n$ is a permutation on the letters $1, 2, \ldots, i-1, i+1, \ldots, n$ that avoid both 1234 and τ such that it satisfies $\pi_n^{-1} < \pi_{n-1}^{-1} < \cdots < \pi_{\max\{n+4-k,i+1\}}^{-1}$ and $\pi_1 = j$.
(4) if $\pi \in S_n(1234, \tau)$ satisfies (12) with $\pi_1\pi_2 = jn$, then $\pi_1\pi_3 \ldots \pi_n \in S_{n-1}(1234, \tau)$ satisfies $\pi_{n-1}^{-1} < \pi_{n-2}^{-1} < \cdots < \pi_{n+1-\ell}^{-1}$ with $\pi_1 = j$.

Adding the above disjoint cases, Cases (1)-(4), we get the desired result, as claimed in (13). ∎

Using Lemma 8 we quickly generate the numbers $|S_n(1234, 12k \ldots 3)|$; the first few of these numbers are given in Table 1.

$k\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
4	1	1	2	6	22	90	394	1806	8558	41586	206098	1037718	5293446	27297738
5	1	1	2	6	23	102	496	2566	13869	77420	442966	2584127	15312730	91914699
6	1	1	2	6	23	103	512	2740	15478	91062	552747	3438323	21809897	140557645

TABLE 1. The number of permutations which avoid both 1234 and $12k \ldots 3$.

Define, $G_n^{(\ell)}(v) = \sum_{t=1}^{n} g_n^{(\ell)}(t)v^{t-1}$ for any $n \ge 1$ and $\ell = 1, 2, \ldots, k-3$. It is natural to extend $G_n^{(\ell)}$ to the case $n = 0$ by setting $G_0^{(\ell)}(v) = 1$. If multiplying $g_n^{(\ell)}(j)$ by v^{j-1} and summing over all the possibilities of j, then Lemma 8 gives the following result.

Proposition 9. *For all $n \ge 2$,*

$$\begin{aligned} G_n^{(1)}(v) &= \frac{1}{1-v}(G_{n-1}^{(1)}(v) - v^{n-3}G_{n-1}^{(1)}(1)) + G_{n-1}^{(1)}(v) \\ &\quad + v^{n-3}(1+v+v^2)G_{n-1}^{(1)}(1) - v^{n-2}G_{n-2}^{(1)}(1) + \sum_{i=2}^{k-3}(G_{n-1}^{(i)}(v) - v^{n-2}G_{n-2}^{(i-1)}(1)) \end{aligned}$$

and

$$\begin{aligned} G_n^{(\ell)}(v) &= \frac{v}{1-v}\left[(G_{n-1}^{(\ell)}(v) - v^{n-2}G_{n-2}^{(\ell-1)}(1)) - v^{n-\ell-1}(G_{n-1}^{(\ell)}(1) - G_{n-2}^{(\ell-1)}(1))\right] \\ &\quad + v^{n-1}G_{n-1}^{(\ell-1)}(1) + \sum_{i=\ell-1}^{k-3}(G_{n-1}^{(i)}(v) - v^{n-2}G_{n-2}^{\max\{i-1,1\}}(1)) \end{aligned}$$

for all $\ell = 2, 3, \ldots, k-3$.

Proof. Multiplying the recurrence relation (13) by v^{j-1} and summing over $j=1,2,\ldots,n$ we arrive at

$$\begin{aligned}
G_n^{(1)}(v) &= v^{n-1}G_{n-1}^{(1)}(1)+\sum_{j=1}^{n-1}g_{n-1}^{(1)}(j)v^{j-1}+\sum_{j=1}^{n-1}v^{j-1}\sum_{i=1}^{j-1}g_{n-1}^{(1)}(i)+\sum_{j=1}^{n-1}\sum_{i=1}^{\min\{k-3,n-j-1\}}g_{n-1}^{(i)}(j)v^{j-1}\\
&= v^{n-1}G_{n-1}^{(1)}(1)+G_{n-1}^{(1)}(v)+\tfrac{1}{1-v}(vG_{n-1}^{(1)}(v)-v^{n-1}G_{n-1}^{(1)}(1))\\
&\quad +(G_{n-1}^{(1)}(v)-v^{n-2}G_{n-2}^{(1)}(1))+\sum_{i=2}^{k-3}(G_{n-1}^{(i)}(v)-v^{n-2}G_{n-2}^{(i-1)}(1))\\
&= v^{n-1}G_{n-1}^{(1)}(1)+G_{n-1}^{(1)}(v)+\tfrac{1}{1-v}(G_{n-1}^{(1)}(v)-v^{n-1}G_{n-1}^{(1)}(1))\\
&\quad -v^{n-2}G_{n-2}^{(1)}(1)+\sum_{i=2}^{k-3}(G_{n-1}^{(i)}(v)-v^{n-2}G_{n-2}^{(i-1)}(1))\\
&= \tfrac{1}{1-v}(G_{n-1}^{(1)}(v)-v^{n-3}G_{n-1}^{(1)}(1))+G_{n-1}^{(1)}(v)+v^{n-3}(1+v+v^2)G_{n-1}^{(1)}(1)\\
&\quad -v^{n-2}G_{n-2}^{(1)}(1)+\sum_{i=2}^{k-3}(G_{n-1}^{(i)}(v)-v^{n-2}G_{n-2}^{(i-1)}(1)).
\end{aligned}$$

The case ℓ, where $2\le\ell\le k-3$, can be obtained by using similar arguments as above. ∎

Define $G^{(\ell)}(x,v)=\sum_{n\ge0}G_n^{(\ell)}(v)x^n$. Multiplying the equations in the statement of Proposition 9 by x^n and summing over all possibilities $n\ge0$, we obtain the following result.

Theorem 10. *Let $k\ge4$. Then*

$$\mathbf{P}_k(x,v)\begin{bmatrix}G^{(1)}(x,v)\\G^{(2)}(x,v)\\\vdots\\G^{(k-3)}(x,v)\end{bmatrix}=\begin{bmatrix}b_{10}(x,v)\\b_{20}(x,v)\\\vdots\\b_{(k-3)0}(x,v)\end{bmatrix}+\mathbf{Q}_k(x,v)\begin{bmatrix}G^{(1)}(xv,1)\\G^{(2)}(xv,1)\\\vdots\\G^{(k-3)}(xv,1)\end{bmatrix}, \tag{14}$$

where $\mathbf{P}_k(x,v)$ is a $(k-3)\times(k-3)$ matrix

$$\begin{bmatrix}
1-x\frac{2-v}{1-v} & -x & -x & \cdots & -x & -x & -x\\
-x & 1-\frac{x}{1-v} & -x & \cdots & -x & -x & -x\\
0 & -x & 1-\frac{x}{1-v} & \cdots & -x & -x & -x\\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 0 & -x & 1-\frac{x}{1-v}
\end{bmatrix},$$

$\mathbf{Q}_k(x,v)$ is a $(k-3)\times(k-3)$ matrix

$$\begin{bmatrix}
u_0+(\delta_{k,4}-1)x^2 & -x^2 & -x^2 & \cdots & -x^2 & -x^2 & 0\\
x-x^2 & u_2 & -x^2 & \cdots & -x^2 & -x^2 & 0\\
-x^2 & w_3 & u_3 & \cdots & -x^2 & -x^2 & 0\\
0 & -x^2 & w_4 & \ddots & -x^2 & -x^2 & 0\\
 & & \ddots & \ddots & \ddots & & \vdots\\
0 & 0 & 0 & \ddots & w_{k-4} & u_{k-4} & 0\\
0 & 0 & 0 & \cdots & -x^2 & w_{k-3} & -\frac{x}{v^{k-4}(1-v)}
\end{bmatrix},$$

where $u_j = -\dfrac{x}{v^{j-1}(1-v)} - x^2$ *and* $w_j = x + \dfrac{x^2(1-v^{j-2})}{v^{j-2}(1-v)}$, *and*

$$\begin{bmatrix} b_{10}(x,v) \\ b_{20}(x,v) \\ b_{30}(x,v) \\ b_{40}(x,v) \\ \vdots \\ b_{(k-3)0}(x,v) \end{bmatrix} = \begin{bmatrix} 1-x \\ -x \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $\delta_{k,4}$ *equals* 1 *if* $k=4$, *and* 0 *otherwise.*

Now let us study the zeros of the equation $\det(\mathbf{P}_{k-3}(x,v/x)) = 0$. Using induction on n it can be obtained that the determinant of the $n \times n$ matrix

$$\begin{bmatrix} c_1 & a & a & a & \cdots & a & a & a \\ c_2 & b & a & a & \cdots & a & a & a \\ c_3 & a & b & a & \cdots & a & a & a \\ c_4 & 0 & a & b & \cdots & a & a & a \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ c_{n-1} & 0 & 0 & 0 & \cdots & a & b & a \\ c_n & 0 & 0 & 0 & \cdots & 0 & a & b \end{bmatrix}$$

is given by

$$-a\left(c_1\sqrt{a(a-b)}^{\,n-2}U_n(t) + \sum_{j=2}^{n} c_j\left((b-a)^{j-2}\sqrt{a(a-b)}^{\,n-j}U_{n-j}(t)\right)\right),$$

where $t = \frac{\sqrt{a-b}}{2\sqrt{a}}$, and U_m is the m-th Chebyshev polynomial of the second kind. Hence, we can state the following lemma.

Lemma 11. *For all* $k \geq 4$,

$$\det(\mathbf{P}_k(x,v/x)) = 0 \textit{ if and only if } \frac{U_{k-5}\left(\frac{1}{2}\sqrt{\frac{x-v-xv}{x(x-v)}}\right)}{U_{k-3}\left(\frac{1}{2}\sqrt{\frac{x-v-xv}{x(x-v)}}\right)} = \frac{1}{x} - \frac{2x-v}{x-v}.$$

For instance, if we denote the set of the solutions $x(v)$ of the equation $\det(\mathbf{P}_k(x,v/x)) = 0$ by $\mathcal{P}_k$, then for $k = 4,5,6$ Lemma 11 gives Table 4.1.

Now, let us change the variable v to v/x in the statement of Theorem 10. Our algorithm as described in the second section gives explicitly the generating function for the number of permutations which avoid 1234 and $12k\ldots3$ for any given k. For instance, we have the following result.

Theorem 12. (1) *The generating function for the number of permutations in* S_n *that avoid both* 1234 *and* 1243 *is given by*

$$\frac{1}{2}(3 - x - \sqrt{1-6x+x^2}).$$

k	The elements of the set $\mathcal{P}_k$
4	$\frac{1}{4}\left(1+v\pm\sqrt{1-6v+v^2}\right)$
5	$\frac{1}{2(\sqrt{5}+1)}\left(2v-1+\sqrt{5}\pm\sqrt{4v^2-20v+6-2\sqrt{5}(1-2v)}\right)$ $\frac{1}{2(\sqrt{5}-1)}\left(1-2v+\sqrt{5}\pm\sqrt{4v^2-20v+6+2\sqrt{5}(1-2v)}\right)$
6	$\frac{v}{1-v}$, $\frac{1}{2}\left(1\pm\sqrt{1-4v}\right)$, $\frac{1}{6}\left(1+2v\pm\sqrt{4v^2-8v+1}\right)$

TABLE 2. the set $\mathcal{P}_k$ where $k=4,5,6$.

(2) *The generating function for the number of permutations in* S_n *that avoid both* 1234 *and* 12543 *is given by*

$$\frac{1}{4}(x-5)(x^2-x-1)+\frac{1}{4}(1-x)\sqrt{(x^2-3x+1)(x^2-7x+1)} -\frac{(1-\sqrt{5})x^2-x(1-3\sqrt{5})+2}{16}\sqrt{6x^2-20x+4+2\sqrt{5}x(x-2)} -\frac{(1+\sqrt{5})x^2-x(1+3\sqrt{5})+2}{16}\sqrt{6x^2-20x+4-2\sqrt{5}x(x-2)}.$$

(3) *The generating function for the number of permutations in* S_n *that avoid both* 1234 *and* 126543 *is given by*

$$\frac{(1-x)(6x^2-6x+5)+(4-3x^2)\sqrt{4x^2-8x+1}+(2+x^2)\sqrt{1-4x}+(1+x)\sqrt{1-4x}\sqrt{4x^2-8x+1}}{8x^2-16x+5+4(1-x)\sqrt{4x^2-8x+1}+2(1-x)\sqrt{1-4x}+\sqrt{1-4x}\sqrt{4x^2-8x+1}}.$$

4.2. **Avoiding** 1243 **and** $12\dots k$**.** In this subsection we present an exact formula to the generating function for the number of permutations of length n that avoid both 1243 and $12\dots k$. To do that we need the following notations. We denote by $h_n(i_1,i_2,\dots,i_m)$ the number of permutations $\pi\in S_n(1243,12\dots k)$ such that $\pi_1\pi_2\dots\pi_m=i_1i_2\dots i_m$. It is natural to extend h_n to the case $m=0$ by setting $h_n(\emptyset)=\#S_n(1243,12\dots k)$. More generally, we denote by $h_n^{(\ell)}(i_1,i_2,\dots,i_m)$ the number of permutations $\pi\in S_n(1243,12\dots k)$ such that $\pi_1\pi_2\dots\pi_m=i_1i_2\dots i_m$ together with

$$\pi^{-1}_{n+1-\ell}<\pi^{-1}_{n+2-\ell}<\cdots<\pi^{-1}_n. \tag{15}$$

It is natural to extend $h_n^{(\ell)}$ to the case $m=0$ as the number of permutations $\pi\in S_n(1243,12\dots k)$ that satisfy (15).

Lemma 13. *Let* $k\geq 4$, $\ell=1,2,\dots,k-3$, *and* $1\leq j\leq n-\ell$. *Then*

$$\begin{cases} h_n^{(\ell)}(j)=\sum_{i=1}^{j-1}h_{n-1}^{(\ell)}(i)+\sum_{i=\ell-1}^{\min\{k-3,n-j-1\}}h_{n-1}^{(\max\{1,i\})}(j),\\ h_n^{(\ell)}(n-\ell+1)=h_{n-1}^{(\max\{1,\ell-1\})}\\ h_n^{(\ell)}(n-\ell+2)=\cdots=h_n^{(\ell)}(n)=0.\end{cases} \tag{16}$$

Proof. Let π be any permutation of length n that avoid both 1243 and $12\dots k$, start at $\pi_1 = j$ and satisfy (15). Then, $h_n^{(\ell)}(j) = 0$ for all $j = n+2-\ell, n+3-\ell, \dots, n$ and $h_n^{(\ell)}(n+1-\ell) = h_{n-1}^{(\ell-1)}$. For all $j = 1, 2, \dots, n-\ell$,

$$\begin{aligned} h_n^{(\ell)}(j) &= \sum_{i=1}^{j-1} h_n^{(\ell)}(j,i) + \sum_{i=j+1}^{n+2-k} g_n^{(\ell)}(j,i) + \sum_{i=n+3-k}^{n-1} g_n^{(\ell)}(j,i) + g_n^{(\ell)}(j,n) \\ &= \sum_{i=1}^{j-1} h_{n-1}^{(\ell)}(i) + \sum_{i=\ell-1}^{\min\{k-3,n-1-j\}} h_{n-1}^{(\max\{1,i\})}(j). \end{aligned}$$

The last equality materializes from the following facts:

(1) $\pi \in S_n(1243, 12\dots k)$ satisfies (15) with $\pi_1\pi_2 = ji$ where $j > i$ if and only if $\pi_2\pi_3\dots\pi_n$ is a permutation on the letters $1, 2, \dots, j-1, j+1, \dots, n$ that avoid both 1243 and $12\dots k$ and satisfies (15) with $\pi_2 = i$, so in this case there are exactly $h_{n-1}^{(\ell)}(i)$ permutations.

(2) if $\pi \in S_n(1243, 12\dots k)$ satisfies (15) with $\pi_1\pi_2 = ji$ where $j < i < n+3-k$, then π contains $12\dots k$.

(3) $\pi \in S_n(1243, 12\dots k)$ satisfies (15) with $\pi_1\pi_2 = ji$ where $j \le n+2-k < i \le n-\ell+1$ if and only if $\pi_1\pi_3\dots\pi_n$ is a permutation on the letters $1, 2, \dots, i-1, i+1, \dots, n$ that avoid both 1243 and $12\dots k$ and hold $\pi_n^{-1} > \pi_{n-1}^{-1} > \cdots > \pi_{\max\{n+4-k,i+1\}}^{-1}$ and $\pi_1 = j$.

Adding the above disjoint cases, Cases (1)-(3), we get the desired result, as claimed in (16). ∎

Define, $H_n^{(\ell)}(v) = \sum_{t=1}^{n} {}_{n}^{(\ell)}(t)v^{t-1}$ for any $n \ge 1$ and $\ell = 1, 2, \dots, k-3$. It is natural to extend $H_n^{(\ell)}$ to the case $n = 0$ by setting $H_0^{(\ell)}(v) = 1$. If multiplying $h_n^{(\ell)}(j)$ by v^{j-1} and summing over all the possibilities of $j = 1, 2, \dots, n$, then Lemma 8 gives the following result.

Proposition 14. *For all $n \ge 3$, we have*

$$\begin{aligned} H_n^{(1)}(v) = \frac{v}{1-v}\left[H_{n-1}^{(1)}(v) - v^{n-2}H_{n-1}^{(1)}(1)\right] + v^{n-1}H_{n-1}^{(1)}(1) + H_{n-1}^{(1)}(v) \\ + \sum_{i=1}^{k-3}\left(H_{n-1}^{(i)}(v) - v^{n-i-1}H_{n-2}^{(\max\{1,i-1\})}(1)\right). \end{aligned}$$

and for $\ell > 1$,

$$\begin{aligned} H_n^{(\ell)}(v) = \frac{v}{1-v}\left[H_{n-1}^{(\ell)}(v) - v^{n-\ell-1}H_{n-1}^{(\ell)}(1)\right] + v^{n-\ell}H_{n-1}^{(\ell-1)}(1) \\ + \sum_{i=\ell-1}^{k-3}\left(H_{n-1}^{(i)}(v) - v^{n-i-1}H_{n-2}^{(\max\{1,i-1\})}(1)\right). \end{aligned}$$

Define $H^{(\ell)}(x,v) = \sum_{n\ge0} H_n^{(\ell)}(v)x^n$. Multiplying the equation in the statement of Proposition 9 by x^n and summing over all possibilities $n \ge 0$ we obtain the following result.

Theorem 15. *Let $k \ge 4$. Then*

$$\mathbf{U}_k(x,v)\begin{bmatrix} H^{(1)}(x,v) \\ H^{(2)}(x,v) \\ \vdots \\ H^{(k-3)}(x,v) \end{bmatrix} = \begin{bmatrix} w_{10}(x,v) \\ w_{20}(x,v) \\ \vdots \\ w_{(k-3)0}(x,v) \end{bmatrix} + \mathbf{W}_k(x,v)\begin{bmatrix} H^{(1)}(xv,1) \\ H^{(2)}(xv,1) \\ \vdots \\ H^{(k-3)}(xv,1) \end{bmatrix}, \qquad (17)$$

where $\mathbf{U}_k(x,v)$ *is a* $(k-3)\times(k-3)$ *matrix*

$$\begin{bmatrix} 1-x\frac{2-v}{1-v} & -x & -x & \cdots & -x & -x & -x \\ -x & 1-\frac{x}{1-v} & -x & \cdots & -x & -x & -x \\ 0 & -x & 1-\frac{x}{1-v} & \cdots & -x & -x & -x \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -x & 1-\frac{x}{1-v} \end{bmatrix},$$

$\mathbf{W}_k(x,v)$ *is a* $(k-3)\times(k-3)$ *matrix*

$$\begin{bmatrix} u_0-(1-\delta_{4,k})\frac{x^2}{v} & -\frac{x^2}{v^2} & -\frac{x^2}{v^3} & \cdots & -\frac{x^2}{v^{k-4}} & 0 \\ w_1-x^2 & u_2 & -\frac{x^2}{v^3} & \cdots & -\frac{x^2}{v^{k-4}} & 0 \\ -\frac{x^2}{v} & w_2 & u_3 & \vdots & \vdots & 0 \\ 0 & \ddots & \ddots & \ddots & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & -\frac{x^2}{v^{k-6}} & w_{k-5} & u_{k-4} & 0 \\ 0 & \cdots & 0 & -\frac{x^2}{v^{k-5}} & w_{k-4} & -\frac{x}{v^{k-4}(1-v)} \end{bmatrix},$$

where

$$u_i = -\frac{x}{v^{i-1}(1-v)} - \frac{x^2}{v^i}, \quad w_i = \frac{x}{v^i} - \frac{x^2}{v^i},$$

and
$$\begin{bmatrix} w_{10}(x,v) \\ w_{20}(x,v) \\ w_{30}(x,v) \\ \vdots \\ w_{(k-3)0}(x,v) \end{bmatrix} = \begin{bmatrix} 1-x+(1-\delta_{4,k})\frac{x^2}{v} \\ -x+x(1-x)/v \\ x^2/v \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Now, let us change the variable x to x/v in the statement of Theorem 15. Our method as described in the second section gives explicitly the generating function for the number of permutations which avoid 1243 and $12\dots k$ for any given k. For instance, we have the following result.

Theorem 16. (1) *The generating function for the number of permutations in* S_n *that avoid both* 1243 *and* 1234 *is given by*

$$\frac{1}{2}(3-x-\sqrt{1-6x+x^2}).$$

(2) *The generating function for the number of permutations in* S_n *that avoid both* 1243 *and* 12345 *is given by*

$$\frac{1}{4}(x-5)(x^2-x-1)+\frac{1}{4}(1-x)\sqrt{(x^2-3x+1)(x^2-7x+1)}$$
$$-\frac{(1-\sqrt{5})x^2-x(1-3\sqrt{5})+2}{16}\sqrt{6x^2-20x+4+2\sqrt{5}x(x-2)}$$
$$-\frac{(1+\sqrt{5})x^2-x(1+3\sqrt{5})+2}{16}\sqrt{6x^2-20x+4-2\sqrt{5}x(x-2)}.$$

(3) *The generating function for the number of permutations in* S_n *that avoid both* 1243 *and* 123456 *is given by*

$$\frac{(1-x)(6x^2-6x+5)+(4-3x^2)\sqrt{4x^2-8x+1}+(2+x^2)\sqrt{1-4x}+(1+x)\sqrt{1-4x}\sqrt{4x^2-8x+1}}{8x^2-16x+5+4(1-x)\sqrt{4x^2-8x+1}+2(1-x)\sqrt{1-4x}+\sqrt{1-4x}\sqrt{4x^2-8x+1}}.$$

We remark that Stankova [15] mentioned that she found a direct bijection between the set of permutations of length n that avoid both 1243 and $123\dots k$ and the set of permutations of length n that avoid both 1234 and $12k\dots 3$, which will appear in a forthcoming paper.

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