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ScienceDirect
Journal of Combinatorial Theory, Series B 97 (2007) 818–826
Journal of Combinatorial Theory Series B
www.elsevier.com/locate/jctb

Primitive half-transitive graphs constructed from the symmetric groups of prime degrees ☆

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Received 14 July 2006
Available online 1 February 2007

Abstract

A graph is said to be half-transitive if its automorphism group acts transitively on the vertex set and edge set but intransitively on the arc set. In this paper, we construct infinitely many primitive half-transitive graphs with automorphism groups being the symmetric groups of prime degrees, and show that there exists at least one primitive half-transitive graph of valency $2p$ for a prime p no less than 7 and $p \neq 13$. As a byproduct of our construction, infinitely many primitive 2-arc-regular Cayley graphs are given.

Keywords: Primitive permutation group; Suborbit; Symmetric graph; Half-transitive graph

1. Introduction

Throughout the present paper graphs are assumed to be finite, simple, undirected and connected unless specified otherwise.

Let Γ be a graph. We use $V\Gamma$, $E\Gamma$ and $\mathsf{Aut}\,\Gamma$ to denote the vertex set, edge set and automorphism group of Γ, respectively. For a subgroup G of $\mathsf{Aut}\,\Gamma$, the graph Γ is said to be G*-vertex-transitive* (respectively G*-edge-transitive* or G*-arc-transitive*) if G acts transitively on $V\Gamma$ (respectively $E\Gamma$ or the set of arcs of Γ), and Γ is said to be G*-primitive* if G acts primitively on $V\Gamma$. The (undirected) graph Γ is said to be a G*-symmetric* graph if it is G-vertex-transitive and G-arc-transitive, and Γ is said to be a G*-half-transitive* graph if it is G-vertex-transitive and G-edge-transitive but not G-arc-transitive. For the case where $G = \mathsf{Aut}\,\Gamma$, a G-vertex-transitive

☆ This project was partially supported by the NSF, the 973 project and PCSIRT of China and the Doctoral Program Foundation of Institutions of Higher Education of China.
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0095-8956/$ – see front matter
doi:10.1016/j.jctb.2007.01.003

(respectively G-edge-transitive, G-arc-transitive, G-half-transitive or G-primitive) graph Γ is simply called a *vertex-transitive* (respectively *edge-transitive*, *arc-transitive*, *half-transitive* or *primitive*) graph.

The study of half-transitive graphs has currently been an active topic, see [5,9,10,12,13] for references. It follows from a result of Tutte [16] that every half-transitive graph has even valency. For each integer $k \geqslant 2$, Bouwer [1] constructed a half-transitive graph of valency $2k$. The smallest graph given by Bouwer's construction has 54 vertices and valency 4. Holt [6] found a half-transitive graph on 27 vertices, which is the unique half-transitive graph of order 27 and valency 4 (see [18]). In 1981, Holt [6] (see also [7]) posted a question on the existence of primitive half-transitive graphs. Praeger and Xu [14] gave the first ten examples of primitive half-transitive graphs: one has valency 24, one has valency 48, and the others have valency 120. It was shown in [5] that the smallest primitive half-transitive graph has order 165 and valency 48. Moreover, Taylor and Xu [15] constructed an infinite class of such graphs with valency 120. Li, Lu and Marušič [9] constructed another infinite class of primitive half-transitive graphs of valencies $2(2^{2m+1}-1)$ for $m \geqslant 1$. Furthermore, they proved that there exist no such graphs with valency less than 10, and posted the following problem:

Problem 1.1. Find all integers k such that there exist primitive half-transitive graphs of valency $2k$.

We denote by $\mathcal{V}_{\mathrm{ph}}$ the set of integers k such that there exists a primitive half-transitive graph of valency $2k$. Then we have the following proposition.

Proposition 1.2. $2, 3, 4 \notin \mathcal{V}_{\mathrm{ph}}$ *and* $12, 24, 60, (2^{2m+1}-1) \in \mathcal{V}_{\mathrm{ph}}$.

The aim of this paper is to find new members of $\mathcal{V}_{\mathrm{ph}}$. We shall show that the set $\mathcal{V}_{\mathrm{ph}}$ contains each prime p except that p is one of 2, 3, 5 and 13.

Theorem 1.3. *Let p be a prime no less than* 7. *If $p \neq 13$ then $p \in \mathcal{V}_{\mathrm{ph}}$.*

This paper is organized as follows. Section 2 collects several results on primitive permutation groups which have a suborbit of prime length. In Section 3, we shall construct new examples of primitive half-transitive graphs and give the proof of Theorem 1.3.

2. Permutation groups with a suborbit of prime length

In this section, we collect some notation and preliminary results about permutation groups, which will be used in the next section. For the group-theoretic concepts and notation not defined here we refer the reader to [4,17].

Let G be a transitive permutation on a finite set Ω. An orbit Δ of G acting on the cartesian product $\Omega \times \Omega$ is called an orbital of G, and for $\alpha \in \Omega$, the set $\Delta(\alpha) = \{\beta \mid (\alpha, \beta) \in \Delta\}$ is an orbit of the stabilizer G_α, called a suborbit of G at α. For an orbital $\Delta = (\alpha, \beta)^G$, the orbital $\Delta^* = (\beta, \alpha)^G$ is called the paired orbital of Δ, and the suborbit $\Delta^*(\alpha)$ is the paired suborbit of $\Delta(\alpha)$. Then an orbital Δ (respectively a suborbit $\Delta(\alpha)$) is said to be self-paired if $\Delta = \Delta^*$ (respectively $\Delta(\alpha) = \Delta^*(\alpha)$). For paired orbitals Δ and Δ^* of G, the digraph $(\Omega, \Delta \cup \Delta^*)$, with vertex set Ω and arc set $\Delta \cup \Delta^*$, is an undirected graph by identifying two arcs (α, β) and (β, α) as an edge $\{\alpha, \beta\}$. Further $(\Omega, \Delta \cup \Delta^*)$ is G-vertex-transitive and G-edge-transitive, and

$(\Omega, \Delta \cup \Delta^*)$ is G-symmetric if and only if Δ (or $\Delta(\alpha)$) is self-paired. Conversely, if a graph Γ is G-vertex-transitive and G-edge-transitive for some $G \leqslant \mathsf{Aut}\,\Gamma$, then $\Gamma \cong (V\Gamma, \Delta \cup \Delta^*)$ for paired orbitals Δ and Δ^* of G.

Let $\Delta(\alpha)$ be a suborbit of G at $\alpha \in \Omega$, and let $\beta \in \Delta(\alpha)$. Since G is transitive, $\beta = \alpha^g$ for some $g \in G$. Thus $G_{\alpha\beta} = G_\alpha \cap G_\beta = G_\alpha \cap G_\alpha^g$ lies in both G_α and G_α^g. It implies that $G_{\alpha\beta}$ and $G_{\alpha\beta}^{g^{-1}}$ are subgroups of G_α. If $G_{\alpha\beta}$ and $G_{\alpha\beta}^{g^{-1}}$ are conjugate in G_α, then $G_{\alpha\beta}^h = G_{\alpha\beta}^{g^{-1}}$ for some $h \in G_\alpha$, and so $x = hg \in \mathbf{N}_G(G_{\alpha\beta})$, where $\mathbf{N}_G(G_{\alpha\beta})$ is the normalizer of $G_{\alpha\beta}$ in G. Then we have the following lemma.

Lemma 2.1. *Let G be a transitive permutation group on Ω with a suborbit $\Delta(\alpha)$ at $\alpha \in \Omega$, and let $\beta = \alpha^g \in \Delta(\alpha)$ for some $g \in G$. If $G_{\alpha\beta}$ and $G_{\alpha\beta}^{g^{-1}}$ are conjugate in G_α, then there is some $x \in \mathbf{N}_G(G_{\alpha\beta})$ such that $\beta = \alpha^x$.*

Then the following result holds.

Corollary 2.2. *Let G be a transitive permutation group on Ω with a suborbit $\Delta(\alpha)$ at $\alpha \in \Omega$, and let $\beta \in \Delta(\alpha)$. Suppose that $\{G_{\alpha\beta}^y \mid G_{\alpha\beta}^y \leqslant G_\alpha,\ y \in G\}$ is a conjugate class in G_α. For a suborbit $\Sigma(\alpha)$ of G at $\alpha \in \Omega$ and for $\gamma \in \Sigma(\alpha)$, if $G_{\alpha\beta}$ is conjugate to $G_{\alpha\gamma}$ in G_α, then $\gamma = \alpha^{xh}$ for some $x \in \mathbf{N}_G(G_{\alpha\beta})$ and $h \in G_\alpha$.*

Proof. By assumption, $G_{\alpha\gamma} = G_{\alpha\beta}^h$ for some $h \in G_\alpha$. Let $\gamma = \alpha^g$ for some $g \in G$. Then $G_{\alpha\gamma} \leqslant G_\gamma = G_\alpha^g$, and so $(G_{\alpha\gamma})^{g^{-1}} \leqslant G_\alpha$. Since $G_{\alpha\gamma}^{g^{-1}} = G_{\alpha\beta}^{hg^{-1}}$ is conjugate to $G_{\alpha\gamma} = G_{\alpha\beta}^h$ in G_α. Thus, by Lemma 2.1, there is $y \in \mathbf{N}_G(G_{\alpha\gamma})$ such that $\gamma = \alpha^y$. Noting that $\mathbf{N}_G(G_{\alpha\gamma}) = \mathbf{N}_G(G_{\alpha\beta}^h) = (\mathbf{N}_G(G_{\alpha\beta}))^h$. It follows that $y = x^h$ for some $x \in \mathbf{N}_G(G_{\alpha\beta})$, and $\gamma = \alpha^y = \alpha^{x^h} = \alpha^{xh}$. $\square$

For the case where G_α acts primitively on $\Delta(\alpha)$ we have the following results.

Lemma 2.3. *Let G be a transitive permutation group on Ω. Let K be a subgroup of G_α with index $l > 1$, where $\alpha \in \Omega$. Set $\Theta := \{\alpha^{zh} \mid z \in \mathbf{N}_G(K) \setminus \mathbf{N}_G(G_\alpha),\ h \in G_\alpha\}$. Suppose that all subgroups with index l of G_α are conjugate in G_α. Then every suborbit of length l at α is a subset of Θ. If further K is maximal in G_α, then Θ is the union of all suborbits of length l at α.*

Proof. Let $\Delta(\alpha)$ be a suborbit of length l and $\gamma \in \Delta(\alpha)$. Then $\Delta(\alpha) = \gamma^{G_\alpha}$ and $l = |G_\alpha : G_{\alpha\gamma}|$. Now $G_{\alpha\gamma}$ is a subgroup of G_α with index l, thus $G_{\alpha\gamma} = K^x$ for some $x \in G_\alpha$. Set $\beta := \gamma^{x^{-1}}$. Then $\beta \in \Delta(\alpha)$ and $G_{\alpha\beta} = G_{\alpha\gamma^{x^{-1}}} = G_\alpha \cap G_{\gamma^{x^{-1}}} = G_\alpha \cap G_\gamma^{x^{-1}} = (G_\alpha \cap G_\gamma)^{x^{-1}} = K$. Then by Corollary 2.2, $\gamma = \alpha^{zh}$ for some $z \in \mathbf{N}_G(K)$ and some $h \in G_\alpha$. If $z \in \mathbf{N}_G(G_\alpha)$, then $G_\gamma = G_{\alpha^{zh}} = G_\alpha^{zh} = G_\alpha$ and $l = |\Delta(\alpha)| = |G_\alpha : G_{\alpha\gamma}| = 1$, a contradiction. Thus $z \notin \mathbf{N}_G(G_\alpha)$. Therefore, $\Delta(\alpha)$ is a subset of Θ.

Now assume that K is a maximal subgroup of G_α. We shall show, for each $z \in \mathbf{N}_G(K) \setminus \mathbf{N}_G(G_\alpha)$, that $\alpha^{zG_\alpha} := \{\alpha^{zh} \mid h \in G_\alpha\}$ is a suborbit of length l. Clearly, $K = K^z \leqslant G_\alpha \cap G_\alpha^z = G_\alpha \cap G_{\alpha^z}$. Since K is maximal in G_α and $z \notin \mathbf{N}_G(G_\alpha)$, we have $K = G_\alpha \cap G_{\alpha^z}$. Then α^{zG_α} is a suborbit of length $|G_\alpha : K| = l$. Thus Θ is the union of suborbits of length l at α. $\square$

Lemma 2.4. *Let G be a primitive permutation group on Ω. Let $\alpha \in \Omega$ and K a maximal subgroup of G_α with index $l > 1$. Suppose that all subgroups with index l of G_α are conjugate in G_α. Suppose further that K is not normal in G_α.*

(1) *If $z \in \mathbf{N}_G(K) \setminus K$, then α^{zG_α} is a suborbit of length l at α.*
(2) *If $\Delta(\alpha)$ is a suborbit of length l at α, then $\Delta(\alpha) = \alpha^{zG_\alpha}$ for some $z \in \mathbf{N}_G(K) \setminus K$.*
(3) *If $z, y \in \mathbf{N}_G(K) \setminus K$, then $\alpha^{zG_\alpha} = \alpha^{yG_\alpha}$ if and only if $yz^{-1} \in K$.*
(4) *The number of suborbits of length l at α is equal to $|\mathbf{N}_G(K) : K| - 1$.*

Proof. Assume that K is not normal in G_α. Then $G_\alpha \cap \mathbf{N}_G(K) = \mathbf{N}_{G_\alpha}(K) = K$ as K is maximal in G_α. Since G is primitive, G_α is a maximal subgroup of G. Further, $G_\alpha \neq 1$ since $l > 1$. It follows that $\mathbf{N}_G(G_\alpha) = G_\alpha$. Thus $\mathbf{N}_G(K) \setminus \mathbf{N}_G(G_\alpha) = \mathbf{N}_G(K) \setminus G_\alpha = \mathbf{N}_G(K) \setminus (G_\alpha \cap \mathbf{N}_G(K)) = \mathbf{N}_G(K) \setminus K$. Let Θ be as in Lemma 2.3. Then $\Theta = \{\alpha^{zh} \mid z \in \mathbf{N}_G(K) \setminus K,\ h \in G_\alpha\}$. If $\mathbf{N}_G(K) = K$, then $\Theta = \emptyset$ and G has no suborbits of length l. In the following we suppose that $\mathbf{N}_G(K) \setminus K \neq \emptyset$.

By Lemma 2.3, we know that (1) and (2) hold.

Let $z, y \in \mathbf{N}_G(K) \setminus K$. Then $G_\alpha \neq G_\alpha^z = G_{\alpha^z}$, $G_\alpha \neq G_\alpha^y = G_{\alpha^y}$, and K fixes both α^z and α^y. Hence $K = G_\alpha \cap G_{\alpha^z} = G_\alpha \cap G_{\alpha^y}$. Assume first that $yz^{-1} \in K$. Then $Kz = Ky$, hence $zK = (zKz^{-1})z = Kz = Ky = y(y^{-1}Ky) = yK$. It implies that $zG_\alpha = yG_\alpha$, and hence $\alpha^{zG_\alpha} = \alpha^{yG_\alpha}$. Conversely, assume that $\alpha^{zG_\alpha} = \alpha^{yG_\alpha}$. Then $\alpha^z = \alpha^{yh}$ for some $h \in G_\alpha$, and so $yhz^{-1} \in G_\alpha$. Moreover, $K = G_\alpha \cap G_{\alpha^y} = G_\alpha \cap G_{\alpha^z} = G_\alpha \cap G_{\alpha^{yh}} = (G_\alpha \cap G_{\alpha^y})^h = K^h$. Since K is maximal and not normal in G_α, we have $h \in K \leqslant \mathbf{N}_G(K)$, and hence $yhz^{-1} \in G_\alpha \cap \mathbf{N}_G(K) = K$. Then $yz^{-1} = (yhz^{-1})(zh^{-1}z^{-1}) \in K$. Thus (3) holds.

It follows from (3) that the number of suborbits of length l at α is equal to the number of the right cosets Kz contained in $\mathbf{N}_G(K) \setminus K$. This lead to (4). □

Remark 2.5. Let G and K be as in Lemma 2.4. Then by [17, Theorem 3.5] $\mathbf{N}_G(K)$ is transitive on the vertices fixed by K and transitive on the arcs in Δ fixed by K. Then $|\mathbf{N}_G(K) : K|$ equals the number of vertices fixed by K, and also equals the number of arcs (of the digraph (Ω, Δ)) fixed by K.

For the case where $|\Delta(\alpha)|$ is a prime, we have the following result. The proof is a routine application of the classification of primitive permutation groups of prime degree.

Lemma 2.6. *Let G be a primitive group with a suborbit $\Delta(\alpha)$ of prime length p. Let $\mathsf{soc}(G_\alpha^{\Delta(\alpha)})$ be the socle of the permutation group $G_\alpha^{\Delta(\alpha)}$ induced by G_α acting on $\Delta(\alpha)$. Then one of the following statements holds*:

(1) $\mathsf{soc}(G_\alpha^{\Delta(\alpha)}) = \mathrm{PSL}(2, 11)$ *and* $p = 11$;
(2) $\mathsf{soc}(G_\alpha^{\Delta(\alpha)}) = \mathrm{PSL}(d, q)$ *and* $p = \frac{q^d-1}{q-1}$ *for some* $d > 2$;
(3) $(\mathsf{soc}(G_\alpha^{\Delta(\alpha)}), p)$ *is one of the pairs listed in Table* 1, *and either $G_\alpha \cong \mathbb{Z}_p$ or the number of suborbits of length p at α is $|\mathbf{N}_G(K) : K| - 1$, where K is a subgroup of G_α with index p.*

Proof. Let $X = G_\alpha^{\Delta(\alpha)}$. Since $|\Delta(\alpha)| = p$ is a prime, X is either solvable or a 2-transitive permutation group on $\Delta(\alpha)$. If X is solvable, then $\mathbb{Z}_p \leqslant G_\alpha^{\Delta(\alpha)} \leqslant \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$. If X is insolvable, then it follows from [2] that $(\mathsf{soc}(X), p)$ is described as in this lemma.

Table 1

$\mathrm{soc}(G_\alpha^{\Delta(\alpha)})$	p
$\mathbb{Z}_p$	p
A_p	p
M_{11}	11
M_{23}	23
$\mathrm{PSL}(2, 2^{2^s})$, $s \geqslant 1$	$2^{2^s}+1$

Let Δ be the orbital relative to $\Delta(\alpha)$, and let $\beta \in \Delta(\alpha)$. Then $\Delta = (\alpha, \beta)^G$ and the paired orbital $\Delta^* = (\beta, \alpha)^G$. Further, $\Delta(\alpha) = \beta^{G_\alpha}$, $\Delta^*(\beta) = \alpha^{G_\beta}$ and

$$\left|\Delta^*(\beta)\right| = |G_\beta : G_{\beta\alpha}| = |G_\beta|/|G_{\beta\alpha}| = |G_\alpha|/|G_{\alpha\beta}| = p = \left|\Delta(\alpha)\right|.$$

Consider the permutation groups $G_{\alpha\beta}^{\Delta(\alpha)}$ and $G_{\alpha\beta}^{\Delta^*(\beta)}$ induced by $G_{\alpha\beta}$ acting on $\Delta(\alpha)$ and on $\Delta^*(\beta)$, respectively. It follows from [4, Theorem 4.4A] that each prime divisor of $|G_{\alpha\beta}|$ divides either $|G_{\alpha\beta}^{\Delta(\alpha)}|$ or $|G_{\alpha\beta}^{\Delta^*(\beta)}|$. Noting that both α and β are fixed by $G_{\alpha\beta}$. Then $G_{\alpha\beta}^{\Delta(\alpha)}$ is isomorphic to a subgroup of S_{p-1}, and so does $G_{\alpha\beta}^{\Delta^*(\beta)}$. It implies that p is not a divisor of $|G_{\alpha\beta}|$. Therefore, p divides $|G_\alpha|$ but p^2 does not.

Set $X = G_\alpha/N$, where N is the kernel of G_α acting on $\Delta(\alpha)$. Then N is a p'-subgroup of G_α.

Assume that $(\mathrm{soc}(X), p)$ is one of the pairs listed in Table 1. By [2], all permutation representations of such an X with degree p are equivalent. It follows that all subgroups of X with index p are conjugate. For $\beta \in \Delta(\alpha)$, $G_{\alpha\beta}$ has index p in G_α, and hence $G_{\alpha\beta}$ is a p'-Hall subgroup of G_α. Noting that, for each p'-Hall subgroup H of G_α, $N \leqslant H$ and H/N is a subgroup of X with index p. It follows that all p'-Hall subgroups of G_α are conjugate to $G_{\alpha\beta}$ in G_α. Then by Lemma 2.4, either K is normal in G_α or the number of suborbits of length p at α is $|\mathbf{N}_G(K) : K| - 1$, where K is a subgroup of G_α with index p. The former case leads to $K = G_{\alpha\beta}$, $K \lhd G_\alpha$ and $K \lhd G_\beta$, so $K \lhd \langle G_\alpha, G_\beta\rangle = G$, hence $K = 1$ and $G_\alpha \cong \mathbb{Z}_p$. This completes the proof. □

Let $\Delta(\alpha)$ be a self-paired suborbit of G at $\alpha \in \Omega$ with $\Delta(\alpha) \neq \{\alpha\}$. Then $\Delta = (\alpha, \beta)^G = (\beta, \alpha)^G = \Delta^*$ for some $\beta \in \Omega \setminus \{\alpha\}$, and there is some $g \in G$ such that $(\alpha, \beta)^g = (\beta, \alpha)$, it further implies that $g \notin G_\alpha$, $g \notin G_\beta$, $g^2 \in G_\alpha \cap G_\beta$ and

$$G_{\alpha\beta}^g = (G_\alpha \cap G_\beta)^g = G_\alpha^g \cap G_\beta^g = G_{\alpha^g} \cap G_{\beta^g} = G_\beta \cap G_\alpha = G_{\alpha\beta}.$$

Thus the following lemma holds (see also [9]).

Lemma 2.7. *Let $\Delta(\alpha)$ be a suborbit of G at $\alpha \in \Omega$ with $\Delta(\alpha) \neq \{\alpha\}$, and let $\beta \in \Delta(\alpha)$. If $\Delta(\alpha)$ is self-paired, then there is some $g \in \mathbf{N}_G(G_{\alpha\beta}) \setminus (G_\alpha \cup G_\beta)$ such that $\beta = \alpha^g$, $g^2 \in G_{\alpha\beta}$; in particular, $G_{\alpha\beta}$ has even index in its normalizer $\mathbf{N}_G(G_{\alpha\beta})$.*

The following result is a direct consequence of Lemma 2.4.

Corollary 2.8. *Let G be a primitive permutation group on Ω. Let $\alpha \in \Omega$, and let K be a maximal subgroup of G_α with index $l > 1$. Suppose that all subgroups of index l of G_α are conjugate in G_α. Assume that K is not normal in G_α and $z \in \mathbf{N}_G(K) \setminus K$. Then α^{zG_α} is a suborbit of length l, and it is self-paired if and only if $z^2 \in K$.*

Proof. Assume that K is not a normal subgroup of G_α. Let $z \in \mathbf{N}_G(K) \setminus K$. Then, by Lemma 2.4, α^{zG_α} is a suborbit of G of length l. So $\Delta = (\alpha, \alpha^z)^G$ is an orbital of G with $\Delta(\alpha) = a^{zG_\alpha}$, and the paired orbital of Δ is $\Delta^* = (\alpha^z, \alpha)^G$. Since $\Delta^* = (\alpha^z, \alpha)^G = (\alpha, \alpha^{z^{-1}})^{zG} = (\alpha, \alpha^{z^{-1}})^G$, we have $\Delta^*(\alpha) = \alpha^{z^{-1}G_\alpha}$. By Lemma 2.4(3), we know that $a^{zG_\alpha} = \Delta(\alpha) = \Delta^*(\alpha) = \alpha^{z^{-1}G_\alpha}$ if and only if $z^2 = z(z^{-1})^{-1} \in K$. Thus α^{zG_α} is self-paired if and only if $z^2 \in K$. □

Recall that, for two paired orbitals Δ and Δ^* of G, the graph $(\Omega, \Delta \cup \Delta^*)$ is G-symmetric if and only if Δ (or $\Delta(\alpha)$, for $\alpha \in \Omega$) is self-paired. Then by Lemma 2.6 and Corollary 2.8 we have:

Corollary 2.9. *Let G be a primitive group with a suborbit $\Delta(\alpha)$ of prime length p. Let K be a subgroup of G_α with index p. If $K \neq 1$ and $\mathsf{soc}(G_\alpha^{\Delta(\alpha)})$ is one of the groups listed in Table* 1, *then $\Delta(\alpha) = \alpha^{zG_\alpha}$ for some $z \in \mathbf{N}_G(K) \setminus K$, and the graph $(\Omega, \Delta \cup \Delta^*)$ is G-symmetric if and only if $z^2 \in K$.*

3. Examples and proof of Theorem 1.3

In this section we shall prove Theorem 1.3 by constructing primitive half-transitive graphs with valency $2p$ and automorphism group S_p. First we need several preliminary properties of the symmetric group S_n of degree n.

Let $x \in \mathrm{S}_n$ be an n-cycle. Then $\langle x\rangle$ is regular on $\{1, 2, \ldots, n\}$. Thus the normalizer $\mathbf{N}_{\mathrm{S}_n}(\langle x\rangle)$ is the holomorph of $\langle x\rangle$, and $\mathbf{N}_{\mathrm{S}_n}(\langle x\rangle) = \langle x\rangle \rtimes \mathsf{Aut}(\langle x\rangle) \cong \mathbb{Z}_n \rtimes \mathsf{Aut}(\mathbb{Z}_n)$. Hence $\mathbf{N}_{\mathrm{S}_n}(\langle x\rangle)/\langle x\rangle \cong \mathsf{Aut}(\langle x\rangle) \cong \mathsf{Aut}(\mathbb{Z}_n)$. Write $n = 2^e p_1^{e_1} \cdots p_k^{e_k}$ for distinct primes $2, p_1, \ldots, p_k$, and set $\Phi = \mathbb{Z}_{\phi(p_1^{e_1})} \times \cdots \times \mathbb{Z}_{\phi(p_k^{e_k})}$, where ϕ is the Euler function. Then $\mathsf{Aut}(\mathbb{Z}_n)$ is isomorphic to Φ for $e \leqslant 1$, or to $\mathbb{Z}_2 \times \mathbb{Z}_{\phi(2^{e-2})} \times \Phi$ for $e \geqslant 2$. It follows that the quotient group $\mathbf{N}_{\mathrm{S}_n}(\langle x\rangle)/\langle x\rangle$ is an elementary abelian 2-group (i.e., every element has order 1 or 2) if and only if n is a divisor of 24. Then we have the following lemma.

Lemma 3.1. *Let p be a prime no less than* 5, *and let x be a p-cycle in S_p. Then*

(1) $\mathbf{N}_{\mathrm{S}_p}(\langle x\rangle) = \langle x\rangle \rtimes \langle y\rangle$ *for a $(p-1)$-cycle y;*
(2) $\mathbf{N}_{\mathrm{S}_p}(\langle y\rangle)/\langle y\rangle$ *is an elementary abelian* 2*-group if and only if $p \in \{5, 7, 13\}$;*
(3) $\mathbf{N}_{\mathrm{S}_p}(\langle x\rangle)$ *is a maximal subgroup of* S_p.

Proof. Let $x \in \mathrm{S}_p$ be a p-cycle. Then $H := \mathbf{N}_{\mathrm{S}_p}(\langle x\rangle) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$, and H is a primitive permutation group on $\{1, 2, \ldots, p\}$. It follows that $H = \mathrm{AGL}(1, p)$. Then $H = \langle x\rangle \rtimes \langle y\rangle$ for a $(p-1)$-cycle y, and hence (1) hold. Without loss of generality, we may assume $y \in \mathrm{S}_{p-1}$. Then $\langle y\rangle$ is a regular subgroup of S_{p-1} on $\{1, 2, \ldots, p-1\}$. Thus $\mathbf{N}_{\mathrm{S}_p}(\langle y\rangle) = \mathbf{N}_{\mathrm{S}_{p-1}}(\langle y\rangle)$ is the holomorph of $\langle y\rangle$, and $\mathbf{N}_{\mathrm{S}_p}(\langle y\rangle)/\langle y\rangle \cong \mathsf{Aut}(\langle y\rangle) \cong \mathsf{Aut}(\mathbb{Z}_{p-1})$. Then $\mathbf{N}_{\mathrm{S}_p}(\langle y\rangle)/\langle y\rangle$ is an elementary abelian 2-group if and only if $p-1$ is a divisor of 24, that is, $p \in \{5, 7, 13\}$. So (2) holds.

All maximal subgroups of the finite symmetric groups are classified in [11] by using the O'Nan–Scott theorem and the classifications of the finite simple groups and their maximal factorizations. It follows that $\mathbf{N}_{\mathrm{S}_p}(\langle x\rangle) = \mathrm{AGL}(1, p)$ is a maximal subgroup of S_p, and so (3) holds. □

Now we are ready to construct primitive half-transitive graphs.

Construction 3.2. Let $p \geqslant 5$ be a prime. Let $x \in \mathrm{S}_p$ be a p-cycle and $y \in \mathrm{S}_{p-1}$ a $(p-1)$-cycle such that y normalizes $\langle x\rangle$. Set $H = \langle x, y\rangle$ and $K = \langle y\rangle$. Then H is a maximal subgroup of S_p by Lemma 3.1. Since the unique fixed point of K is fixed by $\mathbf{N}_{\mathrm{S}_p}(K)$, we have $\mathbf{N}_{\mathrm{S}_p}(K) = \mathbf{N}_{\mathrm{S}_{p-1}}(K)$. Consider the action of S_p by right multiplication on the set $\Omega := \{Hg \mid g \in \mathrm{S}_p\}$ of right cosets of H in S_p. Then S_p is a primitive permutation group of degree $(p-2)!$. For $z \in \mathbf{N}_{\mathrm{S}_{p-1}}(K) \setminus K$, α^{zH} and $\alpha^{z^{-1}H}$ are paired suborbits of length p, where $\alpha = H \in \Omega$. Then the corresponding orbitals are $\Delta = (\alpha, \alpha^z)^{\mathrm{S}_p}$ and $\Delta^* = (\alpha, \alpha^{z^{-1}})^{\mathrm{S}_p}$. We use $\Gamma(p,z)$ to denote the graph $(\Omega, \Delta \cup \Delta^*)$ with vertex set Ω and arc set $\Delta \cup \Delta^*$.

Let $\Gamma(p,z)$ be as above and $p \geqslant 5$. Then $\Gamma(p,z)$ is S_p-vertex-transitive and S_p-edge-transitive. For $\alpha \in V\Gamma(p,z)$, the neighborhood of α in $\Gamma(p,z)$ is $\Delta(\alpha) \cup \Delta^*(\alpha)$. The following lemma implies that $\Delta(\alpha)$ and $\Delta^*(\alpha)$ are paired suborbits of A_p at α.

Lemma 3.3. *Let G be a transitive permutation group on Ω with a suborbit $\Delta(\alpha)$. Let N be a transitive normal subgroup of G with index $|G:N|$ coprime to $|\Delta(\alpha)|$. Then $\Delta(\alpha)$ is an orbit of N_α and $G = NG_{\alpha\beta}$ for some $\beta \in \Delta(\alpha)$.*

Proof. Let $\beta \in \Delta(\alpha)$. Then $|\Delta(\alpha)| = |G_\alpha : G_{\alpha\beta}|$. Since $N_\alpha = G_\alpha \cap N$ is normal in G_α, all orbits of N_α on $\Delta(\alpha)$ have a same length $|N_\alpha : N_{\alpha\beta}|$. Since N is transitive, $|G : G_\alpha| = |N : N_\alpha|$. Moreover, $NG_{\alpha\beta}$ is a subgroup of G and

$$|NG_{\alpha\beta}| = \big(|N||G_{\alpha\beta}|\big)/|N \cap G_{\alpha\beta}| = \big(|N||G_{\alpha\beta}|\big)/|N_{\alpha\beta}|.$$

We have

$$\begin{aligned}|G : NG_{\alpha\beta}| &= |G|/|NG_{\alpha\beta}| = \big(|G||N_{\alpha\beta}|\big)/\big(|N||G_{\alpha\beta}|\big) = |G : G_{\alpha\beta}|/|N : N_{\alpha\beta}| \\ &= \big(|G : G_\alpha||G_\alpha : G_{\alpha\beta}|\big)/\big(|N : N_\alpha||N_\alpha : N_{\alpha\beta}|\big) \\ &= |G_\alpha : G_{\alpha\beta}|/|N_\alpha : N_{\alpha\beta}|.\end{aligned}$$

Thus $|G : NG_{\alpha\beta}||N_\alpha : N_{\alpha\beta}| = |G_\alpha : G_{\alpha\beta}|$. Noting that $|G : NG_{\alpha\beta}|$ is a divisor of $|G : N|$ and $|G_\alpha : G_{\alpha\beta}|$ is coprime to $|G : N|$. It follows that $|G : NG_{\alpha\beta}| = 1$ and $|N_\alpha : N_{\alpha\beta}| = |G_\alpha : G_{\alpha\beta}| = |\Delta(\alpha)|$. Hence $G = NG_{\alpha\beta}$ and $\Delta(\alpha)$ is an orbit of N_α. □

It is easy to see that S_{p-2} acts regularly on $V\Gamma(p,z)$. Thus $\Gamma(p,z)$ is in fact a Cayley graph of S_{p-2}. Let K be as in Construction 3.2. If $p \in \{5,7,13\}$ then $p-1$ is a divisor of 24, and so $\mathbf{N}_{\mathrm{S}_p}(K)/K = \mathbf{N}_{\mathrm{S}_{p-1}}(K)/K$ is an elementary abelian 2-group by Lemma 3.1; in particular, $z^2 \in K$. Then the next lemma follows from Corollary 2.9, Lemmas 3.1 and 3.3.

Lemma 3.4. *Let $p \geqslant 5$ be a prime. Let $\Gamma(p,z)$ and K be as in Construction* 3.2. *Then $\Gamma(p,z)$ is a Cayley graph of* S_{p-2}, *and one of the following statements holds*:

(1) *$p = 5$, and $\Gamma(p,z)$ is the complete graph K_6 on* 6 *vertices*;
(2) *$p = 7$, and $\Gamma(p,z)$ is a primitive* S_7-*symmetric graph of valency* 7;
(3) *$p = 13$, and $\Gamma(p,z)$ is a primitive* S_{13}-*symmetric graph of valency* 13;
(4) *$p \notin \{5,7,13\}$, and either*
   (a) *$z^2 \in K$, $\Gamma(p,z)$ is a primitive S_p-symmetric graph of valency p*; *or*
   (b) *$z^2 \notin K$, $\Gamma(p,z)$ is a primitive S_p-half-transitive graph of valency $2p$.*

Moreover, $\Gamma(p,z)$ is either A_p-symmetric or A_p-half-transitive depending on whether or not $z^2 \in K$.

The following lemma shows that the automorphism group of $\Gamma(p,z)$ is equal to S_p except for $p = 5$.

Lemma 3.5. *If $p > 5$ is a prime, then $\mathsf{Aut}\,\Gamma(p,z) = S_p$.*

Proof. Let $\Gamma = \Gamma(p,z)$. Then $S_p \leqslant \mathsf{Aut}\,\Gamma \leqslant \mathrm{Sym}(\Omega)$, and $\mathsf{Aut}\,\Gamma$ is a primitive permutation group on Ω since S_p is primitive on Ω. Suppose that $\mathsf{soc}(\mathsf{Aut}\,\Gamma) \neq \mathsf{soc}(S_p) = A_p$. Choose a subgroup of $\mathsf{Aut}\,\Gamma$, say V, which is minimal with respect to $S_p \leqslant V \leqslant \mathsf{Aut}\,\Gamma$ and $\mathsf{soc}(V) \neq A_p$. Then $V \neq S_p$. Let U be a maximal subgroup of V with $S_p \leqslant U$. Then $\mathsf{soc}(U) = A_p$ and $U \neq V$ by the choice of V and U, and hence $U = S_p$. That is, S_p is a maximal subgroup of V. Noting that both S_p and V are primitive subgroups of $\mathrm{Sym}(\Omega)$, such pairs (S_p, V) can be read out from [11, Table III]. Since p is a prime and $|\Omega| = (p-2)!$, we conclude that either $p = 7$, $\mathsf{soc}(V) = A_9$ and $\mathsf{soc}(V)_\alpha = \mathrm{P}\Gamma\mathrm{L}(2,8)$, or $\mathsf{soc}(V) = A_{p+1}$ and $\mathsf{soc}(V)_\alpha = \mathrm{PSL}_2(p)$ for a prime $p > 5$, where $\alpha \in \Omega$.

Now we consider the actions of V_α and $\mathsf{soc}(V)_\alpha$ on the neighborhood $\Gamma(\alpha)$ of α in Γ. Let G be such that $A_p \leqslant G \leqslant S_p$. Then, by Construction 3.2 and Lemma 3.4, either $|\Gamma(\alpha)| = p$ and the stabilizer G_α is transitive on $\Gamma(\alpha)$, or $|\Gamma(\alpha)| = 2p$ and $\Gamma(\alpha) = \Delta(\alpha) \cup \Delta^*(\alpha)$, where $\Delta(\alpha)$ and $\Delta^*(\alpha)$ are two G_α-orbits of length p on $\Gamma(\alpha)$. Since $G \leqslant V \leqslant \mathsf{Aut}\,\Gamma$ and $\mathsf{soc}(G) \leqslant \mathsf{soc}(V) \leqslant \mathsf{Aut}\,\Gamma$, both V and $\mathsf{soc}(V)$ are transitive on $V\Gamma$ and on $E\Gamma$. We conclude that V_α has an orbit on $\Gamma(\alpha)$ of length p or $2p$, and so does $\mathsf{soc}(V)_\alpha$. In particular, each one of V_α and $\mathsf{soc}(V)_\alpha$ has a subgroup with index p or $2p$. By checking the subgroups of $\mathrm{P}\Gamma\mathrm{L}(2,8)$ in the Atlas [3], we know that $\mathrm{P}\Gamma\mathrm{L}(2,8)$ has no subgroups of index 7 or 14. It follows that $\mathsf{soc}(V) = A_{p+1}$ and $\mathsf{soc}(V)_\alpha = \mathrm{PSL}_2(p)$.

If $p \neq 7$ and $p \neq 11$, then $\mathsf{soc}(V)_\alpha = \mathrm{PSL}_2(p)$ has no subgroups of index p or $2p$ (see [8], for example), a contradiction. Thus $p = 7$ or 11, and $\mathsf{soc}(V) = A_8$ or A_{12}, respectively. In this case $\mathrm{PSL}_2(p)$ is not a maximal subgroup of A_{p+1}. We have $V = S_8$ or S_{12} since V is primitive on Ω. Then (V, V_α) is one of $(S_8, \mathrm{PGL}_2(7))$ and $(S_{12}, \mathrm{PGL}_2(11))$. Inspecting in [3] the subgroups of $\mathrm{PGL}_2(p)$ for $p = 7$ or 11, we conclude that $\mathrm{PGL}_2(p)$ has no subgroups of index p. It follows that $V_\alpha = \mathrm{PGL}(2,p)$ is transitive on $\Gamma(\alpha) = \Delta(\alpha) \cup \Delta^*(\alpha)$ and $|\Gamma(\alpha)| = 2p$. In particular, $\Delta(\alpha) \neq \Delta^*(\alpha)$. Then $p \neq 7$ by Lemma 3.4(3). Thus $p = 11$. Then $|V_\alpha : V_{\alpha\beta}| = |\Gamma(\alpha)| = 2p = 22$ for any $\beta \in \Gamma(\alpha)$. So $|V_{\alpha\beta}| = 60$. It implies that $A_5 \cong V_{\alpha\beta} \leqslant \mathsf{soc}(V_\alpha) = \mathrm{PSL}(2,11)$. It follows that $\mathsf{soc}(V_\alpha)$ has two orbits of length 11 on $\Gamma(\alpha)$.

Now let $G = S_{11}$. Then $\mathbb{Z}_{11} \rtimes \mathbb{Z}_{10} \cong G_\alpha \leqslant V_\alpha$, and G_α has exactly two orbits $\Delta(\alpha)$ and $\Delta^*(\alpha)$ on $\Gamma(\alpha)$. Let L be a subgroup of G_α with $L \cong \mathbb{Z}_{11}$. Then $\Delta(\alpha)$ and $\Delta^*(\alpha)$ are L-orbits. It is easily shown that $L \leqslant \mathsf{soc}(V_\alpha)$. Then it follows that $\Delta(\alpha)$ and $\Delta^*(\alpha)$ are orbits of $\mathsf{soc}(V_\alpha)$ on $\Gamma(\alpha)$. Thus $\mathbb{Z}_{11} \rtimes \mathbb{Z}_{10} \cong G_\alpha \leqslant (V_\alpha)_{\Delta(\alpha)} = \mathsf{soc}(V_\alpha) = \mathrm{PSL}(2,11)$. In particular, $\mathrm{PSL}(2,11)$ contains an element of order 10, which is impossible, again a contradiction. Therefore, $\mathsf{soc}(\mathsf{Aut}\,\Gamma) = \mathsf{soc}(S_p) = A_p$, and so $\mathsf{Aut}\,\Gamma = S_p$. □

Proof of Theorem 1.3. By [9, Proposition 5.3], there exists one primitive half-transitive graph of valency 14. Thus $7 \in \mathcal{V}_{\mathrm{ph}}$. Assume that $p > 7$ is a prime and $p \neq 13$. Take a p-cycle $x \in S_p$ and a $(p-1)$-cycle $y \in S_{p-1}$ such that y normalizes $\langle x\rangle$. Set $K = \langle y\rangle$. Since $p > 7$ and $p \neq 13$, by Lemma 3.1, $\mathbf{N}_{S_p}(K)/K$ is not an elementary 2-group. In particular, there exists $z \in \mathbf{N}_{S_p}(K) \setminus K$ with $z^2 \notin K$. Then by Lemmas 3.4 and 3.5 the graph $\Gamma(p,z)$ given by Construction 3.2

is a primitive half-transitive graph of valency $2p$. Thus $p \in \mathcal{V}_{\mathrm{ph}}$. This completes the proof of Theorem 1.3. □

Remarks. (1) A graph is said to be 2-arc-transitive if its automorphism group acts transitively on the set of its 2-arcs. It is well known that a vertex-transitive graph Γ is 2-arc-transitive if and only if the stabilizer of $\alpha \in V\Gamma$ in $\mathsf{Aut}\,\Gamma$ is 2-transitive on the neighborhood of α in Γ. It is easily shown that the graphs $\Gamma(p, z)$ with $z^2 \in K$ are 2-arc-transitive; further, if $p \geqslant 7$ then $\mathsf{Aut}\,\Gamma(p, z)$ is regular on the set of 2-arcs.

(2) The automorphism of each primitive half-transitive graph given by Construction 3.2 has a solvable stabilizer. Then an interesting problem arises: Is there a primitive half-transitive graph of valency twice a prime such that its automorphism group has insolvable stabilizers?

(3) We know affirmatively that the first two primes and 4 are not members of $\mathcal{V}_{\mathrm{ph}}$. How about 5, 13 and other small positive integers?

Acknowledgments

The author is grateful to the anonymous referees for helpful comments.

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