A Combinatorial Proof of the Lebesgue Identity

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Abstract We present a combinatorial proof of the Lebesgue identity based on the insertion algorithm of Zeilberger.

Keywords: The Lebesgue identity, Algorithm Z

Assume that $|q| < 1$, and let $(a;q)_\infty = (1-a)(1-aq)\cdots$. We define the q-shifted factorial $(a;q)_n$ by

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

The following relation is referred to as the Lebesgue identity

$$\sum_{n=0}^{\infty} \frac{(-aq;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = (-aq^2;q^2)_\infty(-q;q)_\infty, \tag{1}$$

A partition λ of a nonnegative integer into at most r parts is denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are nonincreasing sequence of nonnegative integers. Let λ, μ be partitions. We define $\lambda \bigcup \mu$ to be the partition whose parts are those of λ and μ, arranged in nonincreasing order.

Based on 2-modular diagrams [2, 3] or MacMahon diagrams, two combinatorial proofs of (1) have been given respectively. In this paper, we shall give a new combinatorial proof of a generalization of the Lebesgue identity in which Algorithm Z due to Zeilberger (see also Andrews and Bressoud [1]) plays a crucial rule.

Theorem 1 *For any $0 \le k \le n$, there is bijection between the set of pairs of partitions (α, β) where α has k distinct parts with the largest part not exceeding n, β has n distinct parts and the set of pairs of partitions (μ, ν) where μ has k even distinct parts, ν has distinct parts with $n-k$ parts exceeding k.*

We may translate Theorem 1 into the following q-identity:

$$\sum_{n=0}^{\infty} \frac{(-aq;q)_n}{(q;q)_n} b^n q^{\binom{n+1}{2}} = \sum_{k=0}^{\infty} \frac{(-q;q)_k}{(q^2;q^2)_k} (ab)^k q^{k(k+1)} (-bq^{k+1};q)_\infty$$
$$= (-bq;q)_\infty \sum_{k=0}^{\infty} \frac{(ab)^k q^{k(k+1)}}{(q;q)_k (-bq;q)_k}. \qquad (2)$$

In view of Euler's identity

$$\sum_{n=0}^{\infty} \frac{a^n q^{\binom{n}{2}}}{(q;q)_n} = (a;q)_\infty,$$

the Lebesgue identity can be recovered by setting $b = 1$ in (2).

Lemma 1 (**Algorithm Z**) *There is a bijection between the set of pairs of partitions (α, β) where α has at most i parts and β has at most j parts, and the set of pairs of partitions (μ, ν), where μ has at most $i+j$ parts and ν has at most j parts with each part not exceeding i.*

Proof. Given a partition α with at most i parts, denoted by $(\alpha_1, \alpha_2, \ldots, \alpha_i)$, and a partition β with at most j parts, denoted by $(\beta_1, \beta_2, \ldots, \beta_j)$, we may insert β into α to form a new partition μ with at most $i+j$ parts and create a new partition ν which has at most j parts with each part not exceeding i.

The insertion algorithm can be described as the following recursive procedure.

1. If $\beta_1 \leq \alpha_i$, we add β_1 to α so that we get a new partition $(\alpha_1, \alpha_2, \ldots, \alpha_{i+1})$, where $\alpha_{i+1} = \beta_1$. Moreover, we put an empty part as a record.

2. If $\beta_1 > \alpha_i$, we recursively insert $\beta_1 - 1$ to the partition $(\alpha_1, \alpha_2, \ldots, \alpha_{i-1})$. Suppose that the recursive procedure ends with $\beta_1 - \nu_1$ being inserted, we use a part ν_1 to record the position of $\beta_1 - \nu_1$. Obviously, we have $0 \leq \nu_1 \leq i$.

Conversely, given a partition $(\alpha_1, \alpha_2, \ldots, \alpha_{i+1})$ and a number ν_1, $0 \leq \nu_1 \leq i$, we may extract the part β_1 from the given partition. It is easy to see that the above procedure is reversible.

If one applies the insertion algorithm successively, we arrive at Lemma 1, where $\mu = \alpha \bigcup (\beta_1 - \nu_1, \beta_2 - \nu_2, \ldots, \beta_j - \nu_j)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_j)$.

As an example, taking $\alpha = (6,5,3,3,1)$, $\beta = (5,4,4,0)$, where $i = 5$, $j = 4$, we have

$$\mu = (6,5,3,3,3,3,3,1,0),\ \nu = (2,1,1,0).$$

$$\begin{array}{ccccccccc}
6 & 5 & 3 & & 3 & & & 1 & \\
\hline
 & & & & & & & & \\
6 & 5 & 3 & 3 & 3 & 3 & 3 & 1 & 0 \\
 & & & 2 & & 1 & 1 & & 0 \\
\hline
 & & & & & & & & \\
 & & & 5 & & 4 & 4 & & 0
\end{array}$$

∎

Lemma 2 *There is a bijection between the set of pairs of partitions (α, β) where α has distinct parts with the largest part not exceeding i, β has at most i even parts, and the set of partitions λ where λ has at most i parts.*

Proof. Given a partition λ into at most i parts, we denote its conjugate by λ'. We decompose λ' into a pair of partitions α and β' obeying the following rule. For part j $(1 \leq j \leq i)$, if the number of part j denoted by m_j is odd, then α has a part j, β' has $m_j - 1$ equal parts j; otherwise, β' has m_j equal parts j. Clearly, both α and β' have the largest parts not exceeding i. Conjugating β', we get a partition β that has at most i even parts. The reverse procedure is straightforward. ∎

The following is an example of Lemma 2. Taking $i = 8$, $\lambda = (20,19,19,18,11,8,8,7)$, we have

$$\begin{aligned}
\lambda \ &\Leftrightarrow\ \lambda' = (8,8,8,8,8,8,8,8,7,5,5,5,4,4,4,4,4,4,4,3,1) \\
&\Leftrightarrow\ \alpha = (8,7,5,4,3,1), \quad \beta' = (8,8,8,8,8,8,5,5,4,4,4,4,4,4) \\
&\Leftrightarrow\ \alpha = (8,7,5,4,3,1), \quad \beta = (14,14,14,14,8,6,6,6).
\end{aligned}$$

Using Lemma 1 and Lemma 2, we present a combinatorial proof of Theorem 1.

Proof of Theorem 1. We shall prove the theorem through the following steps.

Step 1. Removing the staircase partitions $T_1 = (k, k-1, \dots, 1)$ and $T_2 = (n, n-1, \dots, 1)$ from α and β respectively, we obtain a partition $\bar{\alpha}$ containing at most k parts with the largest part not exceeding $n-k$ and a partition $\bar{\beta}$ into at most n parts.

Step 2. Applying Algorithm Z to $\bar{\alpha}$ and $\bar{\beta}$, we get a pair of partitions $(\bar{\bar{\alpha}}, \bar{\bar{\beta}})$ where $\bar{\bar{\alpha}}$ has at most k parts and $\bar{\bar{\beta}}$ has at most $n-k$ parts.

Step 3. By Lemma 2, we can decompose $\bar{\bar{\alpha}}$ into two partitions $\bar{\mu}$ with at most k even parts and $\bar{\nu}$ into distinct parts with the largest part not exceeding k.

Step 4. Divide T_2 into three parts, two staircase partitions $T_{21} = (k, k-1, \dots, 1)$, $T_{23} = (n-k, n-k-1, \dots, 1)$ and a rectangle partition $T_{22} = (\underbrace{k, \dots, k}_{n-k})$. Adding T_1 and T_{21} to $\bar{\mu}$, we get a partition μ into k distinct even parts. Adding T_{23} to $\bar{\bar{\beta}}$, then putting the new partition to the left of T_{22} and $\bar{\nu}$ below T_{22}, we get a partition ν which has distinct parts with $n-k$ parts exceeding k.

The reverse bijection can be easily constructed. Given μ and ν, we get a partition $\bar{\nu}$ from ν by choosing the parts not exceeding k, and a partition $\bar{\bar{\beta}}$ from the remaining parts of ν by removing a rectangle partition $(\underbrace{k, \dots, k}_{n-k})$ and a staircase partition $(n-k, n-k-1, \dots, 1)$, and a partition $\bar{\mu}$ by removing two staircase partitions $(k, k-1, \dots, 1)$ from μ. By Lemma 1 and Lemma 2, we can get a pair of partitions $(\bar{\alpha}, \bar{\beta})$ from $\bar{\mu}$, $\bar{\nu}$ and $\bar{\bar{\beta}}$ where $\bar{\alpha}$ has at most k parts with the largest part not exceeding $n-k$ and $\bar{\beta}$ has at most n parts. Finally, adding the staircase partitions $(k, k-1, \dots, 1)$ and $(n, n-1, \dots, 1)$ to $\bar{\alpha}$ and $\bar{\beta}$ respectively, we get the desired pair of partitions (α, β).

For example, let

$$k = 5, \quad n = 8, \quad \alpha = (8, 7, 5, 4, 2), \quad \beta = (20, 17, 16, 14, 12, 7, 6, 3).$$

We have

$$\begin{aligned}
(\alpha,\beta) \Leftrightarrow \quad & \bar{\alpha} = (3,3,2,2,1),\ \bar{\beta} = (12,10,10,9,8,4,4,2) \\
& T_1 = (5,4,3,2,1),\ T_2 = (8,7,6,5,4,3,2,1) \\
\overset{\text{Lemma 1}}{\Leftrightarrow} \quad & \bar{\bar{\alpha}} = (15,13,11,10,5),\ \bar{\bar{\beta}} = (10,4,2) \\
& T_1 = (5,4,3,2,1),\ T_2 = (8,7,6,5,4,3,2,1) \\
\overset{\text{Lemma 2}}{\Leftrightarrow} \quad & \bar{\mu} = (12,10,8,8,4),\ \bar{\nu} = (5,4,3),\ \bar{\bar{\beta}} = (10,4,2) \\
& T_1 = (5,4,3,2,1),\ T_2 = (8,7,6,5,4,3,2,1) \\
\Leftrightarrow \quad & \mu = (22,18,14,12,6), \nu = (18,11,8,5,4,3).
\end{aligned}$$

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Acknowledgments. We would like to thank the referees for their helpful comments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China.

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