DOUGALL-DIXON FORMULA AND HARMONIC NUMBER IDENTITIES

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Abstract. The Dougall-Dixon summation formula is reformulated in terms of binomial sums. By computing their second derivatives, we establish several harmonic number identities.

For an indeterminate x and two natural numbers n and ℓ, define the generalized harmonic numbers by

$$H_n^{\langle\ell\rangle}(x) := \sum_{k=1}^{n} \frac{1}{(k+x)^\ell} \quad\Longrightarrow\quad H_n^{\langle\ell\rangle} := H_n^{\langle\ell\rangle}(0) = \sum_{k=1}^{n} \frac{1}{k^\ell}.$$

When $\ell = 1$, they will be abbreviated as $H_n(x)$ and H_n respectively.

Given a differentiable function $f(x)$, denote two derivative operators by

$$\mathcal{D}_x f(x) = \frac{d}{dx} f(x) \quad\text{and}\quad \mathcal{D}_0 f(x) = \frac{d}{dx} f(x)\Big|_{x=0}.$$

Then the generalized harmonic numbers satisfy the following recurrence relation

$$H_n^{\langle 2\rangle}(x) = -\mathcal{D}_x H_n(x) \quad\text{and}\quad \mathcal{D}_x H_n^{\langle\ell\rangle}(x) = -\ell H_n^{\langle 1+\ell\rangle}(x).$$

These numbers come naturally from the derivatives of binomial coefficients:

$$\mathcal{D}_x \binom{n+x}{n} = H_n(x)\binom{n+x}{n} \quad\Longrightarrow\quad \mathcal{D}_0 \binom{n+x}{n} = H_n, \tag{1a}$$

$$\mathcal{D}_x \binom{n+x}{n}^{-1} = -H_n(x)\binom{n+x}{n}^{-1} \quad\Longrightarrow\quad \mathcal{D}_0 \binom{n+x}{n}^{-1} = -H_n. \tag{1b}$$

This fact can trace back to Issac Newton [12], as pointed out by Richard Askey (cf. [1]), and has been explored recently in [7] and [13].

Recall the shifted factorial

$$(c)_0 \equiv 1 \quad\text{and}\quad (c)_n = c(c+1)\cdots(c+n-1) \quad\text{for}\quad n = 1, 2, \cdots$$

and the hypergeometric series (cf. [2, §2.1])

$${}_{1+p}F_q\left[\begin{matrix} a_0, & a_1, \cdots, & a_p \\ & b_1, \cdots, & b_q \end{matrix} \,\Big|\, z\right] = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_p)_n}{n!\,(b_1)_n \cdots (b_q)_n}\, z^n.$$

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We can reproduce the Dougall-Dixon summation theorem [2, §4.3] as

$$ {}_5F_4\left[\begin{matrix}a, & 1+a/2, & b, & c, & d\\ & a/2, & 1+a-b, & 1+a-c, & 1+a-d\end{matrix}\,\Big|\,1\right] \tag{2a}$$

$$ = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} \tag{2b}$$

provided that $\Re(1+2a-b-c-d)>0$ for convergence.

Following the hypergeometric method presented in [7], this paper will further investigate the second derivative of the Dougall-Dixon formula and establish several interesting harmonic number identities. In particular, we find a hypergeometric series proof of one of the hardest challenge identities:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\Big\{3(H_k-H_{n-k})^2+\big(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle}\big)\Big\} = 0 \tag{3}$$

conjectured by Weideman [15, Eq 20] and proved subsequently by Schneider [8, Eq 16] (cf. [9, Eq 12] also) and Chu [6, Eq 0.5].

§1. Specifying the Dougall-Dixon formula (2) by

$$a\to\lambda x-n,\quad b\to\theta x-n,\quad c\to(\lambda x-n)/2,\quad d\to -n$$

we may restate it, under $\lambda=\theta+\vartheta$, as

$$ {}_3F_2\left[\begin{matrix}\lambda x-n, & \theta x-n, & -n\\ & 1+\vartheta x, & 1+\lambda x\end{matrix}\,\Big|\,1\right] = \frac{(1+\lambda x-n)_n\big(1+\frac{n+\lambda x-2\theta x}{2}\big)_n}{(1+\vartheta x)_n\big(1+\frac{\lambda x-n}{2}\big)_n}.$$

Dividing both sides by binomial coefficients $\binom{n-\lambda x}{n}\binom{n-\theta x}{n}$, we reformulate the result as the following finite sum identity:

$$\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}^3}{\binom{k+\lambda x}{k}\binom{k+\vartheta x}{k}\binom{n-k-\lambda x}{n-k}\binom{n-k-\theta x}{n-k}} = \frac{4^n\binom{\frac{n-1+\lambda x}{2}}{n}\binom{\frac{3n+(\lambda-2\theta)x}{2}}{n}}{\binom{n+\lambda x}{n}\binom{n-\lambda x}{n}\binom{n+\vartheta x}{n}\binom{n-\theta x}{n}}. \tag{4}$$

According to the parity of n, the right hand side of last equation reads explicitly as

$$\mathrm{RHS}(4) \overset{n=2m}{=\!=} (-1)^m\frac{(3m)!}{(m!)^3}\frac{\binom{3m+\frac{\lambda-2\theta}{2}x}{3m}}{\binom{m+\frac{\lambda-2\theta}{2}x}{m}\binom{m+\frac{\lambda}{2}x}{m}\binom{m-\frac{\lambda}{2}x}{m}\binom{2m+\vartheta x}{2m}\binom{2m-\theta x}{2m}},$$

$$\mathrm{RHS}(4) \overset{n=1+2m}{=\!=} (-1)^m\frac{\lambda x}{2}\frac{(3+6m)!(m!)^3}{(1+3m)!\{(1+2m)!\}^3}\frac{\binom{3+6m+(\lambda-2\theta)x}{3+6m}}{\binom{1+3m+\frac{\lambda-2\theta}{2}x}{1+3m}}$$

$$\times\quad\frac{\binom{m+\frac{\lambda}{2}x}{m}\binom{m-\frac{\lambda}{2}x}{m}\binom{m+\frac{\lambda-2\theta}{2}x}{m}}{\binom{1+2m+(\lambda-2\theta)x}{1+2m}\binom{1+2m+\lambda x}{1+2m}\binom{1+2m-\lambda x}{1+2m}\binom{1+2m+\vartheta x}{1+2m}\binom{1+2m-\theta x}{1+2m}}.$$

The case $x=0$ recovers the following well-known identity:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3 = \begin{cases}(-1)^m\frac{(3m)!}{(m!)^3}, & n=2m;\\ 0, & n=2m+1.\end{cases} \tag{5}$$

By means of (1), we write down the first derivative of (4) with respect to x:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\frac{\Omega_k(x)}{\binom{k+\lambda x}{k}\binom{k+\vartheta x}{k}\binom{n-k-\lambda x}{n-k}\binom{n-k-\theta x}{n-k}} \tag{6a}$$

$$= \mathrm{RHS}(4)\times\begin{cases}U(x), & n=2m;\\ V(x)+\frac{1}{x}, & n=2m+1,\end{cases} \tag{6b}$$

where $\Omega_k(x)$, $U(x)$ and $V(x)$ are given respectively by

$$\begin{aligned}
\Omega_k(x) &= \lambda H_{n-k}(-\lambda x)+\theta H_{n-k}(-\theta x)-\lambda H_k(\lambda x)-\vartheta H_k(\vartheta x),\\
U(x) &= \tfrac{\lambda-2\theta}{2}H_{3m}(\tfrac{\lambda-2\theta}{2}x)-\tfrac{\lambda-2\theta}{2}H_m(\tfrac{\lambda-2\theta}{2}x)-\tfrac{\lambda}{2}H_m(\tfrac{\lambda}{2}x)\\
&+ \tfrac{\lambda}{2}H_m(-\tfrac{\lambda}{2}x)-\vartheta H_{2m}(\vartheta x)+\theta H_{2m}(-\theta x),\\
V(x) &= (\lambda-2\theta)H_{3+6m}(\lambda x-2\theta x)-\tfrac{\lambda-2\theta}{2}H_{1+3m}(\tfrac{\lambda-2\theta}{2}x)+\tfrac{\lambda}{2}H_m(\tfrac{\lambda}{2}x)\\
&- \tfrac{\lambda}{2}H_m(-\tfrac{\lambda}{2}x)+\tfrac{\lambda-2\theta}{2}H_m(\tfrac{\lambda-2\theta}{2}x)-(\lambda-2\theta)H_{1+2m}(\lambda x-2\theta x)\\
&- \lambda H_{1+2m}(\lambda x)+\lambda H_{1+2m}(-\lambda x)-\vartheta H_{1+2m}(\vartheta x)+\theta H_{1+2m}(-\theta x).
\end{aligned}$$

Noting that

$$\begin{aligned}
\Omega_k(0) &= (\lambda+\theta)H_{n-k}-(\lambda+\vartheta)H_k,\\
U(0) &= \Big(\frac{\lambda}{2}-\theta\Big)\big\{H_{3m}-2H_{2m}-H_m\big\},\\
V(0) &= \Big(\frac{\lambda}{2}-\theta\Big)\big\{2H_{3+6m}-H_{1+3m}-4H_{1+2m}+H_m\big\};
\end{aligned}$$

we have the case $x=0$ of (6):

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\Big\{(\lambda+\vartheta)H_k-(\lambda+\theta)H_{n-k}\Big\}$$
$$=\begin{cases}(-1)^m\frac{(3m)!}{(m!)^3}\Big(\frac{\lambda}{2}-\theta\Big)\big\{H_m+2H_{2m}-H_{3m}\big\}, & n=2m;\\ (-1)^{m+1}\frac{\lambda}{2}\frac{(3+6m)!(m!)^3}{(1+3m)!\{(1+2m)!\}^3}, & n=2m+1.\end{cases}$$

Splitting the summand into two terms according to $(\lambda+\vartheta)H_k-(\lambda+\theta)H_{n-k}$ and then keeping the former invariant and performing involution $k\to n-k$ for the latter, we reduce the last identity to the following simplified form.

Example 1 (Harmonic number identity: cf. [9, Eq 2] for $n=2m$)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3H_k=\begin{cases}\frac{(-1)^m}{2}\frac{(3m)!}{(m!)^3}\big\{H_m+2H_{2m}-H_{3m}\big\}, & n=2m;\\ \frac{(-1)^{m+1}}{6}\frac{(3+6m)!(m!)^3}{(1+3m)!\{(1+2m)!\}^3}, & n=2m+1.\end{cases}$$

The derivative of (6), i.e., the second derivative of (4) with respect to x at $x=0$ reads as

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\Big\{\Omega_k^2(0)+\Omega_k'(0)\Big\}=\begin{cases}(-1)^m\frac{(3m)!}{(m!)^3}\Big\{U^2(0)+U'(0)\Big\}, & n=2m;\\ (-1)^m\lambda\frac{(3+6m)!(m!)^3}{(1+3m)!\{(1+2m)!\}^3}V(0), & n=2m+1.\end{cases}$$

Noting further that

$$\begin{aligned}
\Omega_k'(0) &= (\lambda^2+\vartheta^2)H_k^{\langle 2\rangle}+(\lambda^2+\theta^2)H_{n-k}^{\langle 2\rangle},\\
U'(0) &= \frac{\lambda^2}{2}H_m^{\langle 2\rangle}+\frac{\lambda^2}{2}H_{2m}^{\langle 2\rangle}-\Big(\frac{\lambda}{2}-\theta\Big)^2\big\{H_{3m}^{\langle 2\rangle}-2H_{2m}^{\langle 2\rangle}-H_m^{\langle 2\rangle}\big\};
\end{aligned}$$

we obtain the following general formula.

Theorem 1 ($\lambda = \theta + \vartheta$: Harmonic number identity)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\left\{\begin{array}{c}\big\{(\lambda+\vartheta)H_k-(\lambda+\theta)H_{n-k}\big\}^2\\ +(\lambda^2+\vartheta^2)H_k^{\langle 2\rangle}+(\lambda^2+\theta^2)H_{n-k}^{\langle 2\rangle}\end{array}\right\}$$
$$=\begin{cases}\dfrac{(-1)^m}{4}\dfrac{(3m)!}{\{m!\}^3}\left\{\begin{array}{c}2\lambda^2H_m^{\langle 2\rangle}+(\lambda-2\theta)^2(H_{3m}-2H_{2m}-H_m)^2\\ +2\lambda^2H_{2m}^{\langle 2\rangle}-(\lambda-2\theta)^2(H_{3m}^{\langle 2\rangle}-2H_{2m}^{\langle 2\rangle}-H_m^{\langle 2\rangle})\end{array}\right\}, & n=2m;\\ (-1)^m(\frac{\lambda}{2}-\theta)\lambda\frac{(3+6m)!(m!)^3}{(1+3m)!\{(1+2m)!\}^3}\left\{\begin{array}{c}H_m+2H_{3+6m}\\ -H_{1+3m}-4H_{1+2m}\end{array}\right\}, & n=2m+1.\end{cases}$$

Example 2 ($\lambda = 2$ and $\theta = \vartheta = 1$ in Theorem 1)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\Big\{9(H_k-H_{n-k})^2+5(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle})\Big\}$$
$$=\begin{cases}2(-1)^m\frac{(3m)!}{(m!)^3}\big\{H_{2m}^{\langle 2\rangle}+H_m^{\langle 2\rangle}\big\}, & n=2m;\\ 0, & n=2m+1.\end{cases}$$

Example 3 ($\lambda = 0$ and $\theta = -\vartheta = 1$ in Theorem 1)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\Big\{(H_k+H_{n-k})^2+(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle})\Big\}$$
$$=\begin{cases}(-1)^m\frac{(3m)!}{(m!)^3}\left\{\begin{array}{c}(H_m+2H_{2m}-H_{3m})^2\\ +\big(H_m^{\langle 2\rangle}+2H_{2m}^{\langle 2\rangle}-H_{3m}^{\langle 2\rangle}\big)\end{array}\right\}, & n=2m;\\ 0, & n=2m+1.\end{cases}$$

Example 4 ($\lambda = -\theta = 1$ and $\vartheta = 2$ in Theorem 1)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\Big\{9H_k^2+(5H_k^{\langle 2\rangle}+2H_{n-k}^{\langle 2\rangle})\Big\}$$
$$=\begin{cases}\frac{(-1)^m}{4}\frac{(3m)!}{(m!)^3}\left\{\begin{array}{c}9(H_m+2H_{2m}-H_{3m})^2\\ +11H_m^{\langle 2\rangle}+20H_{2m}^{\langle 2\rangle}-9H_{3m}^{\langle 2\rangle}\end{array}\right\}, & n=2m;\\ \frac{3}{2}(-1)^m\frac{(3+6m)!(m!)^3}{(1+3m)!\{(1+2m)!\}^3}\left\{\begin{array}{c}H_m+2H_{3+6m}\\ -H_{1+3m}-4H_{1+2m}\end{array}\right\}, & n=2m+1.\end{cases}$$

§2. Specifying the Dougall-Dixon formula (2) by

$$a\to -n,\quad b\to -\lambda x-n,\quad c\to -n/2,\quad d\to -\theta x-n$$

we may restate it as

$${}_3F_2\left[\begin{matrix}-n, & -\lambda x-n, & -\theta x-n\\ & 1+\lambda x, & 1+\theta x\end{matrix}\,\middle|\,1\right]=\begin{cases}(-4)^m\frac{(\frac{1}{2})_m(1+2m+\lambda x+\theta x)_m}{(1+\lambda x)_m(1+\theta x)_m}, & n=2m;\\ 0, & n=2m+1.\end{cases}$$

Here we consider only the case $n = 2m$ because the case $n = 1 + 2m$ does not produce any interesting result on harmonic numbers.

Dividing both sides by binomial coefficients $\binom{n+\lambda x}{n}\binom{n+\theta x}{n}$, we reformulate the result as the following finite sum identity:

$$\sum_{k=0}^{2m}\frac{(-1)^k\binom{2m}{k}^3}{\binom{k+\lambda x}{k}\binom{k+\theta x}{k}\binom{2m-k+\lambda x}{2m-k}\binom{2m-k+\theta x}{2m-k}}=\frac{(-1)^m\binom{3m}{m,m,m}\binom{3m+\lambda x+\theta x}{3m}}{\binom{m+\lambda x}{m}\binom{2m+\lambda x}{2m}\binom{m+\theta x}{m}\binom{2m+\theta x}{2m}\binom{2m+\lambda x+\theta x}{2m}}. \tag{7}$$

By means of (1), we write down its first derivative with respect to x:

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\frac{\Omega_k(x)}{\binom{k+\lambda x}{k}\binom{k+\theta x}{k}\binom{2m-k+\lambda x}{2m-k}\binom{2m-k+\theta x}{2m-k}} \tag{8a}$$

$$=\frac{(-1)^m\binom{3m}{m,m,m}\binom{3m+\lambda x+\theta x}{3m}W(x)}{\binom{m+\lambda x}{m}\binom{2m+\lambda x}{2m}\binom{m+\theta x}{m}\binom{2m+\theta x}{2m}\binom{2m+\lambda x+\theta x}{2m}}. \tag{8b}$$

where $\Omega_k(x)$ and $W(x)$ are explicitly given by

$$\begin{aligned}\Omega_k(x) &= \lambda\{H_k(\lambda x)+H_{2m-k}(\lambda x)\}+\theta\{H_k(\theta x)+H_{2m-k}(\theta x)\},\\ W(x) &= \lambda\{H_m(\lambda x)+H_{2m}(\lambda x)\}+\theta\{H_m(\theta x)+H_{2m}(\theta x)\}\\ &+ (\lambda+\theta)\{H_{2m}(\lambda x+\theta x)-H_{3m}(\lambda x+\theta x)\}.\end{aligned}$$

Noting that

$$\begin{aligned}\Omega_k(0) &= (\lambda+\theta)\Big\{H_k+H_{2m-k}\Big\},\\ \Omega'_k(0) &= -(\lambda^2+\theta^2)\Big\{H_k^{\langle 2\rangle}+H_{2m-k}^{\langle 2\rangle}\Big\};\\ W(0) &= (\lambda+\theta)\Big\{H_m+2H_{2m}-H_{3m}\Big\},\\ W'(0) &= -(\lambda+\theta)^2\{H_{2m}^{\langle 2\rangle}-H_{3m}^{\langle 2\rangle}\}-(\lambda^2+\theta^2)\{H_m^{\langle 2\rangle}+H_{2m}^{\langle 2\rangle}\};\end{aligned}$$

we may compute the derivative of (8), i.e., the second derivative of (7) with respect to x at $x=0$:

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\Big\{\Omega_k^2(0)-\Omega'_k(0)\Big\}=(-1)^m\frac{(3m)!}{(m!)^3}\Big\{W^2(0)-W'(0)\Big\}$$

from which we derive the following formula.

Theorem 2 (Harmonic number identity)**.**

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\left\{\begin{array}{c}(\lambda+\theta)^2(H_k+H_{2m-k})^2\\ +(\lambda^2+\theta^2)\big(H_k^{\langle 2\rangle}+H_{2m-k}^{\langle 2\rangle}\big)\end{array}\right\}$$

$$=(-1)^m\frac{(3m)!}{(m!)^3}\left\{\begin{array}{c}(\lambda+\theta)^2(H_m+2H_{2m}-H_{3m})^2\\ +(\lambda+\theta)^2\big(H_{2m}^{\langle 2\rangle}-H_{3m}^{\langle 2\rangle}\big)\\ +(\lambda^2+\theta^2)\big(H_m^{\langle 2\rangle}+H_{2m}^{\langle 2\rangle}\big)\end{array}\right\}.$$

Putting $\lambda=0$ and $\theta=1$ in Theorem 2, we recover again the harmonic number identity stated in Example 3.

Example 5 ($\lambda=1$ and $\theta=-1$ in Theorem 2: cf. [9, Eq 3])**.**

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\Big\{H_k^{\langle 2\rangle}+H_{2m-k}^{\langle 2\rangle}\Big\}=(-1)^m\tfrac{(3m)!}{(m!)^3}\{H_m^{\langle 2\rangle}+H_{2m}^{\langle 2\rangle}\}.$$

Example 6 ($\lambda=\theta=1$ in Theorem 2)**.**

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\Big\{2(H_k+H_{2m-k})^2+\Big(H_k^{\langle 2\rangle}+H_{2m-k}^{\langle 2\rangle}\Big)\Big\}$$
$$=(-1)^m\tfrac{(3m)!}{(m!)^3}\left\{\begin{array}{c}2(H_m+2H_{2m}-H_{3m})^2\\ +\big(H_m^{\langle 2\rangle}+3H_{2m}^{\langle 2\rangle}-2H_{3m}^{\langle 2\rangle}\big)\end{array}\right\}.$$

Example 7 ($\lambda=2$ and $\theta=1$ in Theorem 2)**.**

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\Big\{9(H_k+H_{2m-k})^2+5(H_k^{\langle 2\rangle}+H_{2m-k}^{\langle 2\rangle})\Big\}$$
$$=\ (-1)^m\frac{(3m)!}{(m!)^3}\left\{\begin{array}{c}9(H_m+2H_{2m}-H_{3m})^2\\ +5H_m^{\langle 2\rangle}+14H_{2m}^{\langle 2\rangle}-9H_{3m}^{\langle 2\rangle}\end{array}\right\}.$$

§3. Subtracting twice Example 5 from Example 2, we obtain immediately Weideman's identity (3). Combining the formulae established in the previous examples, we further display the harmonic number identities in the following short list.

Example 8 (Example 5 with involution)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\Big\{H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle}\Big\}\ =\begin{cases}(-1)^m\frac{(3m)!}{(m!)^3}\Big\{H_m^{\langle 2\rangle}+H_{2m}^{\langle 2\rangle}\Big\}, & n=2m;\\ 0, & n=2m+1.\end{cases}$$

Example 9 (Example 2 minus five times of Example 5)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(H_k-H_{n-k})^2\ =\begin{cases}\frac{(-1)^{m+1}}{3}\frac{(3m)!}{(m!)^3}\Big\{H_m^{\langle 2\rangle}+H_{2m}^{\langle 2\rangle}\Big\}, & n=2m;\\ 0, & n=2m+1.\end{cases}$$

Example 10 (Example 6 minus Example 5)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(H_k+H_{n-k})^2=\begin{cases}(-1)^m\frac{(3m)!}{(m!)^3}\left\{\begin{array}{c}(H_m+2H_{2m}-H_{3m})^2\\ +\big(H_{2m}^{\langle 2\rangle}-H_{3m}^{\langle 2\rangle}\big)\end{array}\right\}, & n=2m;\\ 0, & n=2m+1.\end{cases}$$

Example 11 (Example 9 plus Example 10: cf. [9, Eq 23])**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(H_k^2+H_{n-k}^2)=\begin{cases}\frac{(-1)^m}{6}\frac{(3m)!}{(m!)^3}\left\{\begin{array}{c}3(H_m+2H_{2m}-H_{3m})^2\\ -\big(H_m^{\langle 2\rangle}-2H_{2m}^{\langle 2\rangle}+3H_{3m}^{\langle 2\rangle}\big)\end{array}\right\}, & n=2m;\\ 0, & n=2m+1.\end{cases}$$

Example 12 (Example 9 minus Example 10: cf. [9, Eq 22])**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3H_kH_{n-k}=\begin{cases}\frac{(-1)^m}{12}\frac{(3m)!}{(m!)^3}\left\{\begin{array}{c}3(H_m+2H_{2m}-H_{3m})^2\\ +\big(H_m^{\langle 2\rangle}+4H_{2m}^{\langle 2\rangle}-3H_{3m}^{\langle 2\rangle}\big)\end{array}\right\}, & n=2m;\\ 0, & n=2m+1.\end{cases}$$

The factor $H_k^2+H_k^{\langle 2\rangle}$ is separatable, in case $n=2m$, as follows.

Example 13 (Example 8 with involution).

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3 H_k^{\langle 2\rangle} \;=\; \frac{(-1)^m}{2}\frac{(3m)!}{(m!)^3}\Big\{H_m^{\langle 2\rangle}+H_{2m}^{\langle 2\rangle}\Big\}.$$

Example 14 (Example 11 with involution).

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3 H_k^2 \;=\; \frac{(-1)^m}{12}\frac{(3m)!}{(m!)^3}\left\{\begin{matrix}3(H_m+2H_{2m}-H_{3m})^2\\ -\big(H_m^{\langle 2\rangle}-2H_{2m}^{\langle 2\rangle}+3H_{3m}^{\langle 2\rangle}\big)\end{matrix}\right\}.$$

When n is odd, we are not able to split the factor $H_k^2+H_k^{\langle 2\rangle}$. However, we do have the following interesting result.

Example 15 (Example 4 with involution).

$$\sum_{k=0}^{1+2m}(-1)^k\binom{1+2m}{k}^3\Big\{3H_k^2+H_k^{\langle 2\rangle}\Big\} \;=\; \frac{(-1)^m(3+6m)!(m!)^3}{2(1+3m)!\big\{(1+2m)!\big\}^3}\left\{\begin{matrix}H_m+2H_{3+6m}\\ -H_{1+3m}-4H_{1+2m}\end{matrix}\right\}.$$

§4. Specifying the Dougall-Dixon formula (2) by

$$a\to\lambda x-n,\quad b\to\theta x-n,\quad c\to(1+\lambda x-n)/2,\quad d\to -n$$

we may restate it, with $\lambda=\theta+\vartheta$, as

$${}_4F_3\left[\begin{matrix}\lambda x-n,\ 1+(\lambda x-n)/2,\ \theta x-n,\quad -n\\ (\lambda x-n)/2,\quad 1+\vartheta x,\ 1+\lambda x\end{matrix}\ \Big|\ 1\right] \;=\; \frac{(1-n+\lambda x)_n\left(\frac{1+n+\lambda x-2\theta x}{2}\right)_n}{(1+\vartheta x)_n\left(\frac{1-n+\lambda x}{2}\right)_n}.$$

Dividing both sides by binomial coefficients $\binom{n-\lambda x}{n}\binom{n-\theta x}{n}$, we reformulate the result as the following finite sum identity:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\frac{(\lambda x+2k-n)}{\binom{k+\lambda x}{k}\binom{k+\vartheta x}{k}\binom{n-k-\lambda x}{n-k}\binom{n-k-\theta x}{n-k}} \;=\; \frac{(-1)^n\lambda x\binom{\frac{3n-1+\lambda x-2\theta x}{2}}{n}}{\binom{n+\vartheta x}{n}\binom{n-\theta x}{n}\binom{\frac{n-1+\lambda x}{2}}{n}}. \tag{9}$$

According to the parity of n, the right hand side of last equation reads explicitly as

$$\begin{aligned}\mathrm{RHS}(9) \;&\overset{n=2m}{=\!=\!=}\; (-1)^m\lambda x\frac{(6m)!(m!)^3}{(3m)!\{(2m)!\}^3}\frac{\binom{m+\frac{\lambda-2\theta}{2}x}{m}\binom{6m+\lambda x-2\theta x}{6m}}{\binom{2m+\lambda x-2\theta x}{2m}\binom{3m+\frac{\lambda-2\theta}{2}x}{3m}}\\ &\times\quad\frac{\binom{m-\frac{\lambda x}{2}}{m}\binom{m+\frac{\lambda x}{2}}{m}}{\binom{2m+\lambda x}{2m}\binom{2m-\lambda x}{2m}\binom{2m+\vartheta x}{2m}\binom{2m-\theta x}{2m}},\end{aligned}$$

$$\begin{aligned}\mathrm{RHS}(9) \;&\overset{n=1+2m}{=\!=\!=}\; 2(-1)^{m+1}\frac{(1+3m)!}{(m!)^3}\frac{\binom{1+3m+\frac{\lambda-2\theta}{2}x}{1+3m}}{\binom{m+\frac{\lambda-2\theta}{2}x}{m}\binom{m+\frac{\lambda x}{2}}{m}\binom{m-\frac{\lambda x}{2}}{m}}\\ &\times\quad\frac{1}{\binom{1+2m+\vartheta x}{1+2m}\binom{1+2m-\theta x}{1+2m}}.\end{aligned}$$

The case $x=0$ reads as follows:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(n-2k) \;=\; \begin{cases}0, & n=2m;\\ 2(-1)^m\frac{(1+3m)!}{(m!)^3}, & n=2m+1.\end{cases} \tag{10}$$

By means of (1), we write down the first derivative of (9) with respect to x:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\frac{\lambda+(\lambda x+2k-n)\Omega_k(x)}{\binom{k+\lambda x}{k}\binom{k+\vartheta x}{k}\binom{n-k-\lambda x}{n-k}\binom{n-k-\theta x}{n-k}} \tag{11a}$$

$$= \text{RHS}(9)\times\begin{cases}U(x)+\frac{1}{x}, & n=2m;\\ V(x), & n=2m+1,\end{cases} \tag{11b}$$

where $\Omega_k(x)$, $U(x)$ and $V(x)$ are given respectively by

$$\begin{aligned}
\Omega_k(x) &= -\lambda H_k(\lambda x)-\vartheta H_k(\vartheta x)+\lambda H_{n-k}(-\lambda x)+\theta H_{n-k}(-\theta x),\\
U(x) &= \tfrac{\lambda-2\theta}{2}H_m(\tfrac{\lambda-2\theta}{2}x)+(\lambda-2\theta)H_{6m}(\lambda x-2\theta x)-\tfrac{\lambda}{2}H_m(-\tfrac{\lambda x}{2})\\
&+ \tfrac{\lambda}{2}H_m(\tfrac{\lambda x}{2})-(\lambda-2\theta)H_{2m}(\lambda x-2\theta x)-\tfrac{\lambda-2\theta}{2}H_{3m}(\tfrac{\lambda-2\theta}{2}x)\\
&- \lambda H_{2m}(\lambda x)+\lambda H_{2m}(-\lambda x)-\vartheta H_{2m}(\vartheta x)+\theta H_{2m}(-\theta x),\\
V(x) &= \tfrac{\lambda-2\theta}{2}H_{1+3m}(\tfrac{\lambda-2\theta}{2}x)-\tfrac{\lambda-2\theta}{2}H_m(\tfrac{\lambda-2\theta}{2}x)\\
&- \tfrac{\lambda}{2}H_m(\tfrac{\lambda}{2}x)+\tfrac{\lambda}{2}H_m(-\tfrac{\lambda}{2}x)-\vartheta H_{1+2m}(\vartheta x)+\theta H_{1+2m}(-\theta x).
\end{aligned}$$

Noting that

$$\begin{aligned}
\Omega_k(0) &= (\lambda+\theta)H_{n-k}-(\lambda+\vartheta)H_k,\\
U(0) &= \Big(\frac{\lambda}{2}-\theta\Big)\Big\{2H_{6m}-H_{3m}-4H_{2m}+H_m\Big\},\\
V(0) &= \Big(\frac{\lambda}{2}-\theta\Big)\Big\{H_{1+3m}-2H_{1+2m}-H_m\Big\};
\end{aligned}$$

and then recalling (5), we have the case $x=0$ of (11) as follows:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(n-2k)\Big\{(\lambda+\vartheta)H_k-(\lambda+\theta)H_{n-k}\Big\} \tag{12a}$$

$$=\begin{cases}(-1)^m\lambda\left\{\frac{(6m)!(m!)^3}{(3m)!\{(2m)!\}^3}-\frac{(3m)!}{(m!)^3}\right\}, & n=2m;\\ (-1)^m(\lambda-2\theta)\frac{(1+3m)!}{(m!)^3}\Big\{H_m+2H_{1+2m}-H_{1+3m}\Big\}, & n=2m+1.\end{cases} \tag{12b}$$

Example 16 ($\lambda=-\theta=1$ and $\vartheta=2$ in (12))**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(n-2k)H_k=\begin{cases}\frac{(-1)^m}{3}\left\{\frac{(6m)!(m!)^3}{(3m)!\{(2m)!\}^3}-\frac{(3m)!}{(m!)^3}\right\}, & n=2m;\\ (-1)^m\frac{(1+3m)!}{(m!)^3}\Big\{H_m+2H_{1+2m}-H_{1+3m}\Big\}, & n=2m+1.\end{cases}$$

The derivative of (11), i.e., the second derivative of (9) with respect to x at $x=0$ reads as

$$2\lambda\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\Omega_k(0)+\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(2k-n)\Big\{\Omega_k^2(0)+\Omega_k'(0)\Big\}$$

$$=\begin{cases}(-1)^m2\lambda\frac{(6m)!(m!)^3}{(3m)!\{(2m)!\}^3}U(0), & n=2m;\\ (-1)^{m+1}2\frac{(1+3m)!}{(m!)^3}\Big\{V^2(0)+V'(0)\Big\}, & n=2m+1.\end{cases}$$

Noting further that

$$\begin{aligned}
\Omega_k'(0) &= (\lambda^2+\vartheta^2)H_k^{\langle 2\rangle}+(\lambda^2+\theta^2)H_{n-k}^{\langle 2\rangle},\\
V'(0) &= -\Big(\frac{\lambda}{2}-\theta\Big)^2\Big\{H_{1+3m}^{\langle 2\rangle}-2H_{1+2m}^{\langle 2\rangle}-H_m^{\langle 2\rangle}\Big\}+\frac{\lambda^2}{2}\Big\{H_m^{\langle 2\rangle}+H_{1+2m}^{\langle 2\rangle}\Big\};
\end{aligned}$$

and then appealing Example 1, we have the following general formula.

Theorem 3 ($\lambda=\theta+\vartheta$: Harmonic number identity)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(n-2k)\left\{\begin{matrix}\big\{(\lambda+\vartheta)H_k-(\lambda+\theta)H_{n-k}\big\}^2\\ +(\lambda^2+\vartheta^2)H_k^{\langle 2\rangle}+(\lambda^2+\theta^2)H_{n-k}^{\langle 2\rangle}\end{matrix}\right\}$$
$$=\begin{cases}(-1)^m\lambda(\lambda-2\theta)\left\{\begin{matrix}\frac{(6m)!m!^3}{(3m)!\{(2m)!\}^3}\big(H_{3m}+4H_{2m}-2H_{6m}-H_m\big)\\ +\frac{(3m)!}{(m!)^3}\big(H_{3m}-2H_{2m}-H_m\big)\end{matrix}\right\}, & n=2m;\\ \frac{(-1)^m}{2}\left\{\begin{matrix}\frac{2\lambda^2(3+6m)!(m!)^3}{(1+3m)!\{(1+2m)!\}^3}+\frac{(1+3m)!}{(m!)^3}\Big\{2\lambda^2(H_m^{\langle 2\rangle}+H_{1+2m}^{\langle 2\rangle})\\ +(\lambda-2\theta)^2(H_{1+3m}-2H_{1+2m}-H_m)^2\\ -(\lambda-2\theta)^2(H_{1+3m}^{\langle 2\rangle}-2H_{1+2m}^{\langle 2\rangle}-H_m^{\langle 2\rangle})\Big\}\end{matrix}\right\}, & n=2m+1.\end{cases}$$

Example 17 ($\theta=\vartheta=1$ and $\lambda=2$ in Theorem 3)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(n-2k)\Big\{9(H_k-H_{n-k})^2+5(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle})\Big\}$$
$$=\begin{cases}0, & n=2m;\\ 4(-1)^m\Big\{\frac{(1+3m)!}{(m!)^3}(H_m^{\langle 2\rangle}+H_{1+2m}^{\langle 2\rangle})+\frac{(3+6m)!(m!)^3}{(1+3m)!\{(1+2m)!\}^3}\Big\}, & n=2m+1.\end{cases}$$

Example 18 ($\theta=-\vartheta=1$ and $\lambda=0$ in Theorem 3)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(n-2k)\Big\{(H_k+H_{n-k})^2+(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle})\Big\}$$
$$=\begin{cases}0, & n=2m;\\ (-1)^m\frac{(1+3m)!}{(m!)^3}\left\{\begin{matrix}2(H_{1+3m}-2H_{1+2m}-H_m)^2\\ +4H_{1+2m}^{\langle 2\rangle}-2H_{1+3m}^{\langle 2\rangle}+2H_m^{\langle 2\rangle}\end{matrix}\right\}, & n=2m+1.\end{cases}$$

§5. Specifying the Dougall-Dixon formula (2) by

$$a\to\lambda x-n,\quad b\to\theta x-n,\quad c\to-n,\quad d\to\infty$$

we may restate it, with $\lambda=\theta+\vartheta$, as

$${}_4F_3\left[\begin{matrix}\lambda x-n,\ 1+(\lambda x-n)/2,\ \theta x-n,\quad -n\\ (\lambda x-n)/2,\quad 1+\vartheta x,\ 1+\lambda x\end{matrix}\ \Big|\ -1\right]=\frac{(1+\lambda x-n)_n}{(1+\vartheta x)_n}.$$

Dividing both sides by binomial coefficients $\binom{n-\lambda x}{n}\binom{n-\theta x}{n}$, we reformulate the result as the following finite sum identity:

$$\sum_{k=0}^{n}\binom{n}{k}^3\frac{(\lambda x+2k-n)}{\binom{k+\lambda x}{k}\binom{k+\vartheta x}{k}\binom{n-k-\lambda x}{n-k}\binom{n-k-\theta x}{n-k}}=\frac{(-1)^n\,\lambda x}{\binom{n-\theta x}{n}\binom{n+\vartheta x}{n}}.\tag{13}$$

By means of (1), we write down the first derivative of (13) with respect to x:

$$\sum_{k=0}^{n}\binom{n}{k}^3\frac{\lambda+(\lambda x+2k-n)\Omega_k(x)}{\binom{k+\lambda x}{k}\binom{k+\vartheta x}{k}\binom{n-k-\lambda x}{n-k}\binom{n-k-\theta x}{n-k}}=\frac{(-1)^n\,\lambda\big\{1+xW(x)\big\}}{\binom{n-\theta x}{n}\binom{n+\vartheta x}{n}}\tag{14}$$

where $\Omega_k(x)$ and $W(x)$ are given respectively by

$$\begin{aligned}\Omega_k(x)&=-\lambda H_k(\lambda x)-\vartheta H_k(\vartheta x)+\lambda H_{n-k}(-\lambda x)+\theta H_{n-k}(-\theta x),\\ W(x)&=\theta H_n(-\theta x)-\vartheta H_n(\vartheta x);\end{aligned}$$

with the following particulalr values:

$$\begin{aligned}\Omega_k(0) &= (\lambda+\theta)H_{n-k}-(\lambda+\vartheta)H_k,\\ \Omega'_k(0) &= (\lambda^2+\vartheta^2)H_k^{\langle 2\rangle}+(\lambda^2+\theta^2)H_{n-k}^{\langle 2\rangle},\\ W(0) &= (\theta-\vartheta)H_n.\end{aligned}$$

The case $x=0$ of (14) reduces to

$$\lambda\sum_{k=0}^{n}\binom{n}{k}^3+\sum_{k=0}^{n}(n-2k)\binom{n}{k}^3\Big\{(\lambda+\vartheta)H_k-(\lambda+\theta)H_{n-k}\Big\}=(-1)^n\lambda.$$

Under involution $k\to n-k$, we may reformulate it as the following identity.

Example 19 (Paule and Schneider [13, Eq 3], cf. also Chu and De Donno [7, I-16])**.**

$$\sum_{k=0}^{n}\binom{n}{k}^3\big\{1+3(n-2k)H_k\big\} = (-1)^n.$$

Similarly, the derivative of (13), i.e., the second derivative of (14) with respect to x at $x=0$ reads as

$$2\lambda\sum_{k=0}^{n}\binom{n}{k}^3\Omega_k(0)+\sum_{k=0}^{n}\binom{n}{k}^3(2k-n)\Big\{\Omega_k^2(0)+\Omega'_k(0)\Big\}=2(-1)^n\lambda W(0),$$

which leads us to the following general formula.

$$\sum_{k=0}^{n}\binom{n}{k}^3\left\{\begin{array}{c}2\lambda\big\{(\lambda+\vartheta)H_k-(\lambda+\theta)H_{n-k}\big\}\\ +(n-2k)\big\{(\lambda+\vartheta)H_k-(\lambda+\theta)H_{n-k}\big\}^2\\ +(n-2k)\big\{(\lambda^2+\vartheta^2)H_k^{\langle 2\rangle}+(\lambda^2+\theta^2)H_{n-k}^{\langle 2\rangle}\big\}\end{array}\right\} = 2(-1)^n\lambda(\vartheta-\theta)H_n.$$

By means of $k\to n-k$, this formula can be simplified as the following identity.

Example 20.

$$\sum_{k=0}^{n}\binom{n}{k}^3\Big\{H_k+(\tfrac{n}{2}-k)(3H_k^2+H_k^{\langle 2\rangle})\Big\} = (-1)^nH_n.$$

§6. Specifying the Dougall-Dixon formula (2) by

$$a\to\lambda x-n,\quad b\to 1+\frac{\lambda x-n}{2},\quad c\to\theta x-n,\quad d\to -n$$

we may restate it, with $\lambda=\theta+\vartheta$, as

$$\begin{aligned}&{}_5F_4\left[\begin{matrix}\lambda x-n,\ 1+(\lambda x-n)/2,\ 1+(\lambda x-n)/2,\ \theta x-n,\quad -n\\ (\lambda x-n)/2,\qquad (\lambda x-n)/2,\quad 1+\vartheta x,\ 1+\lambda x\end{matrix}\,\middle|\,1\right]\\ &= \frac{(1+\lambda x-n)_n(\frac{n+\lambda x-2\theta x}{2})_n}{(1+\vartheta x)_n(\frac{\lambda x-n}{2})_n}.\end{aligned}$$

Dividing both sides by binomial coefficients $\binom{n-\lambda x}{n}\binom{n-\theta x}{n}$, we reformulate the result as the following finite sum identity:

$$\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}^3(\lambda x+2k-n)^2}{\binom{k+\lambda x}{k}\binom{k+\vartheta x}{k}\binom{n-k-\lambda x}{n-k}\binom{n-k-\theta x}{n-k}} = (-1)^n\lambda x\frac{(\lambda x-n)\binom{\frac{3n-2+\lambda x-2\theta x}{2}}{n}}{\binom{n-\theta x}{n}\binom{n+\vartheta x}{n}\binom{\frac{n-2+\lambda x}{2}}{n}}. \tag{15}$$

According to the parity of n, the right hand side of last equation reads explicitly as

$$\text{RHS(15)} \overset{n=2m}{=\!=\!=} \frac{(-1)^m}{3}\frac{(3m)!}{(m!)^3}\frac{(\lambda^2x^2-4m^2)\binom{3m-1+\frac{\lambda-2\theta}{2}x}{3m-1}}{\binom{m-1+\frac{\lambda-2\theta}{2}x}{m-1}\binom{2m-\theta x}{2m}\binom{2m+\vartheta x}{2m}\binom{m+\frac{\lambda}{2}x}{m}\binom{m-\frac{\lambda}{2}x}{m}},$$

$$\begin{aligned}\text{RHS(15)} \overset{n=1+2m}{=\!=\!=} &(-1)^m\frac{\lambda x}{6}\frac{(3+6m)!(m!)^3}{(1+3m)!\{(1+2m)!\}^3}\frac{\binom{2+6m+\lambda x-2\theta x}{2+6m}}{\binom{1+3m+\frac{\lambda-2\theta}{2}x}{1+3m}}\\ \times &\frac{\{\lambda^2x^2-(1+2m)^2\}\binom{m+\frac{\lambda-2\theta}{2}x}{m}\binom{m+\frac{\lambda}{2}x}{m}\binom{m-\frac{\lambda}{2}x}{m}}{\binom{1+2m+\lambda x}{1+2m}\binom{1+2m-\lambda x}{1+2m}\binom{1+2m+\vartheta x}{1+2m}\binom{1+2m-\theta x}{1+2m}\binom{2m+\lambda x-2\theta x}{2m}}.\end{aligned}$$

The case $x=0$ reads as follows:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(n-2k)^2 = \begin{cases}(-1)^{m+1}\frac{4m^2}{3}\frac{(3m)!}{(m!)^3}, & n=2m;\\ 0, & n=2m+1.\end{cases} \tag{16}$$

By means of (1), we write down the first derivative of (15) with respect to x:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\frac{2\lambda(\lambda x+2k-n)+(\lambda x+2k-n)^2\Omega_k(x)}{\binom{k+\lambda x}{k}\binom{k+\vartheta x}{k}\binom{n-k-\lambda x}{n-k}\binom{n-k-\theta x}{n-k}} \tag{17a}$$

$$= \text{RHS(15)}\times\begin{cases}U(x)+\frac{2\lambda^2x}{\lambda^2x^2-4m^2}, & n=2m;\\ V(x)+\frac{1}{x}+\frac{2\lambda^2x}{\lambda^2x^2-(1+2m)^2}, & n=2m+1\end{cases} \tag{17b}$$

where $\Omega_k(x)$, $U(x)$ and $V(x)$ are given respectively by

$$\begin{aligned}\Omega_k(x) &= \lambda H_{n-k}(-\lambda x)+\theta H_{n-k}(-\theta x)-\lambda H_k(\lambda x)-\vartheta H_k(\vartheta x),\\ U(x) &= \tfrac{\lambda-2\theta}{2}H_{3m-1}(\tfrac{\lambda-2\theta}{2}x)+\theta H_{2m}(-\theta x)-\vartheta H_{2m}(\vartheta x)\\ &- \tfrac{\lambda-2\theta}{2}H_{m-1}(\tfrac{\lambda-2\theta}{2}x)-\tfrac{\lambda}{2}H_m(\tfrac{\lambda}{2}x)+\tfrac{\lambda}{2}H_m(-\tfrac{\lambda}{2}x),\\ V(x) &= (\lambda-2\theta)H_{2+6m}(\lambda x-2\theta x)-\tfrac{\lambda-2\theta}{2}H_{1+3m}(\tfrac{\lambda-2\theta}{2}x)\\ &+ \tfrac{\lambda-2\theta}{2}H_m(\tfrac{\lambda-2\theta}{2}x)+\tfrac{\lambda}{2}H_m(\tfrac{\lambda}{2}x)-\tfrac{\lambda}{2}H_m(-\tfrac{\lambda}{2}x)-(\lambda-2\theta)H_{2m}(\lambda x-2\theta x)\\ &- \lambda H_{1+2m}(\lambda x)+\lambda H_{1+2m}(-\lambda x)+\theta H_{1+2m}(-\theta x)-\vartheta H_{1+2m}(\vartheta x).\end{aligned}$$

Noting that

$$\begin{aligned}\Omega_k(0) &= (\lambda+\theta)H_{n-k}-(\lambda+\vartheta)H_k,\\ U(0) &= \Big(\frac{\lambda}{2}-\theta\Big)\Big\{H_{3m-1}-2H_{2m}-H_{m-1}\Big\},\\ V(0) &= \Big(\frac{\lambda}{2}-\theta\Big)\Big\{2(H_{2+6m}-H_{1+2m}-H_{2m})-(H_{1+3m}-H_m)\Big\};\end{aligned}$$

and then applying (10), we have the case $x=0$ of (17):

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(n-2k)^2\Big\{(\lambda+\vartheta)H_k-(\lambda+\theta)H_{n-k}\Big\}\\ &=\begin{cases}(-1)^m(\lambda-2\theta)\frac{2m^2}{3}\frac{(3m)!}{(m!)^3}(H_{3m-1}-2H_{2m}-H_{m-1}), & n=2m;\\ (-1)^m\lambda\Big\{\frac{(1+6m)!(m!)^3}{(3m)!(2m)!^3}-4\frac{(1+3m)!}{(m!)^3}\Big\}, & n=2m+1.\end{cases}\end{aligned}$$

Under involution $k\to n-k$, it simplifies as follows.

Example 21.

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(n-2k)^2H_k=\begin{cases}(-1)^m\frac{2m^2}{3}\frac{(3m)!}{(m!)^3}\big(H_{3m-1}-2H_{2m}-H_{m-1}\big), & n=2m;\\ \frac{(-1)^m}{3}\Big\{\frac{(1+6m)!(m!)^3}{(3m)!(2m)!^3}-4\frac{(1+3m)!}{(m!)^3}\Big\}, & n=2m+1.\end{cases}$$

The derivative of (17), i.e., the second derivative of (15) with respect to x at $x=0$ reads as

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\Big\{2\lambda^2-4\lambda(n-2k)\Omega_k(0)+(n-2k)^2\big\{\Omega_k^2(0)+\Omega_k'(0)\big\}\Big\}$$
$$=\begin{cases}(-1)^m\frac{(3m)!}{(m!)^3}\Big\{\frac{2\lambda^2}{3}-\frac{4m^2}{3}\Big(U^2(0)+U'(0)\Big)\Big\}, & n=2m;\\ (-1)^{m+1}\frac{\lambda(1+2m)^2}{3}\frac{(3+6m)!(m!)^3}{(1+3m)!\{(1+2m)!\}^3}V(0), & n=2m+1.\end{cases}$$

Note further that

$$\Omega_k'(0) = (\lambda^2+\vartheta^2)H_k^{\langle 2\rangle}+(\lambda^2+\theta^2)H_{n-k}^{\langle 2\rangle},$$
$$U'(0) = \frac{\lambda^2}{2}\Big\{H_m^{\langle 2\rangle}+H_{2m}^{\langle 2\rangle}\Big\}-\Big(\frac{\lambda}{2}-\theta\Big)^2\Big\{H_{3m-1}^{\langle 2\rangle}-2H_{2m}^{\langle 2\rangle}-H_{m-1}^{\langle 2\rangle}\Big\};$$

and then appealing (5) and (12), we get the following general formula.

Theorem 4 ($\lambda=\theta+\vartheta$: Harmonic number identity)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(n-2k)^2\left\{\begin{matrix}(\lambda^2+\vartheta^2)H_k^{\langle 2\rangle}+(\lambda^2+\theta^2)H_{n-k}^{\langle 2\rangle}\\ +\big\{(\lambda+\vartheta)H_k-(\lambda+\theta)H_{n-k}\big\}^2\end{matrix}\right\}$$
$$=\begin{cases}(-1)^{m+1}\left\{\begin{matrix}4\lambda^2\frac{(6m)!(m!)^3}{(3m)!\{(2m)!\}^3}-\frac{(3m)!}{3(m!)^3}\Big\{8\lambda^2-2m^2\lambda^2(H_m^{\langle 2\rangle}+H_{2m}^{\langle 2\rangle})\\ -m^2(\lambda-2\theta)^2\big(H_{3m-1}-2H_{2m}-H_{m-1}\big)^2\\ +m^2(\lambda-2\theta)^2\big(H_{3m-1}^{\langle 2\rangle}-2H_{2m}^{\langle 2\rangle}-H_{m-1}^{\langle 2\rangle}\big)\Big\}\end{matrix}\right\}, & n=2m;\\ (-1)^{m+1}\lambda(\lambda-2\theta)\left\{\begin{matrix}\frac{(1+6m)!(m!)^3}{(3m)!\{(2m)!\}^3}\Big\{(H_m-H_{1+3m})\\ +2(H_{2+6m}-H_{1+2m}-H_{2m})\Big\}\\ +4\frac{(1+3m)!}{(m!)^3}(2H_{1+2m}+H_m-H_{1+3m})\end{matrix}\right\}, & n=2m+1.\end{cases}$$

Example 22 ($\theta=\vartheta=1$ and $\lambda=2$ in Theorem 4)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(n-2k)^2\Big\{9(H_k-H_{n-k})^2+5(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle})\Big\}$$
$$=\begin{cases}(-1)^m\Big\{\frac{8}{3}\frac{(3m)!}{(m!)^3}\big\{4-m^2(H_{2m}^{\langle 2\rangle}+H_m^{\langle 2\rangle})\big\}-16\frac{(6m)!(m!)^3}{(3m)!\{(2m)!\}^3}\Big\} & n=2m,\\ 0, & n=2m+1.\end{cases}$$

Example 23 ($\theta=-\vartheta=1$ and $\lambda=0$ in Theorem 4)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3(n-2k)^2\Big\{(H_k+H_{n-k})^2+(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle})\Big\}$$
$$=\begin{cases}(-1)^{m+1}\frac{4m^2}{3}\frac{(3m)!}{(m!)^3}\left\{\begin{matrix}(H_{3m-1}-2H_{2m}-H_{m-1})^2\\ +(H_{m-1}^{\langle 2\rangle}+2H_{2m}^{\langle 2\rangle}-H_{3m-1}^{\langle 2\rangle})\end{matrix}\right\}, & n=2m,\\ 0, & n=2m+1.\end{cases}$$

§7. Specifying the Dougall formula by

$$a \to \lambda x - n, \quad b \to \theta x - n, \quad c \to \epsilon x - n, \quad d \to -n,$$

we restate it, under $\lambda = \theta + \vartheta = \epsilon + \varepsilon$, as

$$\begin{aligned} &{}_5F_4\left[\begin{matrix}\lambda x - n, & 1+\frac{\lambda x - n}{2}, & \theta x - n, & \epsilon x - n, & -n \\ & \frac{\lambda x - n}{2}, & 1+\vartheta x, & 1+\varepsilon x, & 1+\lambda x\end{matrix}\,\middle|\, 1\right] \\ &= \frac{(1+\lambda x - n)_n (1+n+\vartheta x - \epsilon x)_n}{(1+\vartheta x)_n (1+\varepsilon x)_n}. \end{aligned}$$

Dividing both sides by binomial coefficients $\binom{n-\lambda x}{n}\binom{n-\theta x}{n}\binom{n-\epsilon x}{n}$, we reformulate the result as the following finite sum identity:

$$\sum_{k=0}^{n}\binom{n}{k}^4 \frac{(\lambda x + 2k - n)}{\binom{k+\lambda x}{k}\binom{k+\vartheta x}{k}\binom{k+\varepsilon x}{k}\binom{n-k-\lambda x}{n-k}\binom{n-k-\theta x}{n-k}\binom{n-k-\epsilon x}{n-k}} \tag{18a}$$

$$= (-1)^n \lambda x \binom{2n}{n} \frac{\binom{2n+\vartheta x - \epsilon x}{2n}}{\binom{n+\vartheta x - \epsilon x}{n}\binom{n+\vartheta x}{n}\binom{n+\varepsilon x}{n}\binom{n-\theta x}{n}\binom{n-\epsilon x}{n}}. \tag{18b}$$

By means of (1), we write down the first derivative of (18) with respect to x:

$$\sum_{k=0}^{n}\binom{n}{k}^4 \frac{\lambda + (\lambda x + 2k - n)\Omega_k(x)}{\binom{k+\lambda x}{k}\binom{k+\vartheta x}{k}\binom{k+\varepsilon x}{k}\binom{n-k-\lambda x}{n-k}\binom{n-k-\theta x}{n-k}\binom{n-k-\epsilon x}{n-k}} = \mathrm{Eq(18b)}\Big\{W(x) + \tfrac{1}{x}\Big\} \tag{19}$$

where

$$\begin{aligned} \Omega_k(x) &= \lambda H_{n-k}(-\lambda x) + \theta H_{n-k}(-\theta x) + \epsilon H_{n-k}(-\epsilon x) \\ &- \lambda H_k(\lambda x) - \vartheta H_k(\vartheta x) - \varepsilon H_k(\varepsilon x), \\ W(x) &= (\vartheta - \epsilon) H_{2n}(\vartheta x - \epsilon x) - (\vartheta - \epsilon) H_n(\vartheta x - \epsilon x) \\ &+ \theta H_n(-\theta x) - \vartheta H_n(\vartheta x) + \epsilon H_n(-\epsilon x) - \varepsilon H_n(\varepsilon x). \end{aligned}$$

In accordance with

$$\begin{aligned} \Omega_k(0) &= (\lambda + \theta + \epsilon) H_{n-k} - (\lambda + \vartheta + \varepsilon) H_k, \\ W(0) &= (\vartheta - \epsilon) H_{2n} - 3(\vartheta - \epsilon) H_n; \end{aligned}$$

the case $x = 0$ of (19) results in the following formula.

$$\sum_{k=0}^{n}\binom{n}{k}^4 \Big\{\lambda + (n-2k)\big\{(\lambda+\vartheta+\varepsilon)H_k - (\lambda+\theta+\epsilon)H_{n-k}\big\}\Big\} = (-1)^n \lambda \binom{2n}{n}.$$

Under involution $k \to n-k$, it reduces to the following identity.

Example 24 (Paule and Schneider [13, Eq 4], cf. also Chu and De Donno [7, I-17])**.**

$$\sum_{k=0}^{n}\binom{n}{k}^4 \big\{1 + 4(n-2k)H_k\big\} = (-1)^n \binom{2n}{n}.$$

The derivative of (19) i.e., the second derivative of (18) with respect to x reads as

$$\sum_{k=0}^{n}\binom{n}{k}^4 \frac{2\lambda\Omega_k(x) + (\lambda x + 2k - n)\Big\{\Omega_k^2(x) + \Omega_k'(x)\Big\}}{\binom{k+\lambda x}{k}\binom{k+\vartheta x}{k}\binom{k+\varepsilon x}{k}\binom{n-k-\lambda x}{n-k}\binom{n-k-\theta x}{n-k}\binom{n-k-\epsilon x}{n-k}} \tag{20a}$$

$$= \mathrm{Eq(18b)} \times \Big\{\tfrac{2}{x} W(x) + W^2(x) + W'(x)\Big\} \tag{20b}$$

where

$$\begin{aligned}\Omega'_k(x) &= \lambda^2 H^{\langle 2\rangle}_{n-k}(-\lambda x)+\theta^2 H^{\langle 2\rangle}_{n-k}(-\theta x)+\epsilon^2 H^{\langle 2\rangle}_{n-k}(-\epsilon x)\\ &+ \lambda^2 H^{\langle 2\rangle}_{k}(\lambda x)+\vartheta^2 H^{\langle 2\rangle}_{k}(\vartheta x)+\varepsilon^2 H^{\langle 2\rangle}_{k}(\varepsilon x),\\ W'(x) &= -(\vartheta-\epsilon)^2 H^{\langle 2\rangle}_{2n}(\vartheta x-\epsilon x)+(\vartheta-\epsilon)^2 H^{\langle 2\rangle}_{n}(\vartheta x-\epsilon x)\\ &+ \vartheta^2 H^{\langle 2\rangle}_{n}(\vartheta x)+\varepsilon^2 H^{\langle 2\rangle}_{n}(\varepsilon x)+\theta^2 H^{\langle 2\rangle}_{n}(-\theta x)+\epsilon^2 H^{\langle 2\rangle}_{n}(-\epsilon x).\end{aligned}$$

Noting that

$$\begin{aligned}\Omega'_k(0) &= (\lambda^2+\theta^2+\epsilon^2)H^{\langle 2\rangle}_{n-k}+(\lambda^2+\vartheta^2+\varepsilon^2)H^{\langle 2\rangle}_{k},\\ \Omega''_k(0) &= 2(\lambda^3+\theta^3+\epsilon^3)H^{\langle 3\rangle}_{n-k}-2(\lambda^3+\vartheta^3+\varepsilon^3)H^{\langle 3\rangle}_{k},\\ W'(0) &= (\theta^2+\vartheta^2+\epsilon^2+\varepsilon^2)H^{\langle 2\rangle}_{n}-(\vartheta-\epsilon)^2(H^{\langle 2\rangle}_{2n}-H^{\langle 2\rangle}_{n}),\end{aligned}$$

we have from the case $x=0$ of (20) the following general formula.

$$\sum_{k=0}^{n}\binom{n}{k}^4\left\{\begin{array}{c}2\lambda\big\{(\lambda+\vartheta+\varepsilon)H_k-(\lambda+\theta+\epsilon)H_{n-k}\big\}\\ +(n-2k)\big\{(\lambda+\vartheta+\varepsilon)H_k-(\lambda+\theta+\epsilon)H_{n-k}\big\}^2\\ +(n-2k)\big\{(\lambda^2+\vartheta^2+\varepsilon^2)H^{\langle 2\rangle}_k+(\lambda^2+\theta^2+\epsilon^2)H^{\langle 2\rangle}_{n-k}\big\}\end{array}\right\}$$
$$=(-1)^n 2\lambda(\vartheta-\epsilon)\binom{2n}{n}(3H_n-H_{2n}).$$

Under involution $k\to n-k$, this formula becomes the following identity.

Example 25.

$$\sum_{k=0}^{n}\binom{n}{k}^4\Big\{2H_k+(n-2k)\big(4H_k^2+H^{\langle 2\rangle}_k\big)\Big\}=(-1)^n\binom{2n}{n}\big\{3H_n-H_{2n}\big\}.$$

The derivative of (20), i.e., the third derivative of (18) with respect to x at $x=0$ reads as

$$3\lambda\sum_{k=0}^{n}\binom{n}{k}^4\Big\{\Omega_k^2(0)+\Omega'_k(0)\Big\}+\sum_{k=0}^{n}\binom{n}{k}^4(2k-n)\Big\{\Omega_k^3(0)+3\Omega_k(0)\Omega'_k(0)+\Omega''_k(0)\Big\}$$
$$=(-1)^n 3\lambda\binom{2n}{n}\Big\{W^2(0)+W'(0)\Big\}.$$

This leads to the following general formula.

Theorem 5 ($\lambda=\theta+\vartheta=\epsilon+\varepsilon$: Harmonic number identity)**.**

$$\sum_{k=0}^{n}\binom{n}{k}^4\left\{\begin{array}{l}\quad 3\lambda\big\{(\lambda+\vartheta+\varepsilon)H_k-(\lambda+\theta+\epsilon)H_{n-k}\big\}^2\\ +3\lambda\big\{(\lambda^2+\vartheta^2+\varepsilon^2)H^{\langle 2\rangle}_k+(\lambda^2+\theta^2+\epsilon^2)H^{\langle 2\rangle}_{n-k}\big\}\\ +(n-2k)\big\{(\lambda+\vartheta+\varepsilon)H_k-(\lambda+\theta+\epsilon)H_{n-k}\big\}^3\\ +2(n-2k)\big\{(\lambda^3+\vartheta^3+\varepsilon^3)H^{\langle 3\rangle}_k-(\lambda^3+\theta^3+\epsilon^3)H^{\langle 3\rangle}_{n-k}\big\}\\ +3(n-2k)\big\{(\lambda+\vartheta+\varepsilon)H_k-(\lambda+\theta+\epsilon)H_{n-k}\big\}\\ \quad\times\big\{(\lambda^2+\vartheta^2+\varepsilon^2)H^{\langle 2\rangle}_k+(\lambda^2+\theta^2+\epsilon^2)H^{\langle 2\rangle}_{n-k}\big\}\end{array}\right\}$$
$$=(-1)^n\,3\lambda\binom{2n}{n}\left\{\begin{array}{c}(\vartheta-\epsilon)^2\big\{(H_{2n}-3H_n)^2-(H^{\langle 2\rangle}_{2n}-H^{\langle 2\rangle}_n)\big\}\\ +(\theta^2+\vartheta^2+\epsilon^2+\varepsilon^2)H^{\langle 2\rangle}_n\end{array}\right\}.$$

Example 26 ($\theta=\vartheta=\epsilon=\varepsilon=1$ and $\lambda=2$ in Theorem 5)**.**

$$\sum_{k=0}^{n}\binom{n}{k}^4\left\{\begin{array}{c}24(H_{n-k}-H_k)^2+9(H^{\langle 2\rangle}_{n-k}+H^{\langle 2\rangle}_k)\\ +18(n-2k)(H_k-H_{n-k})(H^{\langle 2\rangle}_k+H^{\langle 2\rangle}_{n-k})\\ +(n-2k)\big\{16(H_k-H_{n-k})^3+5(H^{\langle 3\rangle}_k-H^{\langle 3\rangle}_{n-k})\big\}\end{array}\right\}=6(-1)^n\binom{2n}{n}H^{\langle 2\rangle}_n.$$

Example 27 ($\theta=\varepsilon=2$, $\vartheta=\epsilon=-1$ and $\lambda=1$ in Theorem 5).

$$\sum_{k=0}^{n}\binom{n}{k}^4\left\{\begin{array}{c}6(H_k-H_{n-k})^2+9(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle})\\ +18(n-2k)(H_k-H_{n-k})(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle})\\ +4(n-2k)\left\{\scriptstyle (H_k-H_{n-k})^3+2(H_k^{\langle 3\rangle}-H_{n-k}^{\langle 3\rangle})\right\}\end{array}\right\}=15(-1)^n\binom{2n}{n}H_n^{\langle 2\rangle}.$$

Example 28 (Combined difference of Example 26 and Example 27).

$$\sum_{k=0}^{n}\binom{n}{k}^4\left\{\begin{array}{c}(n-2k)\left\{\scriptstyle (H_k^{\langle 3\rangle}-H_{n-k}^{\langle 3\rangle})-4(H_k-H_{n-k})^3\right\}\\ -6(H_k-H_{n-k})^2\end{array}\right\}=3(-1)^n\binom{2n}{n}H_n^{\langle 2\rangle}.$$

Example 29 (Combined difference of Example 26 and Example 27).

$$\sum_{k=0}^{n}\binom{n}{k}^4\left\{\begin{array}{c}(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle})+(n-2k)(H_k^{\langle 3\rangle}-H_{n-k}^{\langle 3\rangle})\\ +2(n-2k)(H_k-H_{n-k})(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle})\end{array}\right\}=2(-1)^n\binom{2n}{n}H_n^{\langle 2\rangle}.$$

Example 30 (Combination of Example 26 and Example 27: Chu [6, Eq 1.15]).

$$\sum_{k=0}^{n}\binom{n}{k}^4\left\{\begin{array}{c}12(H_k-H_{n-k})^2+3(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle})\\ +6(n-2k)(H_k-H_{n-k})(H_k^{\langle 2\rangle}+H_{n-k}^{\langle 2\rangle})\\ +(n-2k)\left\{8\scriptstyle (H_k-H_{n-k})^3+(H_k^{\langle 3\rangle}-H_{n-k}^{\langle 3\rangle})\right\}\end{array}\right\}=0.$$

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