MORE ON THE BEST UPPER BOUND FOR THE RANDIĆ INDEX R_{-1} OF TREES

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Abstract

The Randić index $R_{-1}(G)$ of a graph G is defined as the sum of the $(d(u)d(v))^{-1}$ of all edges (uv) of G, where $d(u)$ denotes the degree of a vertex u in G. Denote by T_n^t a tree with n vertices and $n-1 \equiv t$ (mod 7). When $n-1 \equiv 0$ (mod 7) a tree with maximum Randić index has been found in [11]. In this paper we found trees of every order $n \geq 720$ and $t = 1, 2, \cdots, 6$ for which the Randić index achieves its maximum value. The structure of the Max Tree - the tree with maximum Randić index R_{-1}, is a little bit unexpected. It means that it differs from numerical results obtained by Hu, Jin, Li and Wang, but for $t = 1$, $t = 3$ and $t = 5$ we confirm their conjecture. The Max Tree has one vertex of the maximum degree and all adjacent vertices, except at most three, are center of $(3, 4)$ systems. The remainder vertices can be the centers of $(2, 3)$, $(4, 5)$, $(1, 5)^*$ or $(8_{2,3}, 9)$ systems.

1. INTRODUCTION

Let T_n^t be a tree with order $n = \mid V(T) \mid$ and $n-1 \equiv t \ (mod\ 7)$. The degree $d_T(u)$ of a vertex u is the number of vertices in T adjacent to u. A vertex of degree 1 in a tree is called a *leaf*. A *suspended path* from x to z is a path x, y, z with $d(x) = 1,\ d(y) = 2$ and $d(z) \geq 3$. Let $x_1y_1z, \cdots, x_sy_sz$ be s distinct suspended paths adjacent to z, and $w_1, \cdots, w_{d-s}$ be the vertices of T, other than $y_1, \cdots, y_s$, adjacent to z, then we call such system an (s, d) *system* centered at z. We denote by $(1, 5)^*$ a system which has one suspended path x, y, z, and three (2,3) systems adjacent to z, where $d(z) = 5$. System which has 8 (2,3) systems adjacent to one vertex z, where $d(z) = 9$, is denoted by $(8_{(2,3)}, 9)$. A Max Tree is a tree with maximum value of the Randić index R_{-1} for a given order n.

In 1975 Randić [12] proposed two topological indices $R_{-1/2}(G)$ and $R_{-1}(G)$, suitable for measuring the extent of branching of the carbon-atom skeleton of saturated hydrocarbons. The general Randić index $R_\alpha(G)$ of a graph G is defined [4] by $R_\alpha(G) = \sum_{(uv)\in E(G)}(d(u)d(v))^\alpha$, where the summation extends over all edges (uv) of G and $d(u)$ denotes the degree of a vertex u. Randić himself demonstrated [12] that his indices are well correlated with a variety of physico-chemical properties. They have attracted considerable attention of chemists and mathematicians ([1-11]).

In [4] Clark and Moon gave a lower and upper bound for $R_{-1}(T)$ for trees, $1 \leq R_{-1}(T) \leq \frac{15n+8}{18}$. They asked for $K = \lim_{n\to\infty} \frac{f(n)}{n}$, where $f(n)$ is the maximum value of $R_{-1}(T)$ among all trees of order n.

Rautenbach [13] gave an upper bound for $R_{-1}(T)$ of trees with maximum degree 3. Li and Yang [9] used linear programming to determine the sharp upper bound for $R_{-1}(T)$ of chemical trees (i.e., trees with maximum degree at most 4). Hu, Li and Yuan [7] investigated trees with maximum general Randić index $R_\alpha(T)$ among all trees of order n. They distinguished α in several different intervals and for most of the intervals characterized trees with maximum $R_\alpha(T)$. The same authors Hu, Li and Yuan [8] tried to find sharp upper bound for $R_{-1}(T)$ of trees and K. They partially succeeded in it, but they made some errors which have been found in [10] and corrected in [11]. Hu, Jin, Li and Wang [5] determined the maximum value for R_{-1} of all trees of order $n \leq 102$ and gave one of the trees with maximum value of this index. They gave a conjecture about the structure of the Max Trees for $n \geq 103$. In [11] the authors proved that $R_{-1}(T_n^0) \leq \frac{15n-1}{56}$ for all trees of order $n \geq 103$ and found a Max Tree - it has only one vertex of the maximum degree and every adjacent vertex is the center of a $(3, 4)$ system.

In this paper we found a Max Tree when $n-1 \neq 0\ (mod\ 7)$. The Max Tree has one vertex of the maximum degree and every adjacent vertex, except at most three, is the center of a $(3,4)$ system. Each of the remainder vertices can be the center of a $(2,3)$, $(4,5)$, $(1,5)^*$ or $(8_{2,3},9)$ system. This confirms a conjecture in [5] for $t=1,3$ and 5, but for $t=2,4$ and 6 it differs from the conjecture.

2. THE MAIN RESULT

Theorem 1. *For a tree T_n^t of order $n \geq n_0$ ($n_0 = 720$),*

$$R^*_{-1}(T_n^t) = \begin{cases} \frac{15n-1}{56}, & t=0 \\ \frac{15n-1}{56} - \frac{1}{56} + \frac{7}{4(n+5)}, & t=1 \\ \frac{15n-1}{56} - \frac{3}{5}\cdot\frac{1}{56} - \frac{7}{20(n-3)}, & t=2 \\ \frac{15n-1}{56} - \frac{2}{3}\cdot\frac{1}{56} + \frac{7}{6(n+3)}, & t=3 \\ \frac{15n-1}{56} - \frac{6}{5}\cdot\frac{1}{56} - \frac{7}{20(n-12)}, & t=4 \\ \frac{15n-1}{56} - \frac{1}{3}\cdot\frac{1}{56} + \frac{7}{12(n+1)}, & t=5 \\ \frac{15n-1}{56} - \frac{29}{27}\cdot\frac{1}{56} - \frac{35}{36(n-35)}, & t=6 \end{cases}$$

where $R^*_{-1}(T_n^t)$ is the maximum value of the Randić index.

We use mathematical induction throughout this paper, i.e., we suppose that Theorem 1 holds for all trees of order less than $n > n_0$. We know that $R^*_{-1}(T_n^0) = \frac{15n-1}{56}$ and $R_{-1}(T) \leq \frac{15n-1}{56}$ for $n \geq 92$ [11]. We will prove Theorem 1 at the end of this paper. At first we will describe some properties of the Max Tree.

Denote by $\triangle(n,k) = \max_{1\leq t\leq 6}\{R^*_{-1}(T_{n-k}^{t-k}) - R^*_{-1}(T_n^t)\} = R^*_{-1}(T_{n-k}^0) - R^*_{-1}(T_n^k)$, where $k=1,2,3$. For example, $\triangle(n,1) = R^*_{-1}(T_{n-1}^0) - R^*_{-1}(T_n^1) < -\frac{15}{56} + \frac{1}{56} = -\frac{1}{4}$.

Theorem 2. *Every leaf x in the Max Tree is on a path x,y,z with $d(y)=2$ and $d(z) \geq 3$.*

Proof. Let $x_1, \cdots, x_l$ ($l \geq 1$) be the leaves of T, adjacent to a vertex y and $z_1, \cdots, z_m$ be the other vertices of T adjacent y, where $l+m$ is as large as possible. We have $m \geq 1$ and all $d(z_j) \geq 2$. Then

$$\begin{aligned} R_{-1}(T_n^t) &= R_{-1}(T_n^t - \{x_1\}) + \frac{1}{l+m} + \left(\frac{1}{l+m} - \frac{1}{l+m-1}\right)(l-1) + \\ &\left(\frac{1}{l+m} - \frac{1}{l+m-1}\right)\sum_{j=1}^{m}\frac{1}{d(z_j)} \leq R^*_{-1}(T_n^t) + R^*_{-1}(T_{n-1}^{t-1}) - R^*_{-1}(T_n^t) + \\ &\frac{m}{(l+m)(l+m-1)} - \frac{1}{(l+m)(l+m-1)}\sum_{j=1}^{m}\frac{1}{d(z_j)} \leq R^*_{-1}(T_n^t) + \triangle(n,1) \end{aligned}$$

$$+\frac{m}{(l+m)(l+m-1)}-\frac{1}{(l+m)(l+m-1)}\sum_{j=1}^{m}\frac{1}{d(z_j)} < R^*_{-1}(T^t_n)-\frac{1}{4}+$$
$$\frac{m}{(l+m)(l+m-1)}-\frac{1}{(l+m)(l+m-1)}\sum_{j=1}^{m}\frac{1}{d(z_j)} \qquad (1)$$
$$\leq R^*_{-1}(T^t_n)$$

provided $l+m \geq 4,\ (l \geq 1)$ and hence $l+m \leq 3$. Suppose $l=2, m=1$, then

$$(1) \qquad = R^*_{-1}(T^t_n)-\frac{1}{4}+\frac{1}{6}-\frac{1}{6d(z_1)} < R^*_{-1}(T^t_n)$$

Suppose $l=1, m=2$.

$$R_{-1}(T^t_n) = R_{-1}(T^t_n-\{x_1,y\}+z_1z_2)+\frac{1}{3}+\frac{1}{3d(z_1)}+\frac{1}{3d(z_2)}-\frac{1}{d(z_1)d(z_2)} \leq$$
$$R^*_{-1}(T^t_n)+\triangle(n,2)+\frac{1}{3}+\frac{1}{3d(z_1)}+\frac{1}{3d(z_2)}-\frac{1}{d(z_1)d(z_2)} <$$
$$R^*_{-1}(T^t_n)$$

where $\triangle(n,2)=R^*_{-1}(T^{2-2}_{n-2})-R^*_{-1}(T^2_n)=-\frac{30}{56}+\frac{3}{280}+\frac{7}{20(n-3)} < -\frac{30}{56}+\frac{1}{56}=-\frac{29}{56}$ for $n>52$.

Hence, $l=m=1$. If $d(z_1)=2$, then

$$(1) \quad = \quad R^*_{-1}(T^t_n)-\frac{1}{4}+\frac{1}{2}-\frac{1}{4}=R^*_{-1}(T^t_n) \qquad \square$$

We have shown that every leaf x in the Max Tree is on a path x,y,z with $d(y)=2$ and $d(z)\geq 3$. We call x,y,z a suspended path from x to z. An (s,d) *system* centered at z has s distinct suspended paths $x_1y_1z,\cdots,x_sy_sz$ adjacent to z and $w_1,\cdots,w_{d-s}$ vertices of T, other than $y_1,\cdots,y_s$ adjacent to z. We denote by $(1,5)^*$ a system which has one suspended path x,y,z, and three (2,3) systems adjacent to z, where $d(z)=5$. System which has 8 (2,3) systems adjacent to one vertex z, where $d(z)=9$, is denoted by $(8_{(2,3)},9)$. The proof of the next lemmas is omitted because it is easy.

Lemma 1. Every vertex of degree 2 can only be on a suspended path.

Lemma 2. Every vertex of degree 3 is the center of a $(2,3)$ system of the Max Tree, i.e., it appears only in a $(2,3)$ system.

Theorem 3. *The Max Trees could have* $(2,3),\ (3,4),\ (4,5)$ *and* $(1,5)^*$ *systems.*

Proof. Let $x_1y_1z,\cdots,x_sy_sz$ be s distinct suspended paths adjacent to z and $w_1,\cdots,w_{d-s}$ $(d\geq s+1)$ be the vertices of T, other than $y_1,\cdots,y_s$, adjacent to z. We distinguish several cases.

Case 1. $s \geq 5$. By deleting the vertices $x_1, y_1, \cdots, x_5, y_5$ and adding two (2,3) systems adjacent to z, we get a new tree T'. Then $\mid V(T') \mid = n,\ t' = t$ and

$$\begin{aligned} R_{-1}(T_n^t) &= R_{-1}(T_n^{'t}) + s(\frac{1}{2} + \frac{1}{2d}) + (\frac{1}{d} - \frac{1}{d-3})\sum_{j=1}^{d-s}\frac{1}{d(w_j)} - 4(\frac{1}{2} + \frac{1}{6}) - \frac{2}{3(d-3)} \\ - & (s-5)(\frac{1}{2} + \frac{1}{2(d-3)}) = R_{-1}(T_n^{'t}) - \frac{d^2 - 14d + 9s}{6d(d-3)} - \frac{3}{d(d-3)}\sum_{j=1}^{d-s}\frac{1}{d(w_j)} \\ < & R_{-1}(T_n^{'t}) \end{aligned} \quad (2)$$

for $s \geq 6$ and $d \geq s+1$.

If $s = 5$, then $-d^2 + 14d - 45 \leq 0$ for $d \geq 9$ and $R_{-1}(T_n^t) < R_{-1}(T_n^{'t})$. When $s = 5$ and all $d(w_j) \leq 4$, then $\sum_{j=1}^{d-5} \frac{1}{d(w_j)} \geq \frac{d-5}{4}$ and

$$(2) = R_{-1}(T_n^{'t}) - \frac{d^2 - 14d + 45}{6d(d-3)} - \frac{3(d-5)}{4d(d-3)} = R_{-1}(T_n^{'t}) - \frac{2d^2 - 19d + 45}{12d(d-3)} < R_{-1}(T_n^{'t})$$

for $d \geq 6$. Hence if $s = 5$ and $d = 6, 7$ and 8, there must exist a vertex $\tilde{w}$ with $d(\tilde{w}) = \tilde{d} \geq 5$. Let v_j denote the neighbors of $\tilde{w}$ other than z. In this case by deleting the vertices $x_1, y_1, \cdots, x_5, y_5, z$ and adding two $(2,3)$ systems adjacent to $\tilde{w}$ and $d-6$ edges $\tilde{w}w_1, \cdots, \tilde{w}w_{d-6}$ we get a new tree T'. Then $\mid V(T') \mid = n-1, d_{T'}(\tilde{w}) = \tilde{d} + d - 5$ and

$$\begin{aligned} R_{-1}(T_n^t) &= R_{-1}(T_{n-1}^{'t-1}) + 5(\frac{1}{2} + \frac{1}{2d}) + \frac{1}{\tilde{d}d} + (\frac{1}{d} - \frac{1}{\tilde{d} + d - 5})\sum_{j=1}^{d-6}\frac{1}{d(w_j)} + \\ & (\frac{1}{\tilde{d}} - \frac{1}{\tilde{d} + d - 5})\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} - 4(\frac{1}{2} + \frac{1}{6}) - \frac{2}{3(\tilde{d} + d - 5)} \leq R_{-1}^*(T_n^t) + \\ & \triangle(n,1) - \frac{1}{6} + \frac{5}{2d} + \frac{1}{\tilde{d}d} - \frac{2}{3(\tilde{d} + d - 5)} + \frac{\tilde{d} - 5}{d(\tilde{d} + d - 5)}\sum_{j=1}^{d-6}\frac{1}{d(w_j)} + \\ & \frac{d-5}{\tilde{d}(\tilde{d} + d - 5)}\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \end{aligned}$$

Since $\triangle(n,1) < -\frac{1}{4}$, $\sum_{j=1}^{d-6}\frac{1}{d(w_j)} \leq \frac{d-6}{3}$ $(d(w_j) \geq 3)$ and $\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \leq \frac{\tilde{d}-1}{2}$, we have

$$\begin{aligned} R_{-1}(T_n^t) &\leq R_{-1}^*(T_n^t) - \frac{5}{12} + \frac{5}{2d} + \frac{1}{\tilde{d}d} - \frac{2}{3(\tilde{d} + d - 5)} + \frac{\tilde{d} - 5}{d(\tilde{d} + d - 5)} \cdot \frac{d-6}{3} + \\ & \frac{d-5}{\tilde{d}(\tilde{d} + d - 5)} \cdot \frac{\tilde{d} - 1}{2} < R_{-1}^*(T_n^t) \end{aligned}$$

for $d = 6, 7, 8$ and every $\tilde{d} \geq 5$.

Case 2. $s = 4$. At first, we will show that between vertices w_j there must exist one vertex $\tilde{w}$, such that $d(\tilde{w}) = \tilde{d} \geq 5$. By deleting two vertices x_1 and y_1 we get a new tree T'. Then $\mid V(T') \mid = n-2,\ t' = t-2$ and

$$\begin{aligned} R_{-1}(T_n^t) &= R_{-1}(T'^{t-2}_{n-2}) + \frac{1}{2} + \frac{1}{2d} + (s-1)(\frac{1}{2d} - \frac{1}{2(d-1)}) + (\frac{1}{d} - \frac{1}{d-1})\sum_{j=1}^{d-s}\frac{1}{d(w_j)} \\ &\leq R^*_{-1}(T_n^t) + \triangle(n,2) + \frac{1}{2} + \frac{d-s}{2d(d-1)} - \frac{1}{d(d-1)}\sum_{j=1}^{d-s}\frac{1}{d(w_j)} \end{aligned}$$

Since $\triangle(n,2) = -\frac{30}{56} + \frac{3}{280} + \frac{7}{20(n-3)} \leq -\frac{30}{56} + \frac{4}{280}$, for $n \geq 101$, we have

$$R_{-1}(T_n^t) \leq R^*_{-1}(T_n^t) - \frac{6}{280} + \frac{d-s}{2d(d-1)} - \frac{1}{d(d-1)}\sum_{j=1}^{d-s}\frac{1}{d(w_j)} \quad (3)$$

$$< R^*_{-1}(T_n^t) + \frac{-3d^2 + 73d - 70s}{140d(d-1)} \quad (4)$$

When $s = 4$, then $-3d^2 + 73d - 280 \leq 0$ for $d \geq 20$. Hence $5 \leq d \leq 19$. If all $d(w_j) \leq 4$, $\frac{1}{d(d-1)}\sum_{j=1}^{d-s}\frac{1}{d(w_j)} \geq \frac{d-s}{4d(d-1)}$ and for $s = 4$, we get

$$R_{-1}(T_n^t) \leq R^*_{-1}(T_n^t) + \frac{-6d^2 + 76d - 280}{280d(d-1)} < R^*_{-1}(T_n^t)$$

There must exist a vertex $\tilde{w}$ such that $d(\tilde{w}) = \tilde{d} \geq 5$ for $5 \leq d \leq 19$. Let v_j denote the neighbors of $\tilde{w}$ other than z. By deleting $2s+1$ vertices $x_1, y_1, \cdots, x_s, y_s, z$ and adding one $(s, s+1)$ system adjacent to $\tilde{w}$ and $d-s-1$ edges $\tilde{w}w_1, \cdots, \tilde{w}w_{d-s-1}$ we get a new tree T'. Then $\mid V(T')| = n,\ d_{T'}(\tilde{w}) = \tilde{d} + d - s - 1$ and

$$\begin{aligned} R_{-1}(T_n^t) &= R_{-1}(T'^t_n) + \frac{s}{2} + \frac{s}{2d} + \frac{1}{\tilde{d}d} + (\frac{1}{d} - \frac{1}{\tilde{d}+d-s-1})\sum_{j=1}^{d-s-1}\frac{1}{d(w_j)} + \\ &\quad (\frac{1}{\tilde{d}} - \frac{1}{\tilde{d}+d-s-1})\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} - \frac{s}{2} - \frac{s}{2(s+1)} - \frac{1}{(s+1)(\tilde{d}+d-s-1)} \\ &= R_{-1}(T'^t_n) + \frac{s}{2d} - \frac{s}{2(s+1)} + \frac{1}{\tilde{d}d} - \frac{1}{(s+1)(\tilde{d}+d-s-1)} + \\ &\quad \frac{\tilde{d}-s-1}{d(\tilde{d}+d-s-1)}\sum_{j=1}^{d-s-1}\frac{1}{d(w_j)} + \frac{d-s-1}{\tilde{d}(\tilde{d}+d-s-1)}\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \end{aligned}$$

Since the Max Tree can have only the systems $(1,\tilde{d}), (2,\tilde{d}), \cdots, (s,\tilde{d})$, the vertex $\tilde{w}$ can have at most s suspended path (i.e., $d(v_j) = 2$ for at most s "j"). After replacing $\sum_{j=1}^{d-s-1}\frac{1}{d(w_j)} \leq \frac{d-s-1}{3}$ and $\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \leq \frac{s}{2} + \frac{\tilde{d}-s-1}{3} = \frac{2\tilde{d}+s-2}{6}$, we get

$$\begin{aligned} R_{-1}(T_n^t) &\leq R_{-1}(T'^t_n) + \frac{(d-s-1)(\tilde{d}-s-1)((2-s)(\tilde{d}+d)-6)}{6\tilde{d}d(s+1)(\tilde{d}+d-s-1)} \\ &\leq R_{-1}(T'^t_n) \end{aligned} \quad (5)$$

for $d \geq s+1,\ \tilde{d} \geq s+1$ and for $s \geq 2$. If we use less sharp $\sum_{j=1}^{\tilde{d}-1} \frac{1}{d(v_j)} \leq \frac{s-1}{2} + \frac{\tilde{d}-s}{3} = \frac{2\tilde{d}+s-3}{6} < \frac{2\tilde{d}+s-2}{6}$, we get

$$R_{-1}(T_n^t) \leq R_{-1}(T_n^{'t}) + \frac{(d-s-1)(\tilde{d}-s-1)((2-s)(\tilde{d}+d)-6)}{6\tilde{d}d(s+1)(\tilde{d}+d-s-1)} - \frac{d-s-1}{6\tilde{d}(\tilde{d}+d-s-1)} \tag{6}$$

When $s = 4$, $R_{-1}(T_n^t) < R_{-1}(T_n^{'t})$ for $d \geq 6$ and $\tilde{d} \geq 6$ because of Eq.(5) and because the degree of the vertex $\tilde{w}$ in T' (which is, eventually, now the center of a $(4, \tilde{d}+d-5)$ system) is augmented. If $d \geq 6$, inequality holds for $\tilde{d} = 5$ when $\tilde{w}$ is not the center of a $(4, 5)$ system, because of Eq.(6).

Hence, we have to consider the case $d \geq 6$ and $\tilde{w}$ is the center of a $(4, 5)$ system. By deleting 8 vertices $x_1, y_1, \cdots, x_4, y_4$ and 9 vertices of the (4,5) system centered at $\tilde{w}$ and adding two (2,3) systems and one (3,4) system adjacent to vertex z, we get a new tree T'. Then $\mid V(T') \mid= n$ and

$$\begin{aligned} R_{-1}(T_n^t) &= R_{-1}(T_n^{'t}) + 4(\frac{1}{2} + \frac{1}{2d}) + 4(\frac{1}{2} + \frac{1}{10}) + \frac{1}{5d} + (\frac{1}{d} - \frac{1}{d-2})\sum_{j=1}^{d-5} \frac{1}{d(w_j)} - \\ &\quad 4(\frac{1}{2} + \frac{1}{6}) - 3(\frac{1}{2} + \frac{1}{8}) - \frac{2}{3(d-2)} - \frac{1}{4(d-2)} = R_{-1}(T_n^{'t}) \\ &\quad - \frac{17d^2 - 188d + 528}{120d(d-2)} - \frac{2}{d(d-2)} \sum_{j=1}^{d-5} \frac{1}{d(w_j)} < R_{-1}(T_n^{'t}) \end{aligned}$$

Last inequality holds for all $d \geq 6$.

In such a way the degree of the vertex which is the center of a (4,d) system would be augmented up to 20 or the tree T' will not have $(4, d),\ d \geq 6$ systems except (4,5) systems.

Case 3. $s = 3$. From (4) we have

$$R_{-1}(T_n^t) < R_{-1}^*(T_n^t) - \frac{3d^2 - 73d + 210}{140d(d-1)} \leq R_{-1}^*(T_n^t)$$

for $d \geq 21$. Hence $4 \leq d \leq 20$. Since $n \geq n_0$ for $4 \leq d \leq 20$, there must exist one vertex $\tilde{w}$ such that $\tilde{d} \geq 5$ or $\tilde{d} = 4$ but it is not the center of a (4,5) or (3,4) system. If $d \geq 5$, $\tilde{d} \geq 5$ and $\tilde{w}$ is not the center of a $(4, 5)$ system we get from Eq.(5) that $R_{-1}(T_n^t) < R_{-1}(T_n^{'t})$ and the degree of the vertex $\tilde{w}$ in T' (which is, eventually, now the center of a $(3, \tilde{d} + d - 4)$ system) is augmented. If $d \geq 5$, $\tilde{d} = 4$ and $\tilde{w}$ is not the center of a (3,4) system, we get from Eq.(6) that $R_{-1}(T_n^t) < R_{-1}(T_n^{'t})$. In such a way the degree of the vertex which is the center of a (3,d) system would be augmented up to 21 or the tree T' will not have $(3, d),\ d \geq 5$ systems except (3,4) systems.

Case 4. $s = 2$. From (4) we have

$$R_{-1}(T_n^t) < R_{-1}^*(T_n^t) - \frac{3d^2 - 73d + 140}{140d(d-1)} \leq R_{-1}^*(T_n^t)$$

for $d \geq 23$. Hence $3 \leq d \leq 22$. Since $n \geq n_0$ for $3 \leq d \leq 22$ there must exist one vertex $\tilde{w}$ such that $\tilde{d} \geq 5$, but it is not the center of a $(4,5)$ system or $\tilde{d} = 4$, but it is not the center of a (3,4) system. For $d \geq 4$ in both cases we get from Eq.(5) that $R_{-1}(T_n^t) < R_{-1}(T_n'^t)$ and the degree of the vertex $\tilde{w}$ (which is, eventually, now the center of a $(2, \tilde{d} + d - 3)$ system) in T' will be augmented or T' will not have $(2, d)$ systems except (2,3) systems.

Case 5. $s = 1$. From (4) we have

$$R_{-1}(T_n^t) < R_{-1}^*(T_n^t) - \frac{3d^2 - 73d + 70}{140d(d-1)} \leq R_{-1}^*(T_n^t)$$

for $d \geq 24$. Hence, $4 \leq d \leq 23$ because we proved before (Lemma 2) that the presence of a $(1, 3)$ system is not possible. We differ several cases.

Case 5.1. Among the vertices w_j there is one $\tilde{w}$ which has only one suspended path of length 2 $x_2 y_2 \tilde{w}$. By deleting 5 vertices x_1, y_1, z, x_2, y_2 and adding one $(2, 3)$ system adjacent to $\tilde{w}$ and edges $\tilde{w}w_1, \cdots, \tilde{w}w_{d-2}$ we get a new tree T'. Then $\mid V(T') \mid = n,\ d_{T'}(\tilde{w}) = \tilde{d} + d - 3$ and

$$\begin{aligned} R_{-1}(T_n^t) &= R_{-1}(T_n'^t) + \frac{1}{2} + \frac{1}{2d} + \frac{1}{2} + \frac{1}{2\tilde{d}} - 1 - \frac{1}{3} + (\frac{1}{d} - \frac{1}{\tilde{d} + d - 3}) \sum_{j=1}^{d-2} \frac{1}{d(w_j)} \\ &+ (\frac{1}{\tilde{d}} - \frac{1}{\tilde{d} + d - 3}) \sum_{j=1}^{\tilde{d}-2} \frac{1}{d(v_j)} + \frac{1}{\tilde{d}d} - \frac{1}{3(\tilde{d} + d - 3)} = R_{-1}(T_n'^t) - \frac{1}{3} + \frac{1}{2d} + \\ &\frac{1}{2\tilde{d}} + \frac{1}{\tilde{d}d} - \frac{1}{3(\tilde{d} + d - 3)} + \frac{\tilde{d} - 3}{d(\tilde{d} + d - 3)} \sum_{j=1}^{d-2} \frac{1}{d(w_j)} + \frac{d - 3}{\tilde{d}(\tilde{d} + d - 3)} \sum_{j=1}^{\tilde{d}-2} \frac{1}{d(v_j)} \end{aligned}$$

Since $\sum_{j=1}^{d-2} \frac{1}{d(w_j)} \leq \frac{d-2}{3}$ and $\sum_{j=1}^{\tilde{d}-2} \frac{1}{d(v_j)} \leq \frac{\tilde{d}-2}{3}$, we have

$$\begin{aligned} R_{-1}(T_n^t) &\leq R_{-1}(T_n'^t) - \frac{1}{3} + \frac{1}{2d} + \frac{1}{2\tilde{d}} + \frac{1}{\tilde{d}d} - \frac{1}{3(\tilde{d} + d - 3)} + \frac{(\tilde{d} - 3)(d - 2)}{3d(\tilde{d} + d - 3)} + \\ &\frac{(d - 3)(\tilde{d} - 2)}{3\tilde{d}(\tilde{d} + d - 3)} = R_{-1}(T_n'^t) - \frac{\tilde{d}^2 + (2d - 9)\tilde{d} + d^2 - 9d + 18}{6\tilde{d}d(\tilde{d} + d - 3)} < R_{-1}(T_n'^t) \end{aligned}$$

for $4 \leq d \leq 23$ and $\tilde{d} \geq 4$.

Case 5.2. $d(w_j) \geq 4,\ j = 1, 2, \cdots, d - 1$. It is clear $(d \leq 23,\ n \geq n_0)$ that there must exist a vertex $\tilde{w}$ such that $\tilde{d} \geq 4$, which is not the center of a $(3, 4)$ or $(4, 5)$

system. By deleting vertices x_1, y_1 and z and adding edges $\tilde{w}w_1, \cdots, \tilde{w}w_{d-2}$ we get a new tree T'. Then $\mid V(T') \mid = n-3,\ d_{T'}(\tilde{w}) = \tilde{d}+d-3$ and

$$\begin{aligned} R_{-1}(T_n^t) &= R_{-1}(T'^{t-3}_{n-3}) + \frac{1}{2} + \frac{1}{2d} + \frac{1}{\tilde{d}d} + (\frac{1}{d} - \frac{1}{\tilde{d}+d-3})\sum_{j=1}^{d-2}\frac{1}{d(w_j)} + \\ &(\frac{1}{\tilde{d}} - \frac{1}{\tilde{d}+d-3})\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \le R^*_{-1}(T_n^t) + \triangle(n,3) + \frac{1}{2} + \frac{1}{2d} + \frac{1}{\tilde{d}d} + \\ &\frac{\tilde{d}-3}{d(\tilde{d}+d-3)}\sum_{j=1}^{d-2}\frac{1}{d(w_j)} + \frac{d-3}{\tilde{d}(\tilde{d}+d-3)}\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \end{aligned}$$

Since $\sum_{j=1}^{d-2}\frac{1}{d(w_j)} \le \frac{d-2}{4}$, $\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \le \frac{\tilde{d}-1}{3}$ and $\triangle(n,3) < -\frac{45}{56} + \frac{1}{84}$, we have

$$\begin{aligned} R_{-1}(T_n^t) &< R^*_{-1}(T_n^t) - \frac{7}{24} + \frac{1}{2d} + \frac{1}{\tilde{d}d} + \frac{(\tilde{d}-3)(d-2)}{4d(\tilde{d}+d-3)} + \frac{(d-3)(\tilde{d}-1)}{3\tilde{d}(\tilde{d}+d-3)} = \\ &R^*_{-1}(T_n^t) - \frac{d\tilde{d}^2 + (-d^2+9d-24)\tilde{d} + 8(d^2-6d+9)}{24\tilde{d}d(\tilde{d}+d-3)} < R^*_{-1}(T_n^t) \end{aligned}$$

for $4 \le d \le 23$ and $\tilde{d} \ge 4$.

Case 5.3. There is at least one (2,3) system adjacent to z. By deleting vertices x_1, y_1 and this $(2,3)$ system and adding one (3,4) system adjacent to z we get a new tree T'. Then $\mid V(T') \mid = n,\ d_{T'}(z) = d-1$ and

$$\begin{aligned} R_{-1}(T_n^t) &= R_{-1}(T'^t_n) + \frac{1}{2} + \frac{1}{2d} + 2(\frac{1}{2} + \frac{1}{6}) + \frac{1}{3d} + (\frac{1}{d} - \frac{1}{d-1})\sum_{j=1}^{d-2}\frac{1}{d(w_j)} \\ &-3(\frac{1}{2} + \frac{1}{8}) - \frac{1}{4(d-1)} = R_{-1}(T'^t_n) + \frac{-d^2+15d-20}{24d(d-1)} - \frac{1}{d(d-1)}\sum_{j=1}^{d-2}\frac{1}{d(w_j)} \\ &< R_{-1}(T'^t_n) + \frac{-d^2+15d-20}{24d(d-1)} < R_{-1}(T'^t_n) \end{aligned}$$

for $d \ge 14$ $(-d^2+15d-20 < 0)$. We have to consider $4 \le d \le 13$. It is clear that there must exist one vertex $\tilde{w}$ such that $\tilde{d} \ge 4$, which is not the center of a (3,4) or (4,5) system.

Case 5.3.1. $d(w_j) = 3$ for at most three j. By deleting the vertices x_1, y_1, z and one (2,3) system adjacent to z and adding one (3,4) system adjacent to $\tilde{w}$ and edges $\tilde{w}w_1, \cdots, \tilde{w}w_{d-3}$ we get a new tree T'. Then $\mid V(T') \mid = n-1,\ d_{T'}(\tilde{w}) = \tilde{d}+d-3$ and

$$R_{-1}(T_n^t) = R_{-1}(T'^{t-1}_{n-1}) + \frac{1}{2} + \frac{1}{2d} + 2(\frac{1}{2} + \frac{1}{6}) + \frac{1}{3d} + \frac{1}{\tilde{d}d} - 3(\frac{1}{2} + \frac{1}{8}) - \frac{1}{4(\tilde{d}+d-3)}$$

$$+(\frac{1}{d}-\frac{1}{\tilde{d}+d-3})\sum_{j=1}^{d-3}\frac{1}{d(w_j)}+(\frac{1}{\tilde{d}}-\frac{1}{\tilde{d}+d-3})\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)}\leq R^*_{-1}(T^t_n)+$$
$$\triangle(n,1)-\frac{1}{24}+\frac{5}{6d}+\frac{1}{\tilde{d}d}-\frac{1}{4(\tilde{d}+d-3)}+\frac{\tilde{d}-3}{d(\tilde{d}+d-3)}\sum_{j=1}^{d-3}\frac{1}{d(w_j)}+$$
$$\frac{d-3}{\tilde{d}(\tilde{d}+d-3)}\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \tag{7}$$

Since $\triangle(n,1)<-\frac{1}{4}$, $\sum_{j=1}^{d-3}\frac{1}{d(w_j)}\leq\frac{2}{3}+\frac{d-5}{4}=\frac{3d-7}{12}$ and $\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)}\leq\frac{\tilde{d}-1}{3}$, we have

$$\begin{aligned}R_{-1}(T^t_n)&\leq R^*_{-1}(T^t_n)-\frac{7}{24}+\frac{5}{6d}+\frac{1}{\tilde{d}d}-\frac{1}{4(\tilde{d}+d-3)}+\frac{(\tilde{d}-3)(3d-7)}{12d(\tilde{d}+d-3)}+\\&\frac{(d-3)(\tilde{d}-1)}{3\tilde{d}(\tilde{d}+d-3)}=R^*_{-1}(T^t_n)+\frac{(6-d)\tilde{d}^2+(d-6)(d-1)\tilde{d}-8(d-3)^2}{24\tilde{d}d(\tilde{d}+d-3)}\\&<R^*_{-1}(T^t_n)\end{aligned}$$

for $6\leq d\leq 13$ and $\tilde{d}\geq 4$.

Subcase 5.3.1'. When $d=5$ and $d(w_j)=3$ for at most two j, we use $\sum_{j=1}^{d-3}\frac{1}{d(w_j)}=\sum_{j=1}^{2}\frac{1}{d(w_j)}\leq\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$, $\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j})\leq\frac{\tilde{d}-1}{3}$ and from Eq.(7) we get

$$R_{-1}(T^t_n)\quad<R^*_{-1}(T^t_n)-\frac{\tilde{d}^2-2\tilde{d}+32}{120\tilde{d}(\tilde{d}+2)}<R^*_{-1}(T^t_n)$$

for every $\tilde{d}$.

We leave to the reader to consider the case when $d=4$.

Case 5.3.2. There are four (2,3) systems adjacent to z. By deleting vertices x_1,y_1,z and these four (2,3) systems and adding two (3,4) systems and one (4,5) system adjacent to $\tilde{w}$ and edges $\tilde{w}w_1,\cdots,\tilde{w}w_{d-6}$ we get a new tree T'. Then $\mid V(T')\mid=n$, $d_{T'}(\tilde{w})=\tilde{d}+d-4$ and

$$\begin{aligned}R_{-1}(T^t_n)&=R_{-1}(T'^t_n)+\frac{1}{2}+\frac{1}{2d}+8(\frac{1}{2}+\frac{1}{6})+\frac{4}{3d}+\frac{1}{\tilde{d}d}-6(\frac{1}{2}+\frac{1}{8})-4(\frac{1}{2}+\frac{1}{10})\\&-\frac{2}{4(\tilde{d}+d-4)}-\frac{1}{5(\tilde{d}+d-4)}+(\frac{1}{d}-\frac{1}{\tilde{d}+d-4})\sum_{j=1}^{d-6}\frac{1}{d(w_j)}+\\&(\frac{1}{\tilde{d}}-\frac{1}{\tilde{d}+d-4})\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)}=R_{-1}(T'^t_n)-\frac{19}{60}+\frac{11}{6d}+\frac{1}{\tilde{d}d}-\frac{7}{10(\tilde{d}+d-4)}\\&+\frac{\tilde{d}-4}{d(\tilde{d}+d-4)}\sum_{j=1}^{d-6}\frac{1}{d(w_j)}+\frac{d-4}{\tilde{d}(\tilde{d}+d-4)}\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)}\end{aligned}$$

Since $\sum_{j=1}^{d-6}\frac{1}{d(w_j)} \le \frac{d-6}{4}$, $\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \le \frac{\tilde{d}-1}{3}$, we have

$$\begin{aligned} R_{-1}(T_n^t) &\le R_{-1}(T_n^{'t}) + \frac{(20-4d)\tilde{d}^2 + (d^2+4d-20)\tilde{d} - 20(d^2-7d+12)}{60\tilde{d}d(\tilde{d}+d-4)} \\ &< R_{-1}(T_n^{'t}) \end{aligned}$$

for $6 \le d \le 13$ and $\tilde{d} \ge 4$.

Case 5.3.3. There are at least five (2,3) systems adjacent to z. By deleting vertices x_1, y_1, z and these five (2,3) systems and adding four (3,4) systems adjacent to $\tilde{w}$ and edges $\tilde{w}w_1, \cdots, \tilde{w}w_{d-7}$ we get a new tree T'. Then $\mid V(T') \mid = n$, $d_{T'}(\tilde{w}) = \tilde{d} + d - 4$ and

$$\begin{aligned} R_{-1}(T_n^t) &= R_{-1}(T_n^{'t}) + \frac{1}{2} + \frac{1}{2d} + 10(\frac{1}{2}+\frac{1}{6}) + \frac{5}{3d} + \frac{1}{\tilde{d}d} - 12(\frac{1}{2}+\frac{1}{8}) - \frac{4}{4(\tilde{d}+d-4)} \\ &+ (\frac{1}{d} - \frac{1}{\tilde{d}+d-4})\sum_{j=1}^{d-7}\frac{1}{d(w_j)} + (\frac{1}{\tilde{d}} - \frac{1}{\tilde{d}+d-4})\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} = R_{-1}(T_n^{'t}) - \frac{1}{3} \\ &+ \frac{13}{6d} + \frac{1}{\tilde{d}d} - \frac{1}{\tilde{d}+d-4} + \frac{\tilde{d}-4}{d(\tilde{d}+d-4)}\sum_{j=1}^{d-7}\frac{1}{d(w_j)} + \frac{d-4}{\tilde{d}(\tilde{d}+d-4)}\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \end{aligned}$$

Since $\sum_{j=1}^{d-7}\frac{1}{d(w_j)} \le \frac{d-7}{3}$, $\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \le \frac{\tilde{d}-1}{3}$, we have

$$R_{-1}(T_n^t) \le R_{-1}(T_n^{'t}) + \frac{-\tilde{d}^2 + (-d+10)\tilde{d} - 2(d^2-7d+12)}{6\tilde{d}d(\tilde{d}+d-4)} < R_{-1}(T_n^{'t})$$

for $7 \le d \le 13$ and $\tilde{d} \ge 4$.

The only left possible case is the presence of a $(1,5)^*$ system.

We have shown, up to now, that the "suspected" Max Trees can have only $(2,3), (3,4), (4,5)$ and $(1,5)^*$ systems. Now we will show that some of these systems are centered at only one vertex. The proofs of the next lemmas are easy and they are omitted.

Lemma 3. Every vertex of degree 4 is the center of a $(3,4)$ system of the Max Tree, i.e., it appears only in a $(3,4)$ system.

Lemma 4.([11]) The Max Tree can not have the vertex with more than 12 $(2,3)$ systems centered at it.

Theorem 4. *The Max Tree can have only one vertex with maximum degree and all other systems can be centered at it.*

Proof. Let z and $\tilde{z}$ be two adjacent vertices of the Max Tree T such that $d(z) = d \geq 5$ and $d(\tilde{z}) = \tilde{d} \geq 5$. We differ several cases depending on the number of $(2,3)$ systems centered at z and $\tilde{z}$. We take that the number of $(2,3)$ systems centered at z is greater than or equal to the number of $(2,3)$ systems centered at $\tilde{z}$.

Case 1. The number of $(2,3)$ systems centered at z is less than or equal to 3 (and the same for $\tilde{z}$). Let $w_1, \cdots, w_{d-1}$ be the vertices of T, other than $\tilde{z}$, adjacent to z and let $v_1, \cdots, v_{\tilde{d}-1}$ be the vertices of T, other than z, adjacent to $\tilde{z}$. By deleting the vertex $\tilde{z}$ and connecting zv_j, $j = 1, \cdots, \tilde{d}-1$, we get a new tree T'. Then $\mid V(T') \mid = \mid V(T) \mid -1$, $d_{T'}(z) = \tilde{d} + d - 2$ and

$$\begin{aligned} R_{-1}(T_n^t) &= R_{-1}(T_{n-1}^{\prime t-1}) + \frac{1}{d\tilde{d}} + (\frac{1}{d} - \frac{1}{\tilde{d}+d-2})\sum_{i=1}^{d-1}\frac{1}{d(w_i)} + (\frac{1}{\tilde{d}} - \frac{1}{\tilde{d}+d-2}) \\ &\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \leq R_{-1}^*(T_n^t) + \triangle(n,1) + \frac{1}{d\tilde{d}} + \frac{\tilde{d}-2}{d(\tilde{d}+d-2)}\sum_{i=1}^{d-1}\frac{1}{d(w_i)} + \\ &\frac{d-2}{\tilde{d}(\tilde{d}+d-2)}\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \end{aligned}$$

Since $\triangle(n,1) < -\frac{1}{4}$ and $\sum_{i=1}^{d-1}\frac{1}{d(w_i)} \leq \frac{m}{3} + \frac{d-m-1}{4} = \frac{3d+m-3}{12}$, $m = 0,1,2,3$ and $\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \leq \frac{k}{3} + \frac{\tilde{d}-k-1}{4} = \frac{3\tilde{d}+k-3}{12}$, $k = 0,1,2,3$, we have

$$\begin{aligned} R_{-1}(T_n^t) &\leq R_{-1}^*(T_n^t) + \frac{(k-3)d^2 - (6\tilde{d}+2(k-9))d + (m-3)\tilde{d}^2 - 2(m-9)\tilde{d} - 24}{12\tilde{d}d(\tilde{d}+d-2)} \\ &\leq R_{-1}^*(T_n^t) \end{aligned}$$

because $\frac{(k-3)d^2-(6\tilde{d}+2(k-9))d+(m-3)\tilde{d}^2-2(m-9)\tilde{d}-24}{12\tilde{d}d(\tilde{d}+d-2)} \leq 0$ for $m = 0, \cdots, 3$, $d \geq 5$ and $k = 0, \cdots, 3$, $\tilde{d} \geq 5$.

Case 2. The number of $(2,3)$ systems centered at z is $4 (m = 0)$, 5 $(m = 1)$, 6 $(m = 2)$ and 7 $(m = 3)$, where m is the difference between this number and 4. The number of $(2,3)$ systems centered at $\tilde{z}$ is $k = 0, 1, \cdots, 4+m$, $\tilde{d} \geq max\{5, k+1\}$. Let at least 4 $(2,3)$ systems be centered at z, and let $w_1, \cdots, w_{d-5}$ be the vertices of T, other than $\tilde{z}$ and 4 $(2,3)$ systems, adjacent to z and let $v_1, \cdots, v_{\tilde{d}-1}$ be the vertices of T, other than z, adjacent to $\tilde{z}$. By deleting the vertex $\tilde{z}$ and connecting zv_j, $j = 1, \cdots, \tilde{d}-1$, deleting 4 $(2,3)$ systems adjacent to z and adding 3 $(3,4)$ systems adjacent to z we get a new tree T'. Then $\mid V(T') \mid = \mid V(T) \mid$, $d_{T'}(z) = \tilde{d} + d - 3$ and

$$R_{-1}(T_n^t) = R_{-1}(T_n^{\prime t}) + 8(\frac{1}{2} + \frac{1}{6}) + \frac{4}{3d} + \frac{1}{d\tilde{d}} + (\frac{1}{d} - \frac{1}{\tilde{d}+d-3})\sum_{i=1}^{d-5}\frac{1}{d(w_i)} +$$

$$(\frac{1}{\tilde d}-\frac{1}{\tilde d+d-3})\sum_{j=1}^{\tilde d-1}\frac{1}{d(v_j)}-\frac{3}{4(d+\tilde d-3)}-9(\frac{1}{2}+\frac{1}{8})=R_{-1}(T_n^{'t})$$
$$-\frac{7}{24}+\frac{4}{3d}+\frac{1}{d\tilde d}-\frac{3}{4(d+\tilde d-3)}+\frac{\tilde d-3}{d(\tilde d+d-3)}\sum_{i=1}^{d-5}\frac{1}{d(w_i)}$$
$$+\frac{d-3}{\tilde d(\tilde d+d-3)}\sum_{j=1}^{\tilde d-1}\frac{1}{d(v_j)}$$

Since $\sum_{i=1}^{d-5}\frac{1}{d(w_i)}\le\frac{m}{3}+\frac{d-m-5}{4}=\frac{3d+m-15}{12}$, $m=0,1,2,3,\ d\ge 5+m$ and $\sum_{j=1}^{\tilde d-1}\frac{1}{d(v_j)}\le\frac{k}{3}+\frac{\tilde d-k-1}{4}=\frac{3\tilde d+k-3}{12}$, $k=0,1,\cdots,4+m$, $\tilde d\ge\max\{5,k+1\}$, we have

$$\begin{aligned}R_{-1}(T_n^t) &\le R_{-1}(T_n^{'t})-\frac{[\tilde d-2(k-3)]d^2+[\tilde d^2+\tilde d+6(k-7)]d-2(m+1)\tilde d^2}{24\tilde d d(\tilde d+d-3)}\\&\quad-\frac{6(m-3)\tilde d+72}{24\tilde d d(\tilde d+d-3)}\le R_{-1}(T_n^{'t})\end{aligned}$$

for $m=0,1,2,3,\ d\ge 5+m$ and $k=0,1,\cdots,4+m,\ \tilde d\ge\max\{5,k+1\}$.

Case 3. The number of $(2,3)$ systems centered at z $(\tilde z)$ is 8 (is less or equal to 3), $d\ge 9$ and $\tilde d\ge 9$. Let $w_1,\cdots,w_{d-9}$ be the vertices of T, other than $\tilde z$ and 8 $(2,3)$ systems, adjacent to z and let $v_1,\cdots,v_{\tilde d-1}$ be the vertices of T, other than z, adjacent to $\tilde z$. By deleting edges $zw_1,\cdots,zw_{d-9}$ and adding new edges $\tilde z w_1,\cdots,\tilde z w_{d-9}$, we get a new tree T'. Then $\mid V(T')\mid=\mid V(T)\mid$, $d_{T'}(z)=9$, $d_{T'}(\tilde z)=\tilde d+d-9$ and

$$\begin{aligned}R_{-1}(T_n^t) &= R_{-1}(T_n^{'t})+\frac{8}{3d}+\frac{1}{d\tilde d}+(\frac{1}{d}-\frac{1}{\tilde d+d-9})\sum_{i=1}^{d-9}\frac{1}{d(w_i)}-\frac{8}{27}-\frac{1}{9(\tilde d+d-9)}\\&\quad+(\frac{1}{\tilde d}-\frac{1}{\tilde d+d-9})\sum_{j=1}^{\tilde d-1}\frac{1}{d(v_j)}=R_{-1}(T_n^{'t})-\frac{8}{27}+\frac{8}{3d}+\frac{1}{d\tilde d}-\frac{1}{9(\tilde d+d-9)}\\&\quad+\frac{\tilde d-9}{d(\tilde d+d-9)}\sum_{i=1}^{d-9}\frac{1}{d(w_i)}+\frac{d-9}{\tilde d(\tilde d+d-9)}\sum_{j=1}^{\tilde d-1}\frac{1}{d(v_j)}\end{aligned}$$

Since $\sum_{i=1}^{d-9}\frac{1}{d(w_i)}\le\frac{d-9}{4}$ and $\sum_{j=1}^{\tilde d-1}\frac{1}{d(v_j)}\le\frac{k}{3}+\frac{\tilde d-k-1}{4}=\frac{3\tilde d+k-3}{12}$, $k=0,1,2,3$, we have

$$\begin{aligned}R_{-1}(T_n^t) &\le R_{-1}(T_n^{'t})+\frac{[9(k-3)-5\tilde d]d^2+[-5\tilde d^2+78\tilde d-27(3k-13)]d+45\tilde d^2}{108\tilde d d(\tilde d+d-9)}\\&\quad+\frac{-297\tilde d-972}{108\tilde d d(\tilde d+d-9)}\le R_{-1}(T_n^{'t})\end{aligned}$$

for $k=0,1,2,3$ and $d\ge 9,\ \tilde d\ge 9$.

Subcase 3'. The number of $(2,3)$ systems centered at z $(\tilde z)$ is 8 (is less or equal to 3), $d\ge 9$ and $5\le\tilde d\le 8$. By deleting the vertex $\tilde z$ and connecting zv_j, $j=1,\cdots,\tilde d-1$, deleting 7 (2,3) systems adjacent to z and adding 5 (3,4) systems adjacent to z, we get a new tree T'. $\mid V(T')\mid=\mid V(T)\mid -1$, $d_{T'}(z)=\tilde{d}+d-4$ and

$$\begin{aligned} R_{-1}(T_n^t) &= R_{-1}(T_{n-1}^{\prime t-1})+14(\frac{1}{2}+\frac{1}{6})+\frac{8}{3d}+\frac{1}{d\tilde{d}}+(\frac{1}{d}-\frac{1}{\tilde{d}+d-4})\sum_{i=1}^{d-9}\frac{1}{d(w_i)}+ \\ & (\frac{1}{\tilde{d}}-\frac{1}{\tilde{d}+d-4})\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)}-15(\frac{1}{2}+\frac{1}{8})-\frac{5}{4(d+\tilde{d}-4)}-\frac{1}{3(d+\tilde{d}-4)} \\ & = R^*_{-1}(T_n^t)+\triangle(n,1)-\frac{1}{24}+\frac{8}{3d}+\frac{1}{d\tilde{d}}-\frac{19}{12(d+\tilde{d}-4)}+ \\ & \frac{\tilde{d}-4}{d(\tilde{d}+d-4)}\sum_{i=1}^{d-9}\frac{1}{d(w_i)}+\frac{d-4}{\tilde{d}(\tilde{d}+d-4)}\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \end{aligned}$$

Since $\triangle(n,1)<-\frac{1}{4}$, $\sum_{i=1}^{d-9}\frac{1}{d(w_i)}\leq\frac{d-9}{4}$ and $\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)}\leq\frac{k}{3}+\frac{\tilde{d}-k-1}{4}=\frac{3\tilde{d}+k-3}{12}$, $k=0,1,2,3$, we have

$$\begin{aligned} R_{-1}(T_n^t) &\leq R^*_{-1}(T_n^t)-\frac{[\tilde{d}-2(k-3)]d^2+[\tilde{d}^2-6\tilde{d}+8(k-6)]d-10\tilde{d}^2+16\tilde{d}+96}{24\tilde{d}d(\tilde{d}+d-4)} \\ &\leq R^*_{-1}(T_n^t) \end{aligned}$$

for $k=0,1,2,3,\ d\geq 9,\ 5\leq\tilde{d}\leq 8$.

Case 4. The number of $(2,3)$ systems centered at z ($\tilde{z}$) is 9 (is less or equal to 3), $d\geq 10$ and $\tilde{d}\geq 5$. Let $w_1,\cdots,w_{d-10}$ be the vertices of T, other than $\tilde{z}$ and 9 $(2,3)$ systems, adjacent to z and let $v_1,\cdots,v_{\tilde{d}-1}$ be the vertices of T, other than z, adjacent to $\tilde{z}$. By deleting vertex z and 9 (2,3) systems adjacent to z and adding 4 (3,4) systems and one $(1,5)^*$ system adjacent to $\tilde{z}$ and edges $\tilde{z}w_1,\cdots,\tilde{z}w_{d-10}$, we get a new tree T'. Then $\mid V(T')\mid=\mid V(T)\mid$, $d_{T'}(\tilde{z})=\tilde{d}+d-6$ and

$$\begin{aligned} R_{-1}(T_n^t) &= R_{-1}(T_n^{\prime t})+12(\frac{1}{2}+\frac{1}{6})+\frac{9}{3d}+\frac{1}{\tilde{d}d}+(\frac{1}{d}-\frac{1}{\tilde{d}+d-6})\sum_{i=1}^{d-10}\frac{1}{d(w_i)}+ \\ & (\frac{1}{\tilde{d}}-\frac{1}{\tilde{d}+d-6})\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)}-12(\frac{1}{2}+\frac{1}{8})-\frac{4}{4(\tilde{d}+d-6)}-\frac{3}{15}-\frac{1}{2}-\frac{1}{10} \\ & -\frac{1}{5(\tilde{d}+d-6)}=R_{-1}(T_n^{\prime t})-\frac{3}{10}+\frac{3}{d}+\frac{1}{\tilde{d}d}-\frac{6}{5(\tilde{d}+d-6)}+ \\ & \frac{\tilde{d}-6}{d(\tilde{d}+d-6)}\sum_{i=1}^{d-10}\frac{1}{d(w_i)}+\frac{d-6}{\tilde{d}(\tilde{d}+d-6)}\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)} \end{aligned}$$

Since $\sum_{i=1}^{d-10}\frac{1}{d(w_i)}\leq\frac{d-10}{4}$, $\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)}\leq\frac{k}{3}+\frac{\tilde{d}-k-1}{4}=\frac{3\tilde{d}+k-3}{12}$, $k=0,1,2,3$, we have

$$\begin{aligned} R_{-1}(T_n^t) &\leq R_{-1}(T_n^{\prime t})+\frac{[5(k-3)-3\tilde{d}]d^2+[-3\tilde{d}^2+36\tilde{d}-30(k-5)]d+30\tilde{d}^2}{60\tilde{d}d(\tilde{d}+d-6)} \\ & -\frac{120\tilde{d}+360}{60\tilde{d}d(\tilde{d}+d-6)}\leq R_{-1}(T_n^{\prime t}) \end{aligned}$$

for $k=0,\cdots,3,\ \tilde{d}\geq 6$ and $d\geq 10$. When $\tilde{d}=5$ we maximize $\frac{\tilde{d}-6}{d(\tilde{d}+d-6)}\sum_{i=1}^{d-10}\frac{1}{d(w_i)}\leq 0$ and we get

$$R_{-1}(T_n^t)\ \leq\ R_{-1}(T_n^{\prime t})+\frac{(k-6)d^2+6(11-k)d-192}{60d(d-1)}<R_{-1}(T_n^{\prime t})$$

for $k=0,\cdots,3$ and $d\geq 10$.

Case 5. The number of $(2,3)$ systems centered at z is 10 $(m=0)$, $11(m=1)$ and 12 $(m=2)$, where m is the difference between this number and 10, while the number of $(2,3)$ systems centered at $\tilde{z}$ is less or equal 3, $d\geq 11+m,\ \ \tilde{d}\geq 5$. Let $w_1,\cdots,w_{d-11}$ be the vertices of T, other than $\tilde{z}$ and 10 $(2,3)$ systems, adjacent to z and let $v_1,\cdots,v_{\tilde{d}-1}$ be the vertices of T, other than z, adjacent to $\tilde{z}$. By deleting vertex z and these 10 (2,3) systems adjacent to z and adding 6 (3,4) systems and one (4,5) system adjacent to $\tilde{z}$ and adding edges $\tilde{z}w_1,\cdots,\tilde{z}w_{d-11}$, we get a new tree T'. Then $|V(T')|=|V(T)|$, $d_{T'}(\tilde{z})=\tilde{d}+d-5$ and

$$\begin{aligned}R_{-1}(T_n^t)\ =\ & R_{-1}(T_n^{\prime t})+20(\frac{1}{2}+\frac{1}{6})+\frac{10}{3d}+\frac{1}{\tilde{d}d}+(\frac{1}{d}-\frac{1}{\tilde{d}+d-5})\sum_{i=1}^{d-11}\frac{1}{d(w_i)}+\\ & (\frac{1}{\tilde{d}}-\frac{1}{\tilde{d}+d-5})\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)}-18(\frac{1}{2}+\frac{1}{8})-4(\frac{1}{2}+\frac{1}{10})-\frac{6}{4(\tilde{d}+d-5)}-\\ & \frac{1}{5(\tilde{d}+d-5)}=R_{-1}(T_n^{\prime t})-\frac{19}{60}+\frac{10}{3d}+\frac{1}{\tilde{d}d}-\frac{17}{10(\tilde{d}+d-5)}+\\ & \frac{\tilde{d}-5}{d(\tilde{d}+d-5)}\sum_{i=1}^{d-11}\frac{1}{d(w_i)}+\frac{d-5}{\tilde{d}(\tilde{d}+d-5)}\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)}\end{aligned}$$

Since $\sum_{i=1}^{d-11}\frac{1}{d(w_i)}\leq\frac{m}{3}+\frac{d-m-11}{4}=\frac{3d+m-33}{12},\ m=0,1,2,\ d\geq 11+m$ and $\sum_{j=1}^{\tilde{d}-1}\frac{1}{d(v_j)}\leq\frac{k}{3}+\frac{\tilde{d}-k-1}{4}=\frac{3\tilde{d}+k-3}{12}, k=0,1,2,3,\ \tilde{d}\geq 5$, we have

$$\begin{aligned}R_{-1}(T_n^t)\ \leq\ & R_{-1}(T_n^{\prime t})+\frac{[5(k-3)-4\tilde{d}]d^2+[-4\tilde{d}^2+43\tilde{d}-5(5k-27)]d+5(m+7)\tilde{d}^2}{60\tilde{d}d(\tilde{d}+d-5)}\\ & -\frac{5(5m+23)\tilde{d}+300}{60\tilde{d}d(\tilde{d}+d-5)}\leq R_{-1}(T_n^{\prime t})\end{aligned}$$

for $m=0,1,2,\ d\geq 11+m$ and $\ k=0,1,2,3,\ \tilde{d}\geq 5$.

Case 6. The number of $(2,3)$ systems centered at z is $8+m,\ m=0,1,2,3,4,\ (d\geq 9+m)$, while the number of $(2,3)$ systems centered at $\tilde{z}$ is $4+k$, $k=0,1,2,3,\cdots,4+m$, $\tilde{d}\geq 5+k$. Let $w_1,\cdots,w_{d-9}$ be the vertices of T, other than $\tilde{z}$ and 8 $(2,3)$ systems, adjacent to z and let $v_1,\cdots,v_{\tilde{d}-5}$ be the vertices of T, other than z and 4 (2,3) systems adjacent to $\tilde{z}$. By deleting vertex z, 8 (2,3) systems adjacent to z, 3 (2,3) systems adjacent to $\tilde{z}$ and adding 8 (3,4) systems adjacent to $\tilde{z}$ and new edges $\tilde{z}w_1, \cdots, \tilde{z}w_{d-9}$, we get a new tree T'. Then $\mid V(T') \mid = \mid V(T) \mid$, $d_{T'}(\tilde{z}) = \tilde{d} + d - 5$ and

$$\begin{aligned}
R_{-1}(T_n^t) &= R_{-1}(T_n^{'t}) + 16(\frac{1}{2} + \frac{1}{6}) + \frac{8}{3d} + \frac{1}{\tilde{d}d} + (\frac{1}{d} - \frac{1}{\tilde{d} + d - 5})\sum_{i=1}^{d-9} \frac{1}{d(w_i)} + \frac{4}{3\tilde{d}} + \\
&\quad +6(\frac{1}{2} + \frac{1}{6}) + (\frac{1}{\tilde{d}} - \frac{1}{\tilde{d} + d - 5})\sum_{j=1}^{\tilde{d}-1} \frac{1}{d(v_j)} - 24(\frac{1}{2} + \frac{1}{8}) - \frac{8}{4(\tilde{d} + d - 5)} - \\
&\quad -\frac{1}{3(\tilde{d} + d - 5)} = R_{-1}(T_n^{'t}) - \frac{1}{3} + \frac{8}{3d} + \frac{1}{\tilde{d}d} + \frac{4}{3\tilde{d}} - \frac{7}{3(\tilde{d} + d - 5)} + \\
&\quad \frac{\tilde{d} - 5}{d(\tilde{d} + d - 5)}\sum_{i=1}^{d-9} \frac{1}{d(w_i)} + \frac{d - 5}{\tilde{d}(\tilde{d} + d - 5)}\sum_{j=1}^{\tilde{d}-1} \frac{1}{d(v_j)}
\end{aligned}$$

Since $\sum_{i=1}^{d-9} \frac{1}{d(w_i)} \leq \frac{m}{3} + \frac{d-m-9}{4} = \frac{3d+m-27}{12}$, $m = 0, 1, \cdots, 4$, $d \geq 9+m$ and $\sum_{j=1}^{\tilde{d}-1} \frac{1}{d(v_j)} \leq \frac{k}{3} + \frac{\tilde{d}-k-5}{4} = \frac{3\tilde{d}+k-15}{12}$, $k = 0, 1, 2, 3, \cdots, 4 + m$, $\tilde{d} \geq 5 + k$, we have

$$\begin{aligned}
R_{-1}(T_n^t) &\leq R_{-1}(T_n^{'t}) + \frac{[k + 1 - \tilde{d}]d^2 + [-\tilde{d}^2 + 10\tilde{d} + 7 - 5k]d + (m + 5)\tilde{d}^2}{12\tilde{d}d(\tilde{d} + d - 5)} \\
&\quad -\frac{(5m + 13)\tilde{d} + 60}{12\tilde{d}d(\tilde{d} + d - 5)} \leq R_{-1}(T_n^{'t})
\end{aligned}$$

for $m = 0, 1, \cdots, 4$, $d \geq 9 + m$ and $k = 0, 1, 2, 3, 4 + m$, $\tilde{d} \geq 5 + k$. □

We conclude that the Max Tree can have a $(8_{(2,3)}, 9)$ system. The proof of the next lemma is omitted, because it is easy.

Lemma 5. The Max Tree can not have the next systems adjacent to the same vertex:

1. $(2, 3)$ and $(4, 5)$, $(2, 3)$ and $(1, 5)^*$, $(2, 3)$ and $(8_{(2,3)}, 9)$;
2. $(4, 5)$ and $(4, 5)$, $(4, 5)$ and $(1, 5)^*$, $(4, 5)$ and $(8_{(2,3)}, 9)$;
3. $(1, 5)^*$ and $(1, 5)^*$, $(1, 5)^*$ and $(8_{(2,3)}, 9)$;
4. $(8_{(2,3)}, 9)$ and $(8_{(2,3)}, 9)$.

Now we can prove Theorem 1. Hence, the Max Tree can have $(3, 4)$ systems and one of $(1, 5)^*$ $(t = 4)$, $(4, 5)$ $(t = 2)$ or $(8_{(2,3)}, 9)$ $(t = 6)$ systems. The Max Tree can have $(3, 4)$ systems and one $(2, 3)$ $(t = 5)$ system, two $(2, 3)$ $(t = 3)$ systems and three $(2, 3)$ $(t = 1)$ systems. □

Denote by p the number of $(2, 3)$ systems, by q the number of $(3, 4)$ systems, by r the number of $(4, 5)$ systems, by s the number of $(1, 5)^*$ systems and by v the number of $(8_{(2,3)}, 9)$ systems. In the next table we give the structure of the Max Tree when $n \geq 92$. We gave the proof for it when $n \geq 385$. Exact proof for this table for $92 \leq n \leq 385$, needs considerations of some new cases. We took that the numerical results obtained in [5] are true for our inductional hypothesis which starts later ($n \geq 720$).

<table>
<tr><td>$n-1 \pmod 7$</td><td>$n \geq 92$</td><td>$n \geq 145$</td><td>$n \geq 385$</td><td>$n \geq 720$</td></tr>
<tr><td>0</td><td colspan="4">$(0, \frac{n-1}{7}, 0, 0, 0)$</td></tr>
<tr><td>1</td><td colspan="4">$(3, \frac{n-16}{7}, 0, 0, 0)$</td></tr>
<tr><td>2</td><td>$(6, \frac{n-31}{7}, 0, 0, 0)$</td><td colspan="3">$(0, \frac{n-10}{7}, 1, 0, 0)$</td></tr>
<tr><td>3</td><td colspan="4">$(2, \frac{n-11}{7}, 0, 0, 0)$</td></tr>
<tr><td>4</td><td colspan="2">$(5, \frac{n-26}{7}, 0, 0, 0)$</td><td colspan="2">$(0, \frac{n-19}{7}, 0, 1, 0)$</td></tr>
<tr><td>5</td><td colspan="4">$(1, \frac{n-6}{7}, 0, 0, 0)$</td></tr>
<tr><td>6</td><td colspan="3">$(4, \frac{n-21}{7}, 0, 0, 0)$</td><td>$(0, \frac{n-42}{7}, 0, 0, 1)$</td></tr>
</table>

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